On the sum of the entries in a character table: Extended abstract

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Introduction

- A linear (complex) representation of a finite group G is a pair (ρ, V), where V is a finite dimensional vector space over $\mathbb{C}$ and $\rho : G \to GL(V)$ is a group homomorphism.
- Let $\mathrm{Irr}(G)$ denotes the set of irreducible representations of G, upto isomorphism and $\mathrm{Conj}(G)$ denotes set of conjugacy classes of G. Fact: $|\mathrm{Conj}(G)| = |\mathrm{Irr}(G)|$.
- Character: $\chi_V : G \to \mathbb{C}$ defined by $g \mapsto trace(\rho(g))$. '
- Character table: the matrix whose rows are indexed by $\mathrm{Irr}(G)$, columns by $\mathrm{Conj}(G)$, and whose (V, C)'th entry is $\chi_V(C)$.
- Character table of the symmetric group S_3 :

	$(1)(2)(3)$	$(1,2)(3)$	$(1,2,3)$
$\chi_{(3)}$	1	1	1
$\chi_{(2,1)}$	2	0	-1
$\chi_{(1,1,1)}$	1	-1	1

Some properties of the character table of a finite group:

- Row sums are always nonnegative integers.
- Column sums are always integers and may be negative, e.g., alternating groups.
- The first column sum is at least as large as any other column sum.

How large is the character degree sum $\Gamma_e(G)$ compared to the character table sum $s(G)$ for a finite group G ?

Strengthening Field's forgotten conjecture, we have the following:

Conjecture. *For all finite groups G, we have $s(G) \geq \Gamma_e(G)$. The equality holds if and only if G is abelian.*

- We verified the conjecture for groups of order ≤ 200 using the `SmallGroups` Library in `GAP`.
- Valid for abelian groups G: here $s(G) = |G|$, since all row sums, except the first one, are 0.
- Valid for any finite group of nilpotency class two.

Recall finite irreducible Coxeter groups: consist of four one-parameter families: symmetric groups S_n, hyperoctahedral groups B_n, demihyperoctahedral groups D_n, dihedral groups and a few exceptional groups.

Theorem (Ayyer-Dey-Paul, 2024). *Conjecture is valid for*

- *finite Coxeter groups.*
- *Complex reflection groups $G(r, q, n)$ if $gcd(q, n) \leq 2$.*

For many natural families of finite groups, it seems that the first column sum in the character table of G dominates the sum of the remaining column sums.

Property S. For a finite group G, we have $2\Gamma_e(G) \geq s(G)$.

- Property S holds for Abelian groups.
- Suppose G satisfies Property S and H is an abelian group. Then $G \times H$ satisfies Property S.
- Let G be a finite group such that all the irreducible characters of G have degrees at most 2. Then G satisfies Property S.
- Property S will not generally hold. Counterexamples occur in groups of orders 64, 125, 128, 160, and 192.
- In fact, for an integer $m > 1$, there exists a group G for which $s(G) > m\Gamma_e(G)$.

Our main result is the following:

Theorem (Ayyer–Dey–Paul, 2024). *Property S holds for all finite irreducible Coxeter groups.*

Conjecture. *Property S holds for all alternating groups.*

Main ingredient: column sums and square roots

Column sums of the character table of finite Coxeter groups are given by the number of square roots of conjugacy class representatives.

Theorem (Frobenius–Schur). *Let G be a finite Coxeter group. For each $g \in G$, we have*

$$|\{x \in G \mid x^2 = g\}| = \sum_{V \in \mathrm{Irr}(G)} \chi_V(g).$$

Involutions, derangements, and character table sum for symmetric groups

Definition. Let g_n be the sum of the columns indexed by conjugacy classes corresponding to derangements in the character table of S_n.

Proposition (Ayyer–Dey–Paul, 2024). *Let s_n be the character table sum and i_n be the number of involutions in S_n. For a positive integer n, we have*

$$s_n = \sum_{k=0}^{n} i_k \, g_{n-k}.$$

Remark. A similar relation holds for B_n.

Remark. The character table sum is asymptotically the same as the number of involutions for S_n, B_n, D_n.

Generating functions

Key observation: An element of S_n with cycle type λ has a square root if only if each even part of λ has even multiplicity.

Proposition (Bessenrodt–Olsson, 2004). *The generating function for the number of columns of the character table of S_n with nonzero sum is*

$$\prod_{i=1}^{\infty} \frac{1}{(1-q^{2i-1})(1-q^{4i})}.$$

We extend the result for generalized symmetric groups. The generalized symmetric group is defined by

$$G(r,1,n) = \mathbb{Z}_r \wr S_n := \{(z_1, \ldots, z_n; \sigma) \mid z_i \in \mathbb{Z}_r, \sigma \in S_n\}.$$

Facts

- $G(1,1,n) = S_n, G(2,1,n) = B_n$.
- The conjugacy classes of $G(r,1,n)$ are indexed by r-partite partitions.
- Example: The conjugacy class of $(\bar{2}, \bar{1}, \bar{1}, \bar{1}, \bar{0}, \bar{2}; (123)(45)(6)) \in G(3,1,6)$ is indexed by $(\emptyset \mid (3,2) \mid (1))$.

Given $\pi = (z_1, z_2, \ldots, z_n; \sigma) \in G(r,1,n)$, define the bar operation as $\overline{\pi} := (-z_1, \ldots, -z_n; \sigma)$.

Theorem (Adin–Postnikov–Roichman, 2010).

$$\sum_{V \in \mathrm{Irr}(G(r,1,n))} \chi_V(g) = |\{\pi \in G(r,1,n) \mid \pi\overline{\pi} = g\}| \quad \forall g \in G(r,1,n).$$

Theorem (Ayyer–Dey–Paul, 2024). *The generating function for the number of conjugacy classes of $G(r,1,n)$ with nonzero column sum is*

$$\begin{cases} \displaystyle\prod_{i=1}^{\infty} \frac{1}{(1-q^{2i-1})(1-q^{4i})(1-q^{i})^{(r-1)/2}} & r \text{ odd}, \\ \displaystyle\prod_{i=1}^{\infty} \frac{1}{(1-q^{2i-1})(1-q^{4i})(1-q^{2i})(1-q^{i})^{(r-2)/2}} & r \text{ even}. \end{cases}$$

Generating function for character table sum

Let

$$o_r(m) = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} (2k-1)!! \, r^k.$$

Theorem (Flajolet 1980, Theorem 2(iib)). *We have*

$$\mathcal{D}(x) = \sum_{n \geq 0} (2n-1)!! \, x^n = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{2x}{\ddots}}},$$

$$\mathcal{R}_r(x) = \sum_{n \geq 0} o_r(n) x^n = \cfrac{1}{1 - x - \cfrac{rx^2}{1 - x - \cfrac{2rx^2}{\ddots}}}.$$

Let $x, x_1, x_2, \ldots$ be a family of commuting indeterminates.

Theorem (Ayyer–Dey–Paul, 2024). *The column sum indexed by $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$ in the character table of S_n is the coefficient of $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ in*

$$\prod_{k \geq 1} \mathcal{D}(2k x_{2k}^2) \mathcal{R}_{2k-1}(x_{2k-1}).$$

Consequently, the generating function of the character table sum is

$$\mathcal{S}(x) = \prod_{k \geq 1} \mathcal{D}(2k x^{4k}) \mathcal{R}_{2k-1}(x^{2k-1}).$$

To extend the above result for $G(r,1,n)$ we need the following:

Theorem (Euler, 1760).

$$\mathcal{F}(x) = \sum_{n \geq 0} n! x^n = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{2x}{1 - \cfrac{2x}{\ddots}}}}}.$$

Theorem (Ayyer–Dey–Paul, 2024). *The generating function (in n) of the character table sum of $G(r,1,n)$ is*

$$\begin{cases} \displaystyle\prod_{k \geq 1} \left(\mathcal{F}(krx^{2k})^{(r-1)/2} \, \mathcal{D}(2krx^{4k}) \, \mathcal{R}_{(2k-1)/r}(rx^{2k-1}) \right) & r \text{ odd}, \\ \displaystyle\prod_{k \geq 1} \left(\mathcal{F}(krx^{2k})^{(r-2)/2} \, \mathcal{D}(2krx^{4k}) \, \mathcal{D}(rkx^{2k}) \, \mathcal{R}_{(2k-1)/r}(rx^{2k-1}) \right) & r \text{ even}. \end{cases}$$

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