Toric and Permutoric Promotion

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Abstract

We introduce *toric promotion* as a cyclic analogue of Schützenberger's promotion operator. Toric promotion acts on the set of labelings of a graph G; it is defined as the composition of certain toggle operators, listed in a natural cyclic order. We provide a surprisingly simple description of the orbit structure of toric promotion when G is a forest. We then consider more general *permutoric promotion* operators, which are defined as compositions of the same toggle operators, but in permuted orders. When G is a path graph, we provide a complete description of the orbit structures of all permutoric promotion operators, showing that they satisfy the cyclic sieving phenomenon.

Toric Promotion

Let $G = (V, E)$ be a graph with n vertices. A *labeling* of G is a bijection $V \to \mathbb{Z}/n\mathbb{Z}$. Let Λ_G be the set of labelings of G. Let $(a\ b)$ be the transposition that swaps a and b. For $i \in \mathbb{Z}/n\mathbb{Z}$, the *toggle operator* $\tau_i \colon \Lambda_G \to \Lambda_G$ is defined by

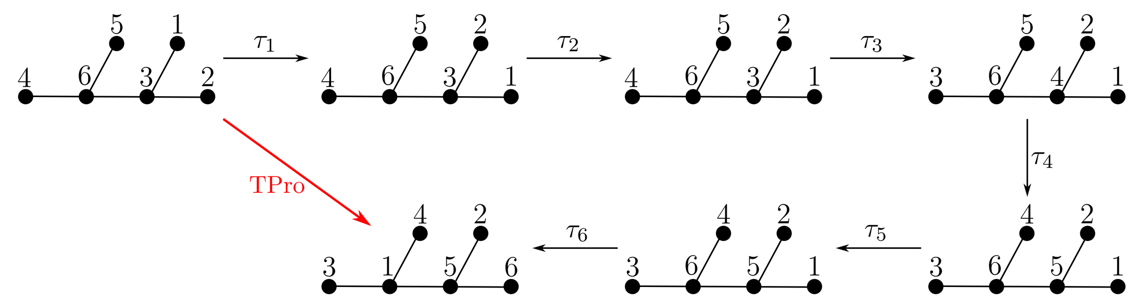
$$\tau_i(\sigma) = \begin{cases} (i\ i+1) \circ \sigma & \text{if } \{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \notin E; \\ \sigma & \text{if } \{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \in E. \end{cases}$$

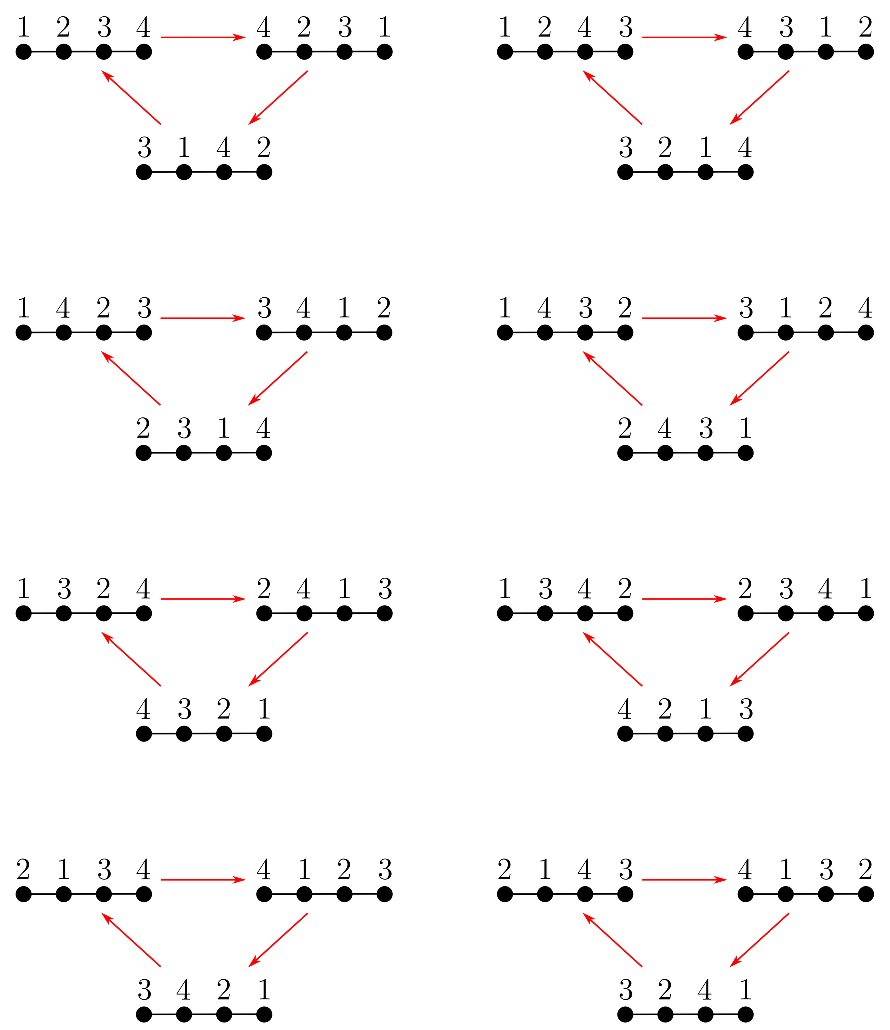
Define *promotion* to be the operator $\text{Pro} \colon \Lambda_G \to \Lambda_G$ given by

$$\text{Pro} = \tau_{n-1} \cdots \tau_2 \tau_1.$$

Toric promotion is the operator $\text{TPro} \colon \Lambda_G \to \Lambda_G$ given by

$$\text{TPro} = \tau_n \tau_{n-1} \cdots \tau_2 \tau_1 = \tau_n \,\text{Pro}.$$





Theorem ([2]). *Let G be a forest with $n \geq 2$ vertices, and let $\sigma \in \Lambda_G$ be a labeling. The orbit of toric promotion containing σ has size $(n-1)t/\gcd(t, n)$, where t is the number of vertices in the connected component of G containing $\sigma^{-1}(1)$. In particular, if G is a tree, then every orbit of* $\text{TPro} \colon \Lambda_G \to \Lambda_G$ *has size $n-1$.*

Permutoric Promotion

Fix a bijection $\pi \colon [n] \to \mathbb{Z}/n\mathbb{Z}$. *Permutoric promotion* is the operator $\text{TPro}_\pi \colon \Lambda_G \to \Lambda_G$ defined by

$$\text{TPro}_\pi = \tau_{\pi(n)} \cdots \tau_{\pi(2)} \tau_{\pi(1)}.$$

We focus on analyzing permutoric promotion when G is the n-vertex path graph Path_n.

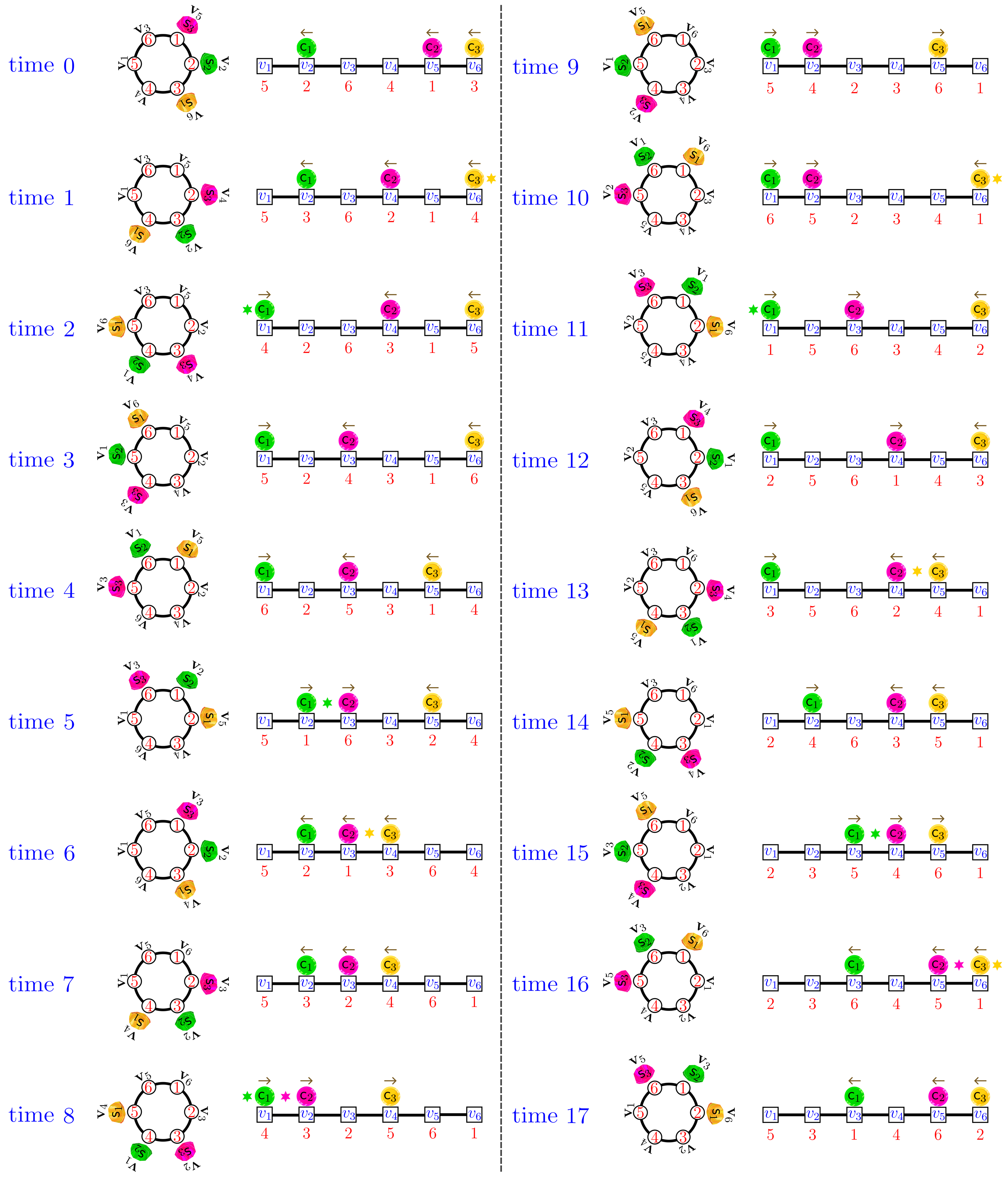
Let X be a finite set, and let $f \colon X \to X$ be an invertible map of order ω. Let $F(q) \in \mathbb{C}[q]$ be a polynomial in the variable q. Following [4], we say the triple $(X, f, F(q))$ *exhibits the cyclic sieving phenomenon* if for every integer k, the number of elements of X fixed by f^k is $F(e^{2\pi i k/\omega})$.

A *cyclic descent* of π^{-1} is an element $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\pi^{-1}(i) > \pi^{-1}(i+1)$ (note that n is permitted to be a cyclic descent).

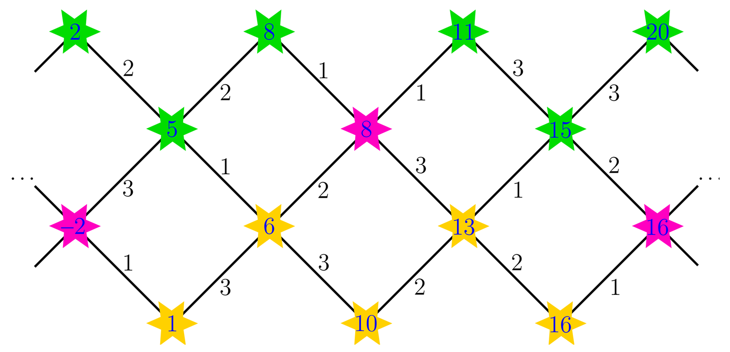
Theorem ([3]). *Let d be the number of cyclic descents of π^{-1}. The order of the permutoric promotion operator $\text{TPro}_\pi \colon \Lambda_{\text{Path}_n} \to \Lambda_{\text{Path}_n}$ is $d(n-d)$. Moreover, the following triple exhibits the cyclic sieving phenomenon:*

$$\left(\Lambda_{\text{Path}_n}, \text{TPro}_\pi, n(d-1)!(n-d-1)![n-d]_{q^d} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q \right).$$

Sliding Stones and Colliding Coins



Collisions



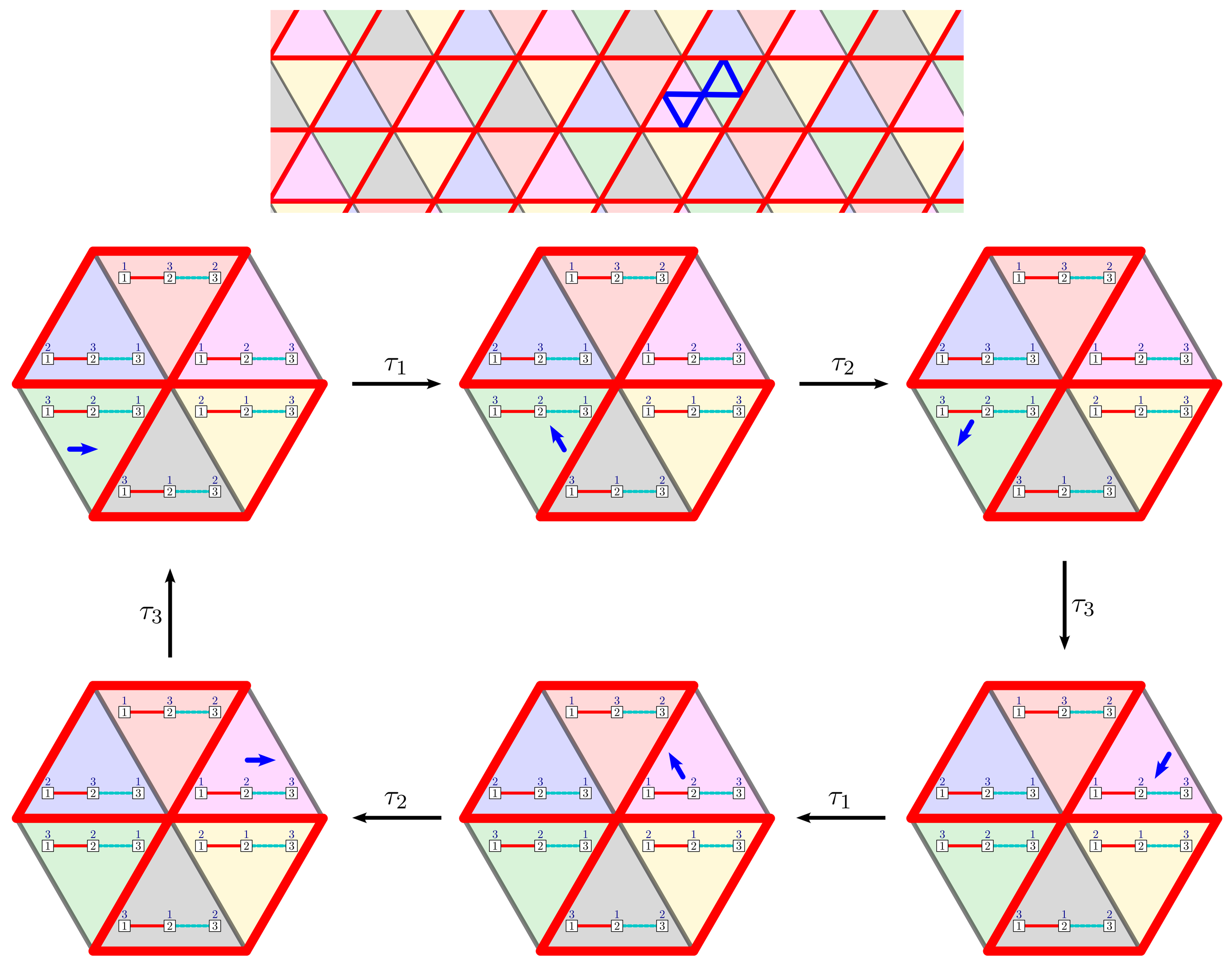
The Hasse diagram of collisions, drawn sideways so that later collisions appear to the right of earlier collisions. Each collision is marked by the time when it occurs. Each edge $\kappa \lessdot \kappa'$ is labeled by the distance between the locations where the collisions κ and κ' occurred.

Lemma ([3]). *In any half-diamond in the Hasse diagram, the two edges have the same label. In any diamond in the Hasse diagram, opposite edges have the same label.*

From an orbit of stones diagrams and coin diagrams, we read transversals of the Hasse diagram of collisions to get an orbit of compositions of n into d parts under rotation. This allows us to relate orbits of labelings under permutoric promotion to orbits of compositions under rotation; the dynamics of the latter are already understood (see, e.g., [4]).

Other Directions

The article [1] shows that toric promotion is a *combinatorial billiard system* in a torus. To be more precise, let $\mathrm{U} = \{(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n : \gamma_1 + \cdots + \gamma_n = 0\}$, and let $\mathcal{H}$ be the set of hyperplanes in U of the form $\{(\gamma_1, \ldots, \gamma_n) \in \mathrm{U} : \gamma_i - \gamma_j = k\}$ for $\{i, j\} \in E$ and $k \in \mathbb{Z}$. Shine a beam of light in the direction of the vector $(1, \ldots, 1, -n+1)$; the beam of light reflects whenever it hits a hyperplane in $\mathcal{H}$. We can discretize the beam of light by only keeping track of the alcove of $\widetilde{\mathfrak{S}}_n$ containing it and the direction it is traveling; thus, we encode the state of the beam of light as an element of $\widetilde{\mathfrak{S}}_n \times \mathbb{Z}/n\mathbb{Z}$. Applying the natural quotient map $\widetilde{\mathfrak{S}}_n \times \mathbb{Z}/n\mathbb{Z} \to \mathfrak{S}_n \times \mathbb{Z}/n\mathbb{Z}$, we recover toric promotion.



Question. *What if we initially shine the beam of light in some direction other than $(1, \ldots, 1, -n+1)$?*

References

[1] A. Adams, C. Defant, and J. Striker. Toric promotion with reflections and refractions. arXiv:2404.03649.

[2] C. Defant. Toric promotion. *Proc. Amer. Math. Soc.*, **151** (2023), 45–57.

[3] C. Defant, R. Madhukara, and H. Thomas. Permutoric promotion: gliding globs, sliding stones, and colliding coins. arXiv:2305.19961.

[4] V. Reiner, D. Stanton, and D. White, The cyclic sieving phenomenon. *J. Combin. Theory Ser. A*, **108** (2004), 17–50.