Counting unicellular maps under cyclic symmetries

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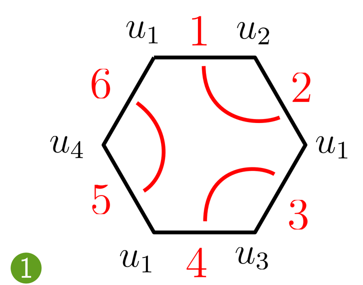
Abstract

The Harer-Zagier formula is a remarkable encoding of the distribution of genus in unicellular maps. It gives a simple product form for the corresponding generating function after a change of the monomial basis $(X^i)_{i=0}^n$ involving Eulerian polynomials.

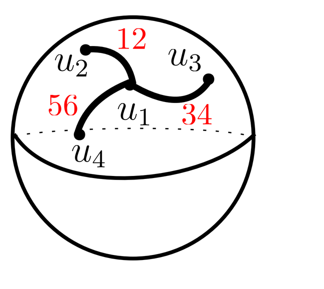
Unicellular maps correspond to matchings of a polygon and carry a natural cyclic action induced by rotation. We prove generalizations of the Harer-Zagier formula counting maps that are fixed by given rotations; we use Steingrimmson's colored Eulerian polynomials and representation theoretic techniques. The leading terms of our formulas recover a known cyclic sieving phenomenon (CSP) for noncrossing matchings.

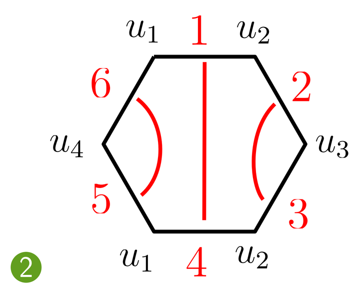
Unicellular maps are matchings

There are 5 orbits of the 15 matchings of a hexagon:

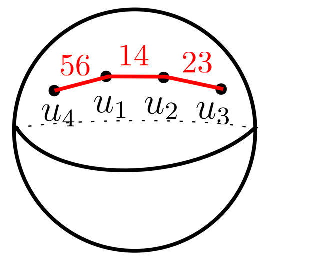


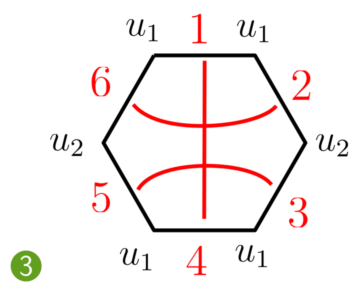
1. $\mathrm{Sym} = \mathbb{Z}_3,\ \# = 2$, $g = 0,\ V = 4$, $d_3 = 0$, $v_3 = 1$, $\{u_1\}, \{u_2, u_3, u_4\}$



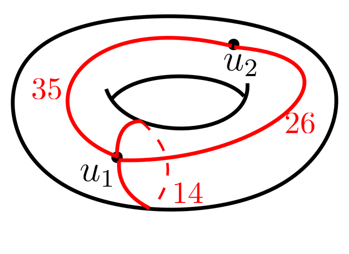


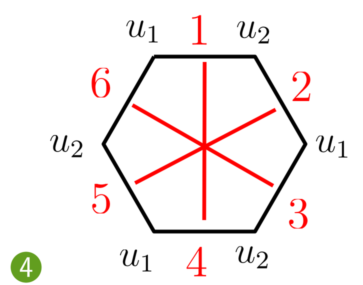
2. $\mathrm{Sym} = \mathbb{Z}_2,\ \# = 3$, $g = 0,\ V = 4$, $d_2 = 1$, $v_2 = 2$, $\{u_1, u_2\}, \{u_3, u_4\}$



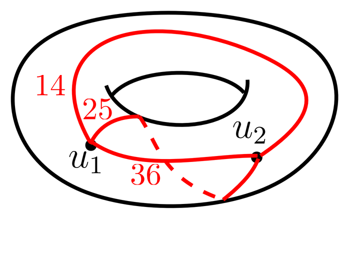


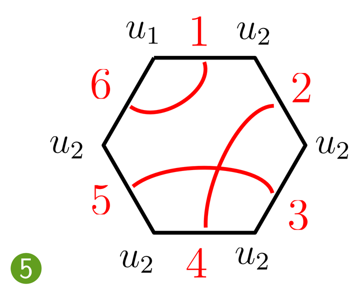
3. $\mathrm{Sym} = \mathbb{Z}_2,\ \# = 3$, $g = 1,\ V = 2$, $d_2 = 1$, $v_2 = 0$, $\{u_1\}, \{u_2\}$



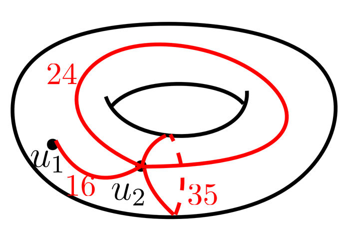


4. $\mathrm{Sym} = \mathbb{Z}_6,\ \# = 1$, $g = 1,\ V = 2$, $d_2 = 3,\ d_3 = 0$, $v_2 = 1,\ v_3 = 0$, $[\{u_1, u_2\}]\ [\{u_1\}, \{u_2\}]$





5. $\mathrm{Sym} = \mathbb{Z}_1,\ \# = 6$, $g = 1,\ V = 2$



The Harer-Zagier formula

- A map is an embedding of a graph G on a surface such that its complement is a union of *cells*.
- A unicellular map with n edges corresponds to a complete matching of the sides of a $2n$-gon.
- The genus of the resulting surface is related to the number V of vertices of G via $V = n + 1 - 2g$.
- The Harer-Zagier numbers $\varepsilon_g(n)$ count (classes of) unicellular maps with n edges and of genus g.
- The normalized Eulerian polynomials $\Phi_n(X)$ are defined in the following two equivalent ways:
$$\Phi_n(X) := \frac{\sum_{k=0}^{n-1} A(n,k)X^k}{(1-X)^{n+1}} = \sum_{k=0}^{\infty}(k+1)^n X^k,$$
where $A(n,k)$ are the *Eulerian numbers* (i.e. the numbers of permutations in S_n with k descents).
- The Harer-Zagier formula below gives a simpler form for the generating function
$$\sum_{g\geq 0}\varepsilon_g(n)\cdot X^{n+1-2g}$$
after the substitution (base change) $X^k \leftrightarrow \Phi_k(X)$.

Theorem (Harer-Zagier) *The numbers $\varepsilon_g(n)$ that count unicellular maps with n edges satisfy*
$$\sum_{g\geq 0}\varepsilon_g(n)\cdot\Phi_{n+1-2g}(X) = (2n-1)!!\cdot\frac{(1+X)^n}{(1-X)^{n+2}}.$$

Unicellular maps with $n = 3$ edges

- In the first column we have listed all 15 matchings of a $(2\cdot 3)$-gon. From that data we can write:
$$\sum_{g\geq 0}\varepsilon_g(3)\cdot X^{3+1-2g} = 5X^4 + 10X^2.$$
- The Eulerian polynomials give us
$$\Phi_2(X) = \frac{1+X}{(1-X)^3},\ \Phi_4(X) = \frac{1+11X+11X^2+X^3}{(1-X)^5}.$$
- To verify the formula we compute
$$5\cdot\Phi_4(X) + 10\cdot\Phi_2(X) =$$
$$\frac{5+55X+55X^2+5X^3}{(1-X)^5} + \frac{10-10X-10X^2+10X^3}{(1-X)^5} =$$
$$15\cdot\frac{1+3X+3X^2+X^3}{(1-X)^5}.$$

Generalized Harer-Zagier numbers

We wish to study symmetric unicellular maps; i.e. (complete) matchings $\boldsymbol{\pi}$ of a $2n$-gon that are invariant under some power of the rotation $\Psi := e^{2\pi i/2n}$.

For any pair of numbers $m > 1, N \geq 1$ such that $mN = 2n$ and $\Psi^N(\boldsymbol{\pi}) = \boldsymbol{\pi}$ (i.e. $\boldsymbol{\pi}$ has at least m-fold symmetry) we define statistics:

- $v_m(\boldsymbol{\pi})$ is the number of *free* (hence, of size m) Ψ^N-orbits of vertices of the $2n$-gon.
- For m even, $d_m(\boldsymbol{\pi})$ is the number of *size*$=m/2$ Ψ^N-orbits of centrally symmetric (i.e. connecting sides i and $n+i$ of the $2n$-gon) pairs of the matching. [[For m odd, $d_m(\boldsymbol{\pi}) = 0$]]

The generalized Harer-Zagier numbers $\varepsilon_{d,v}(m,N)$ count the (complete) matchings $\boldsymbol{\pi}$ of the $2n$-gon such that $\Psi^N(\boldsymbol{\pi}) = \boldsymbol{\pi}$, $v_m(\boldsymbol{\pi}) = v$, and $d_m(\boldsymbol{\pi}) = d$.

Main Theorem and Example

The normalized, colored Eulerian polynomials $\Phi_{m,N}(X)$ are defined in the following two equivalent ways:
$$\Phi_{m,N}(X) = \frac{\sum_{k=0}^{N} A(m,N,k)X^k}{(1-X)^{N+1}} = \sum_{k=0}^{\infty}(mk+1)^N X^k,$$
where $A(m,N,k)$ is the number of Steingrimmson's m-colored N-permutations with k descents.

Theorem (D.) *With notations (m,N), $\Phi_{m,N}(X)$, the numbers $\varepsilon_{d,v}(m,N)$ may be calculated via*
$$\sum_{v\geq 0}\varepsilon_{d,v}(m,N)\cdot\Phi_{m,v}(X) =$$
$$\binom{N}{d}(N-d-1)!!\,m^{\frac{N-d}{2}}\cdot\frac{1}{1-X}\left(\frac{1+X}{1-X}\right)^{\frac{N-d}{2}+\mathbf{1}_{m=2}},$$

For example, for $m = 2, N = 3$ the first column lists 7 matchings $\boldsymbol{\pi}$ that satisfy $\Psi^N(\boldsymbol{\pi}) = \boldsymbol{\pi}$; six with $d_2(\boldsymbol{\pi}) = 1$ and one with $d_2(\boldsymbol{\pi}) = 3$.

- For $d = 1$, we have $\sum_{v\geq 0}\varepsilon_{1,v}(2,3)\cdot X^v = 3X^2 + 3.$
- The colored Eulerian polynomials give us
$$\Phi_{2,2}(X) = \frac{1+6X+X^2}{(1-X)^3},\ \text{while}\ \Phi_{2,0}(X) = \frac{1}{1-X}.$$
- $$3\cdot\Phi_{2,2}(X) + 3\cdot\Phi_{2,0}(X) = \frac{6}{1-X}\left(\frac{1+X}{1-X}\right)^2.$$

Proof Ideas building on Zagier

- A unicellular map $\boldsymbol{\pi}$ can be identified with a factorization $\sigma\alpha c = \mathbf{1}$ in S_{2n}, where σ is the *fixed point free* involution corresponding to the matching, c is the long cycle $(123\cdots 2n)$ and the number of vertices V equals the number of cycles of α.
- The third map in the first column corresponds to
$$(14)(26)(35)\cdot(1245)(36)\cdot(123456) = \mathbf{1}.$$
- Zagier's second proof of the Harer-Zagier formula utilizes the Frobenius lemma, which turns our generating function into a character sum in S_{2n}:
$$\sum_{g\geq 0}\varepsilon_g(n)X^{n+1-2g} =$$
$$\frac{(2n-1)!!}{(2n)!}\cdot\sum_{\chi\in\widehat{S_{2n}}}\chi(\sigma)\chi(c)\cdot\widetilde{\chi}\left(\sum_{w\in S_{2n}} wX^{\mathrm{cyc}(w)}\right).$$
- It can be rewritten via Jucys-Murphy elements, and the sum
$$\sum_{w\in S_{2n}} wX^{\mathrm{cyc}(w)} = X(X+J_2)(X+J_3)\cdots(X+J_{2n}),$$
has normalized character values that are binomials in X.
- Such binomials in X are related via a change of basis to the Eulerian polynomials or $\Phi_n(X)$:
$$\sum_{k=1}^{n}\varepsilon_k X^k = \sum_{k=1}^{n} b_k\binom{X+n-k}{k}\quad\text{iff}$$
$$(1-X)^{n+1}\sum_{k=1}^{n}\varepsilon_k\Phi_k(X) = \sum_{k=1}^{n} b_k X^{k-1}.$$
- Only the hook characters χ_k are non-zero on c and
$$\sum_{\chi\in\widehat{S_{2n}}}\chi(\sigma)\chi(c)X^{\frac{c_\chi}{2n}} = \sum_{k=0}^{2n-1}\chi_k(\sigma)(-1)^k X^k,$$
equals the characteristic polynomial of a fixed point free involution σ, namely $(1-X)^{n-1}\cdot(1+X)^n$.
- Symmetric unicellular maps correspond to factorizations $\sigma\alpha c = \mathbf{1}$ fixed under *simultaneous* conjugation by c^N. These become factorizations of a Coxeter element in the group $G(m,1,N)$ of m-colored N-permutations.
- All above ideas generalize almost directly in $G(m,1,N)$.
- Fixed point free involutions break in many conjugacy classes in $G(m,1,N)$ and must be distinguished by the statistic $d_m(\boldsymbol{\pi})$.
- The topological genus g must be replaced by the combinatorial statistic $v_m(\boldsymbol{\pi})$.

References

[1] S. K. Lando, A. K. Zvonkin. *Graphs on Surfaces and Their Applications.* Springer, 2004

[2] A. R. Miller. *On Foulkes characters.* Math. Ann. 381.3-4 (2021).