Generalized Heawood Graphs and Triangulations of Tori

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Colorings on Surfaces

The four color theorem, dating back to 1852, was proved in 1976 by Appel and Haken after many false proofs and false counterexamples. It is the first major result in mathematics that was proved using a computer.

Theorem (The Four Color Theorem, Appel–Haken 1976)

Any map on a sphere can be colored with four colors such that any two adjacent regions have different colors.

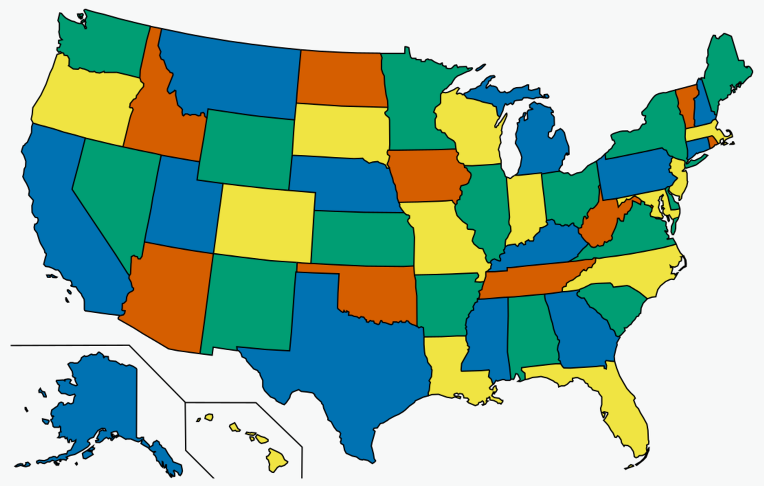


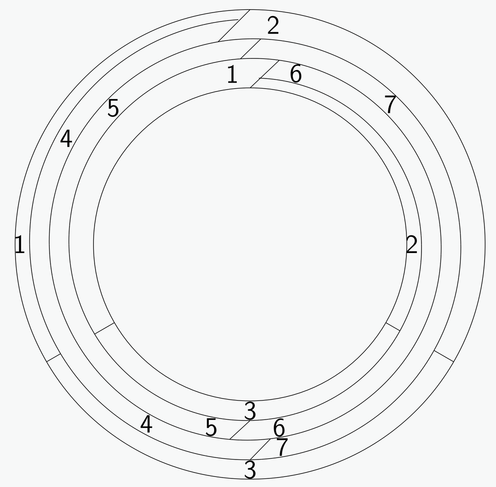
Figure: inductiveload, Wikipedia

In 1890, Heawood found a gap in Kempe's famous false proof from 1879, showed that five colors are sufficient to color a map on a sphere, and started a systematic study of map colorings on surfaces.

Theorem (The Seven Color Theorem, Heawood 1890)

Any map on a torus can be properly colored using seven colors. The number seven is tight.

Heawood's map:
A map on the torus (inner and outer circle are identified) where seven colors are necessary (seven regions where any two are adjacent to each other).



The Heawood Graph

The *Heawood graph* is the 1-skeleton of Heawood's map.
It is a toroidal graph with 14 vertices and 21 edges.

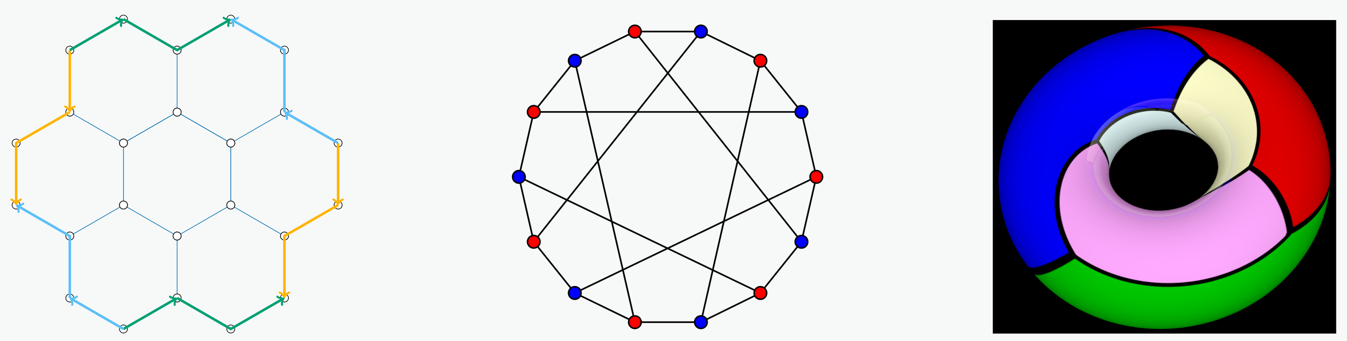
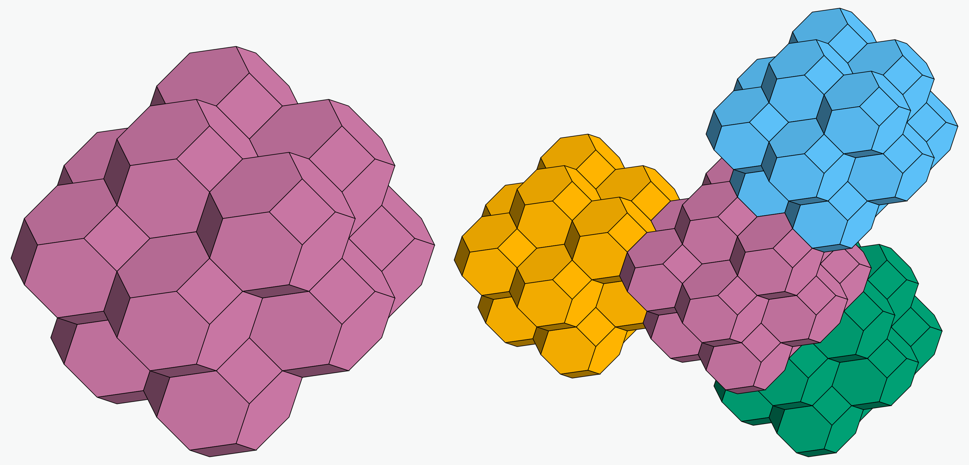


Figure middle: Koko90, Wikipedia, Figure right: PSL27, Wikipedia

Goal of this Presentation

Present generalizations $H_{\mathbf{k}}$ of the Heawood graph in higher dimensions, using permutahedral tilings. The case $\mathbf{k} = (1,1,1)$ recovers the classical Heawood graph.

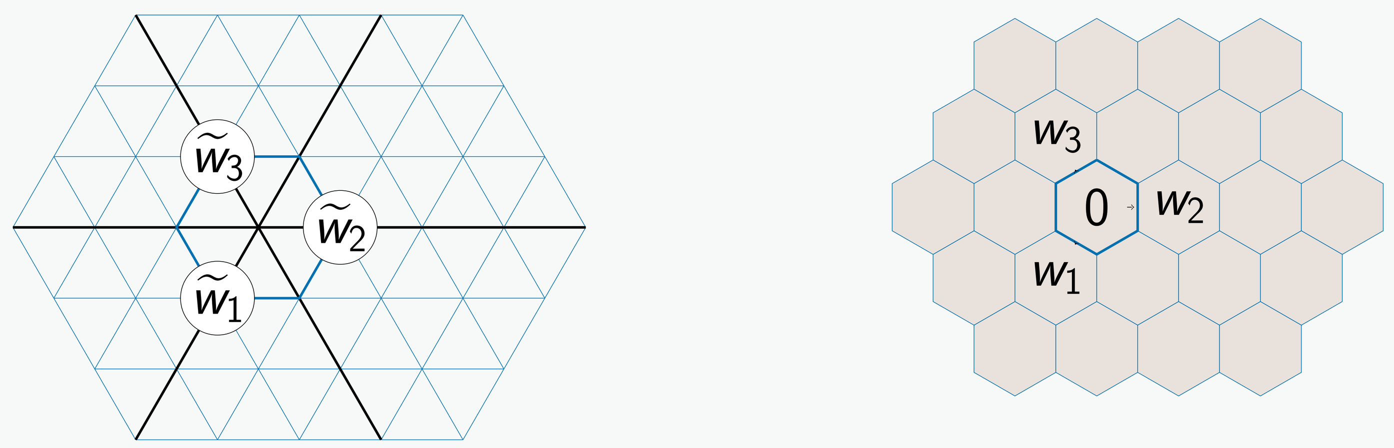


Affine Arrangement and Permutahedral Tiling

For $1 \leq i < j \leq d+1$ and $k \in \mathbb{Z}$, consider the affine hyperplanes

$$H_{ij}^k = \{\mathbf{x} \in \mathbb{R}^{d+1} : \; x_j - x_i = k\}$$

Affine Coxeter arrangement $\widetilde{\mathcal{H}}_d$: restrict this arrangement to $\sum x_i = 0$.



The *permutahedron* Perm_d is the convex hull of permutations of $[d+1]$:

$$\mathrm{Perm}_d = \mathrm{conv}\{(i_1, \dots, i_{d+1}) : \text{ for } \{i_1, \dots, i_{d+1}\} = [d+1]\} \subseteq \mathbb{R}^{d+1}.$$

Let $\mathcal{L}_d$ be lattice of integer linear combinations of $w_1, \dots, w_{d+1}$, where

$$w_i := (d+1)\widetilde{w}_i = (-1, \dots, -1, d, -1, \dots, -1).$$

The *permutahedral tiling* $\mathcal{PT}_d$ is the infinite tiling whose tiles are translates $\mathrm{Perm}_d + v$ of the permutahedron, for $v \in \mathcal{L}_d$.

The Heawood Complex and Graph

For $\mathbf{k} = (k_1, \dots, k_{d+1}) \in \mathbb{N}^{d+1}$, we denote by $M_{\mathbf{k}}$ the matrix

$$M_{\mathbf{k}} = \begin{pmatrix} k_1+1 & -k_2 & & & \\ & k_2+1 & -k_3 & & \\ & & \ddots & \ddots & \\ -k_1 & & & & k_{d+1}+1 \end{pmatrix}$$

The sub-lattice $\mathcal{S}_{\mathbf{k}} \subset \mathcal{L}_d$: linear combinations $a_1 w_1 + \cdots + a_{d+1} w_{d+1}$ such that $(a_1, \dots, a_{d+1})$ is an integer linear combination of the rows of $M_{\mathbf{k}}$.

The *Heawood complex* $\mathcal{HC}_{\mathbf{k}}$ is the polytopal complex of faces of the permutahedral tiling identified up to translation by vectors in $\mathcal{S}_{\mathbf{k}}$.

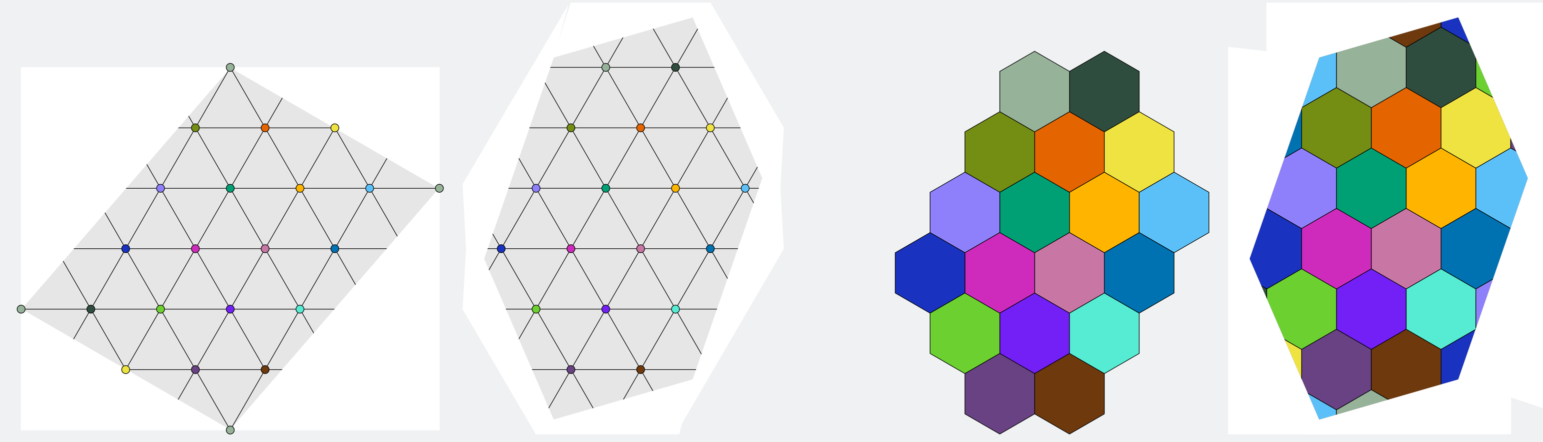
$$\mathcal{HC}_{\mathbf{k}} = \mathcal{PT}_d / \mathcal{S}_{\mathbf{k}}.$$

The *Heawood graph* $H_{\mathbf{k}}$ is the 1-skeleton of the Heawood complex.

Results

Theorem

The generalized Heawood graph $H_{\mathbf{k}}$ is the dual graph of a triangulation of a d-dimensional torus $\mathcal{T}_{\mathbf{k}}$.

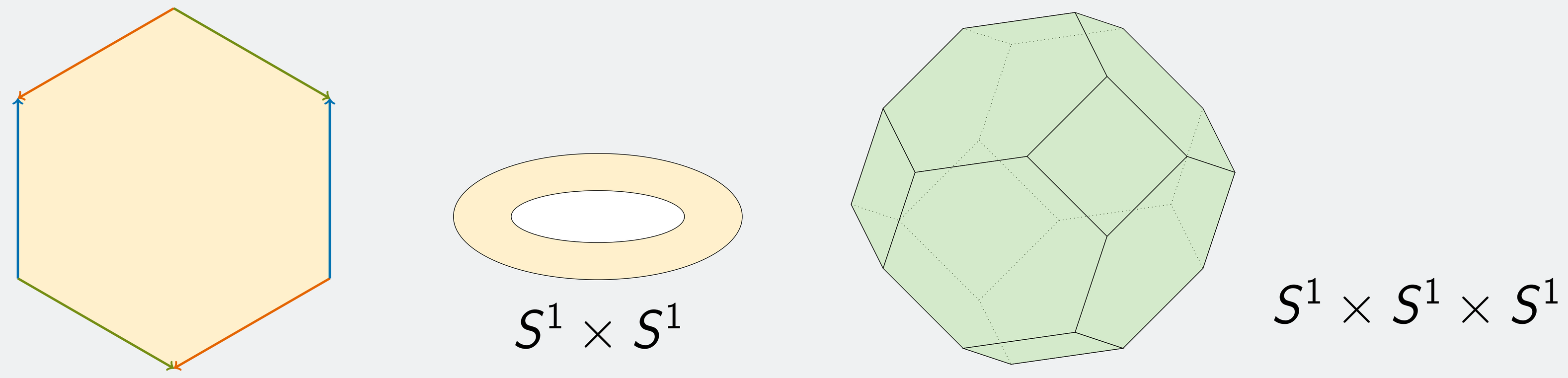


The f-vector $(f_0, f_1, \dots, f_d)$ of $\mathcal{T}_{\mathbf{k}}$ is determined by $f_i = i! \left\{ {d+1 \atop i+1} \right\} D_{\mathbf{k}}$, where $D_{\mathbf{k}} = \det M_{\mathbf{k}} = \prod(k_i+1) - \prod k_i$. and $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind. In particular,

$$f_0 = D_{\mathbf{k}}, \qquad f_d = d!\, D_{\mathbf{k}}, \qquad f_{d-1} = \frac{(d+1)!}{2} D_{\mathbf{k}}.$$

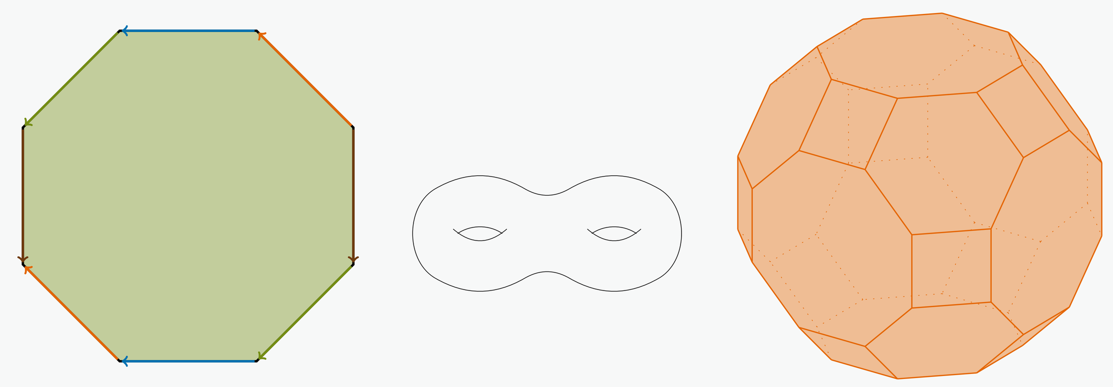
Corollary

The permutahedron Perm_d with opposite facets identified by translation is a topological d-dimensional torus.



An Open Question

What is the topology of other families of polytopes with opposite facets identified by translation? For example permutahedra of type B_n.



Fundamental group?
Homology?
Topology?

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