The Poincaré-extended ab-index

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Motivation: Zeta Functions

Zeta functions are used in group theory and can be used to capture discrete information about groups. There are many kinds of zeta functions, one is an *Igusa local zeta functions*.

- Maglione–Voll studied Igusa local zeta functions defined by products of linear polynomials, and proved that these zeta functions have a simple combinatorial form.
- They were interested in counting the poles of a specialization of this function and their combinatorial form showed that the only pole was $t = 1$.
- They conjectured that the multiplicity of the pole was the rank of the associated hyperplane arrangement [4].

Main Theorem (Version 1)

If P is an R-labeled poset, then the coefficients of the Poincaré-extended $\mathbf{ab}$-index are nonnegative.

Corollary (DBSM)

Maglione-Voll's conjecture is true.

Poincaré-extended $\mathbf{ab}$-index

Let $\mathcal{L}$ be a graded poset (i.e. a ranked poset with $\hat{0}$ and $\hat{1}$) and μ the Möbius function of $\mathcal{L}$. The **weight** of a chain records the ranks of elements of $\mathcal{C}$ as a polynomial in noncommuting variables $\mathbf{a}$ and $\mathbf{b}$, i.e. if $\mathcal{C}$ is a chain with $\mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b}) = m_1 \dots m_k$ where

$$m_i = \begin{cases} \mathbf{b} & \text{if there is an element of rank } i \text{ in } \mathcal{C} \\ (\mathbf{a} - \mathbf{b}) & \text{else}. \end{cases}$$

The **(Poincaré-)extended $\mathbf{ab}$-index** of a graded poset $\mathcal{L}$ is a polynomial in $\mathbb{Z}[t]\langle \mathbf{a}, \mathbf{b} \rangle$ defined by

$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}:\text{chain in } \mathcal{L}\setminus\{\hat{1}\}} \begin{pmatrix} \text{product of Poincaré} \\ \text{polynomials of} \\ \text{intervals in } \mathcal{C} \end{pmatrix} \cdot \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b}).$$

Note. If $\mathcal{L}$ is the lattice of flats of a real hyperplane arrangement, this product of Poincaré polynomials is a y-refinement of the *Bayer-Sturmfels relation*.

Generalized Descents. Let P be an R-labeled poset of rank n. For a maximal chain M with edge labels $v = v_1 \dots v_n$ and subset of the edges E, let $v' = v_1' \dots v_n'$ be the sequence we get by multiplying v_i by -1 if $i \in E$. We can record the descents of this new sequence of edge labels as a monomial in $\mathbf{a}$'s and $\mathbf{b}$'s (where $\mathbf{b}$ in position i indicates a descent from i to $i + 1$). We call this $\mathbf{ab}$ monomial $\mathrm{mon}(\mathcal{M}, E)$.

Example 1

The extended $\mathbf{ab}$-index of the lattice of flats of the Type A reflection arrangement has nonnegative coefficients. In rank 2, the lattice of flats has extended $\mathbf{ab}$-index

$$\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (2 + 3y)\mathbf{ab} + y^2\mathbf{bb}.$$

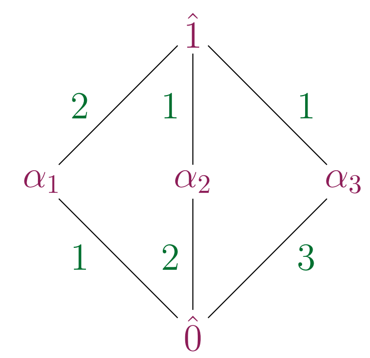
Our main theorem provides a combiantorial interpretation for the coefficients of the Poincaré-extended $\mathbf{ab}$-index. It states:

Main Theorem (Version 2)

Let P be an R-labeled poset of rank n. Then

$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{(\mathcal{M},E)} y^{\#E} \cdot \mathrm{mon}(\mathcal{M}, E)$$

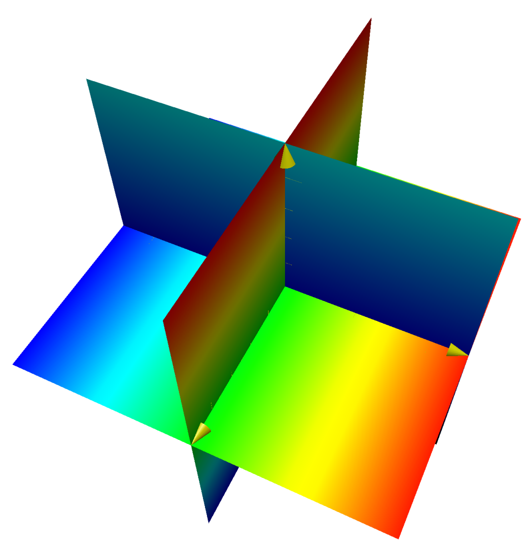
where the sum ranges over maximal chains $\mathcal{M}$ subsets $E \subseteq \{1, \dots, n\}$.

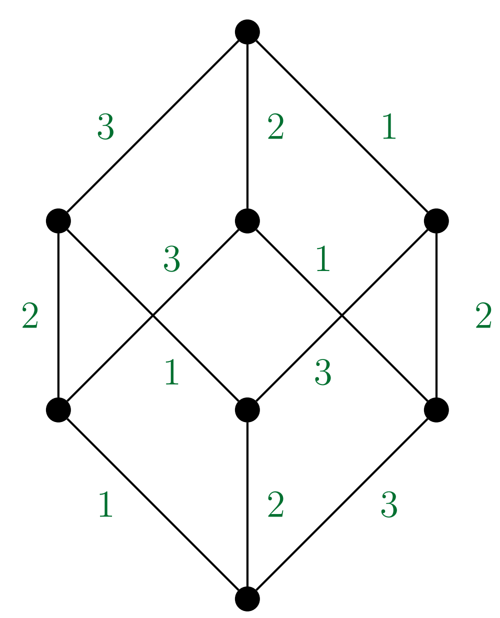


E	$y^{\#E}$	$\hat{0} \lessdot \alpha_1 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_2 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_3 \lessdot \hat{1}$
$\{\}$	1	$\mathbf{aa}$	$\mathbf{ab}$	$\mathbf{ab}$
$\{1\}$	y	$\mathbf{ba}$	$\mathbf{ba}$	$\mathbf{ba}$
$\{2\}$	y	$\mathbf{ab}$	$\mathbf{ab}$	$\mathbf{ab}$
$\{1,2\}$	y^2	$\mathbf{bb}$	$\mathbf{ba}$	$\mathbf{ba}$

Example 2

Here is the Boolean arrangement in three dimensions (left) with its lattice of flats (right). A construction of Björner tells us that every geometric lattice admits an R-labeling, so the coefficients of the Poincaré-extended $\mathbf{ab}$-index are nonnegative.





The extended $\mathbf{ab}$-index of this poset is

$$\mathbf{aaa} + (3y + 2)\mathbf{aab} + (3y^2 + 6y + 2)\mathbf{aba} + (3y^2 + 3y + 1)\mathbf{abb} + (y^3 + 3y^2 + 3y)\mathbf{baa} + (2y^3 + 6y^2 + 3y)\mathbf{bab} + (2y^3 + 3y^2)\mathbf{bba} + y^3\mathbf{bbb}$$

Connection to the $\mathbf{cd}$-index

Let m be a monomial in $\mathbf{a}$ and $\mathbf{b}$. Define a transformation ω that first sends $\mathbf{ab}$ to $\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb}$, then all remaining $\mathbf{a}$'s to $\mathbf{a} + y\mathbf{b}$ and all remaining $\mathbf{b}$'s to $\mathbf{b} + y\mathbf{a}$.

Theorem (DBSM)

The ω map sends the $\mathbf{ab}$-index of $\mathcal{L}$ to its Poincaré-extended $\mathbf{ab}$-index.

- When P is the lattice of flats of an *oriented matroid*, setting $y = 1$ recovers Billera–Ehrenborg–Readdy's ω map relating the face poset of an oriented matroid to its lattice of flats [1],
- When P is a *distributive lattice*, setting $y = r+1$ recovers the ω_r map of Ehrenborg (related to the "r-Signed Birkoff poset" from Hsiao) [2], and
- When P is the lattice of flats of an *oriented interval greedoid*, setting $y = 1$ recovers the ω map of Saliola-Thomas [5].

Theorem (DBSM)

For an R-labeled poset P, there exists a polynomial $\Phi(P; \mathbf{c}_1, \mathbf{c}_2, \mathbf{d})$ in noncommuting variables $\mathbf{c}_1, \mathbf{c}_2, \mathbf{d}$ such that

$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \Phi(P;\ \mathbf{a} + y\mathbf{b},\ \mathbf{b} + y\mathbf{a},\ \mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{ba}).$$

Connection to QSym

If S is the set of positions of $\mathbf{b}$'s in $m \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$, then send m to F_S, *Gessel's fundamental quasisymmetric function.* Since the ring of symmetric functions sits inside QSym, we there is an natural extension of ω to the ring of symmetric functions.

Together with Darij Grinberg, we conjectured that ω preserves Schur positivity. Last month, Ricky Liu proved and a strengthened version of our conjecture using Kronecker products (denoted by $*$).

Theorem (Liu)

For any partition $\lambda \vdash n$, $\omega(s_\lambda) = \sum_{k=0}^{n-1} (s_\lambda * s_{(n-k,1^k)})\, y^k$.

Note. This is closely-related to the q-refinement of QSYM studied by Grinberg–Vassilieva [3].

References

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