Rowmotion Markov Chains

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Abstract

Rowmotion is a certain well-studied bijective operator on the distributive lattice $J(P)$ of order ideals of a finite poset P. We introduce the *rowmotion Markov chain* $\mathbf{M}_{J(P)}$ by assigning a probability p_x to each $x \in P$ and using these probabilities to insert randomness into the original definition of rowmotion. More generally, we introduce a very broad family of *toggle Markov chains* inspired by Striker's notion of generalized toggling. We characterize when toggle Markov chains are irreducible, and we show that each toggle Markov chain has a remarkably simple stationary distribution.

We also provide a second generalization of rowmotion Markov chains to the context of semidistrim lattices. Given a semidistrim lattice L, we assign a probability p_j to each join-irreducible element j of L and use these probabilities to construct a rowmotion Markov chain $\mathbf{M}_L$. Under the assumption that each probability p_j is strictly between 0 and 1, we prove that $\mathbf{M}_L$ is irreducible. We also compute the stationary distribution of the rowmotion Markov chain of a lattice obtained by adding a minimum element and a maximum element to a disjoint union of two chains.

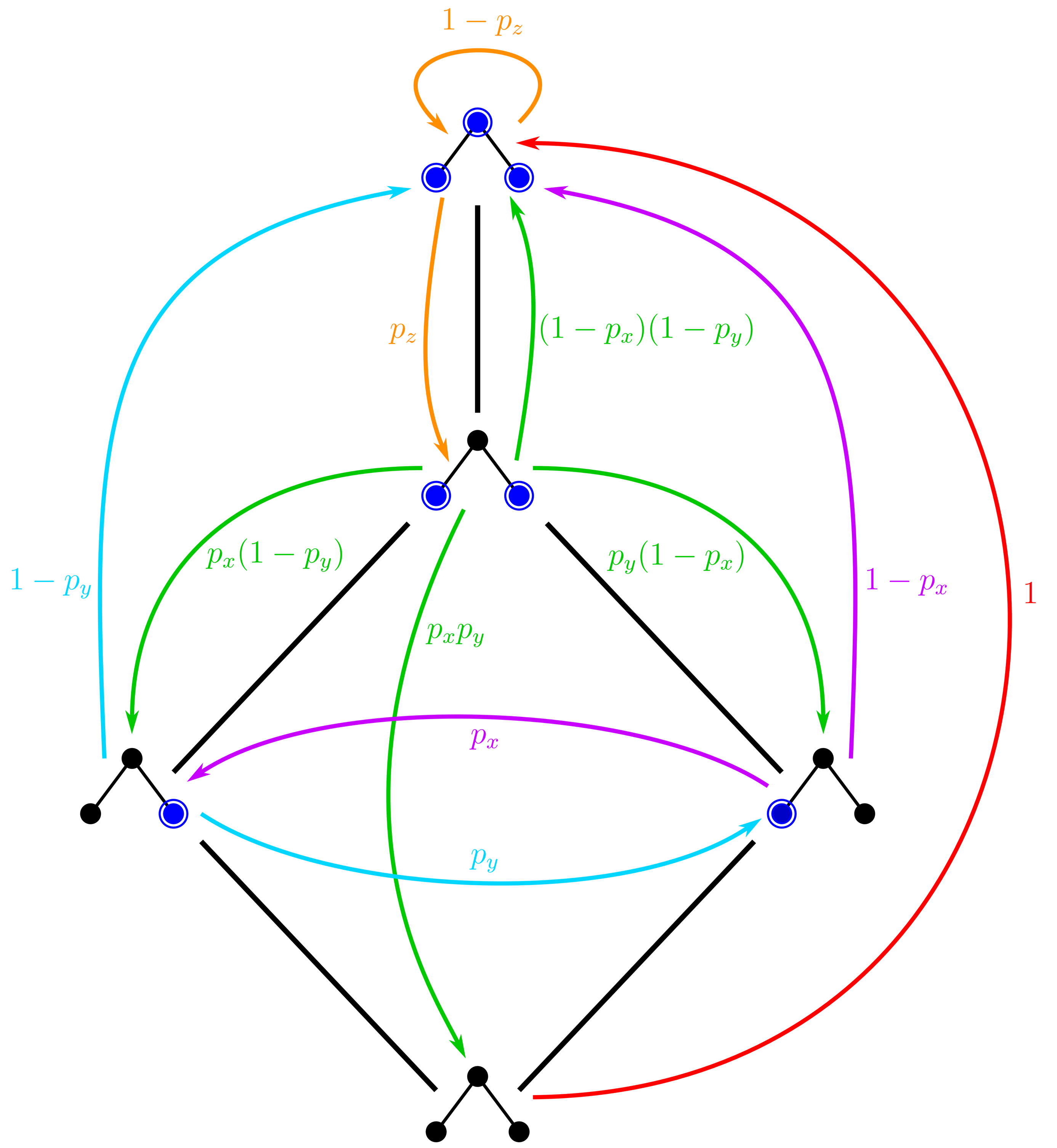
We bound the mixing time of $\mathbf{M}_L$ for an arbitrary semidistrim lattice L. In the special case when L is a Boolean lattice, we use spectral methods to obtain much stronger estimates on the mixing time, showing that rowmotion Markov chains of Boolean lattices exhibit the cutoff phenomenon.

Rowmotion Markov Chains

Let P be a finite poset, and let $J(P)$ be the lattice of order ideals of P. For $S \subseteq P$, let

$$\nabla(S) = \{x \in P : x \geq s \text{ for some } s \in S\}.$$

For each $x \in P$, fix a probability $p_x \in [0,1]$. Define the *rowmotion Markov chain* $\mathbf{M}_{J(P)}$ with state space $J(P)$ as follows. Starting from a state $I \in J(P)$, select a random subset S of $\max(I)$ by adding each element $x \in \max(I)$ into S with probability p_x; then transition to the new state $P \setminus \nabla(S)$. If $p_x = 1$ for all $x \in P$, then $\mathbf{M}_{J(P)}$ is deterministic and agrees with the *rowmotion operator*.



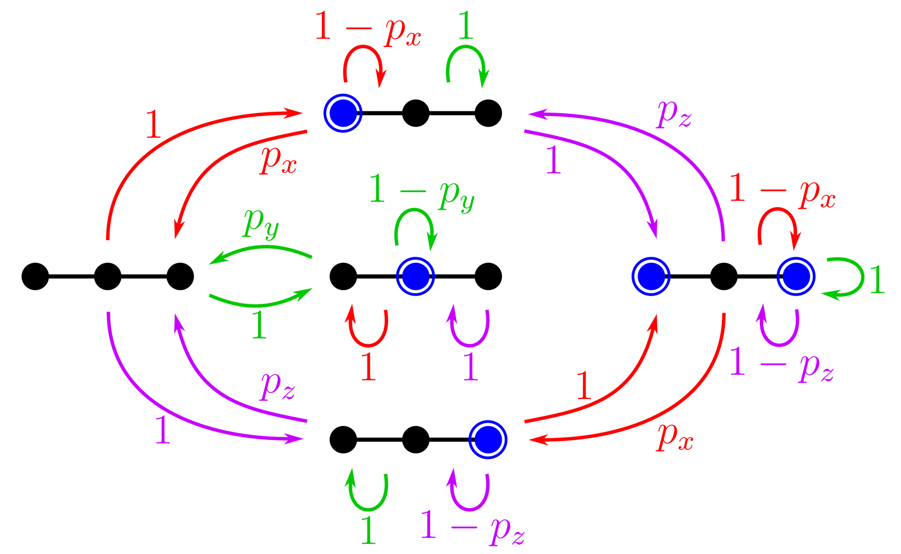
Toggle Markov Chains

Let P be an n-element set, and let $\mathcal{K} \subseteq 2^P$. For $x \in P$, define the *toggle operator* $\tau_x : \mathcal{K} \to \mathcal{K}$ by

$$\tau_x(A) = \begin{cases} A \triangle \{x\} & \text{if } A \triangle \{x\} \in \mathcal{K} \\ A & \text{otherwise,} \end{cases}$$

where $\triangle$ denotes symmetric difference. Fix an ordering $\mathbf{x} = (x_1, \ldots, x_n)$ of P. Given $Y \subseteq P$, let $\tau_Y = \tau_{y_r} \circ \cdots \circ \tau_{y_1}$, where $y_1, \ldots, y_r$ is the list of elements of Y in the order that they appear within the list $x_1, \ldots, x_n$.

For each $x \in P$, fix a probability p_x. Define the *toggle Markov chain* $\mathbf{T} = \mathbf{T}(\mathcal{K}, \mathbf{x})$ as follows. The state space of $\mathbf{T}$ is $\mathcal{K}$. Suppose the Markov chain is in a state $A \in \mathcal{K}$. Choose a subset $T \subseteq A$ randomly so that each element $x \in A$ is included in T with probability p_x, and then transition from A to the new state $\tau_T(A)$.



If P is a finite poset and $\mathbf{x}$ is a linear extension of P, then one can show that $\mathbf{T}(J(P), \mathbf{x})$ coincides with the rowmotion Markov chain $\mathbf{M}_{J(P)}$.

Theorem ([1]). *Suppose $0 < p_x < 1$ for each $x \in P$. Let $\mathcal{H}^P|_{\mathcal{K}}$ be the graph with vertex set $\mathcal{K}$, where $A, A' \in \mathcal{K}$ are adjacent if and only if $|A \triangle A'| = 1$. Then $\mathbf{T}(\mathcal{K}, \mathbf{x})$ is irreducible if and only if the graph $\mathcal{H}^P|_{\mathcal{K}}$ is connected.*

Theorem ([1]). *Suppose the toggle Markov chain $\mathbf{T}(\mathcal{K}, \mathbf{x})$ is irreducible and $p_x > 0$ for every $x \in P$. For $A \in \mathcal{K}$, the probability of the state A in the stationary distribution of $\mathbf{T}(\mathcal{K}, \mathbf{x})$ is*

$$\frac{1}{Z(\mathcal{K})} \prod_{x \in A} p_x^{-1},$$

where $Z(\mathcal{K}) = \sum_{A' \in \mathcal{K}} \prod_{x' \in A'} p_{x'}^{-1}$.

Mixing Times

For $\varepsilon > 0$, let $t^{\mathrm{mix}}_{\mathbf{M}}(\varepsilon)$ denote the *mixing time* of a Markov chain $\mathbf{M}$. The *width* of a finite poset P, denoted $\mathrm{width}(P)$, is the maximum size of an antichain in P.

Theorem ([1]). *Let $\overline{p} = \max_{x \in P} p_x$. For each $\varepsilon > 0$, we have*

$$t^{\mathrm{mix}}_{\mathbf{M}_{J(P)}}(\varepsilon) \leq \log \varepsilon \Big/ \log\left(1 - (1 - \overline{p})^{\mathrm{width}(P)}\right).$$

When P is an n-element antichain (so $J(P)$ is a Boolean lattice), we show that $\mathbf{M}_{J(P)}$ exhibits the *cutoff phenomenon* at time $\frac{1}{2}\log_{1/p} n$. Roughly speaking, this means that $\mathbf{M}_{J(P)}$ stays relatively "unmixed" until around time $\frac{1}{2}\log_{1/p} n$, when it quickly becomes "well mixed."

References

[1] C. Defant, R. Li, and E. Nestoridi. Rowmotion Markov chains. *Adv. Appl. Math.*, **155** (2024).

[2] C. Defant and N. Williams. Semidistrim lattices. *Forum. Math. Sigma*, **11** (2023).

Semidistrim Lattices

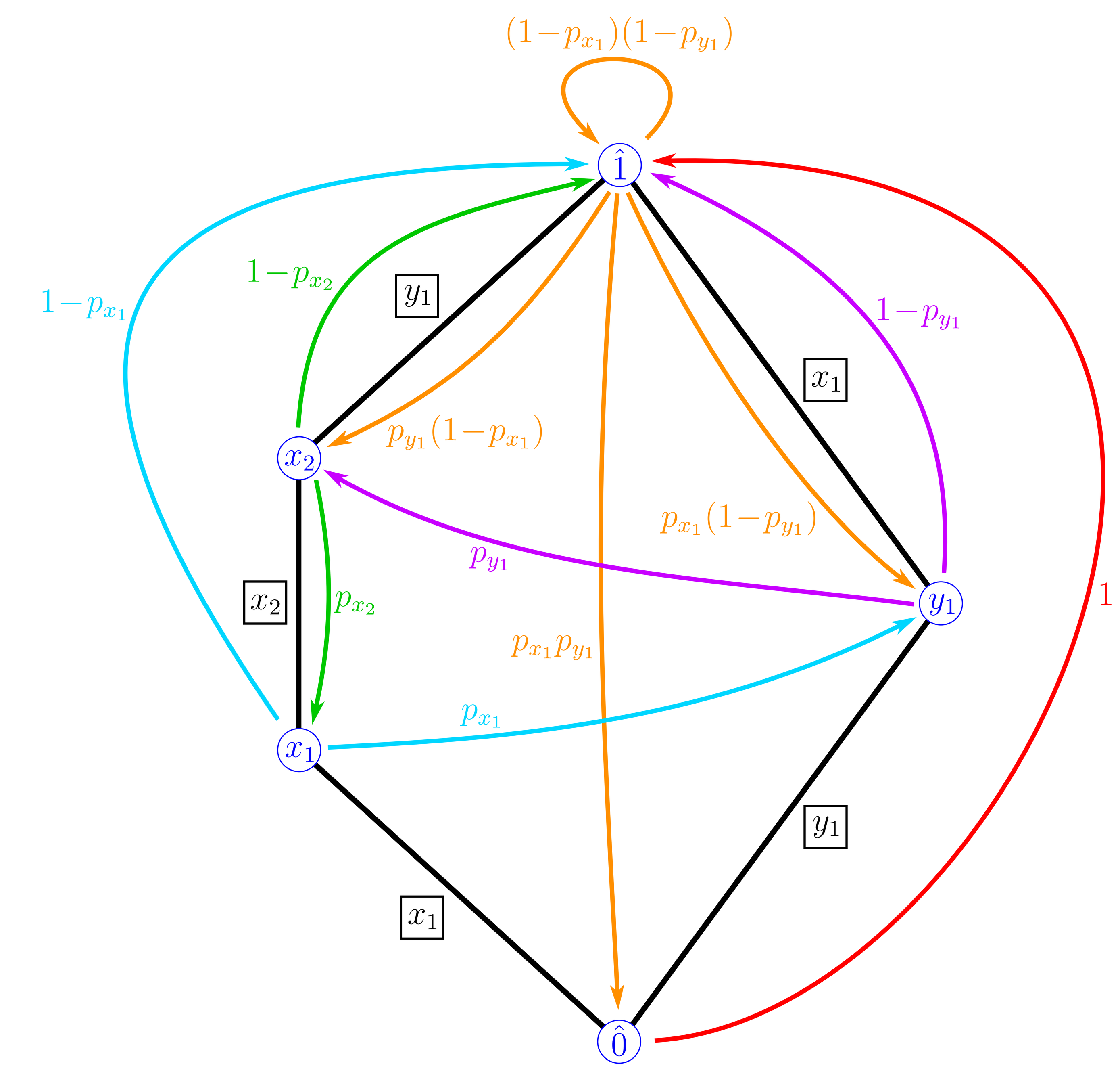
We can extend our definition of rowmotion Markov chains from distributive lattices to *semidistrim lattices* (see [2]); this provides a generalization that differs from toggle Markov chains.

Let L be a semidistrim lattice, and let $\mathcal{J}_L$ and $\mathcal{M}_L$ be the set of join-irreducible elements of L and the set of meet-irreducible elements of L, respectively. There is a specific bijection $\kappa_L : \mathcal{J}_L \to \mathcal{M}_L$ satisfying certain properties. The *Galois graph* of L is the loopless directed graph G_L with vertex set $\mathcal{J}_L$ such that for all distinct $j, j' \in \mathcal{J}_L$, there is an arrow $j \to j'$ if and only if $j \not\leq \kappa_L(j')$. Let $\mathrm{Ind}(G_L)$ be the set of independent sets of G_L. There is a particular way to label the edges of the Hasse diagram of L with elements of $\mathcal{J}_L$; we write j_{uv} for the label of the edge $u \lessdot v$. For $w \in L$, let $\mathcal{D}_L(w)$ be the set of labels of the edges of the form $u \lessdot w$, and let $\mathcal{U}_L(w)$ be the set of labels of the edges of the form $w \lessdot v$. Then $\mathcal{D}_L(w)$ and $\mathcal{U}_L(w)$ are actually independent sets of G_L. Moreover, the maps $\mathcal{D}_L, \mathcal{U}_L : L \to \mathrm{Ind}(G_L)$ are bijections. The *rowmotion* operator $\mathrm{Row} : L \to L$ is defined by $\mathrm{Row} = \mathcal{U}_L^{-1} \circ \mathcal{D}_L$.

The *rowmotion Markov chain* $\mathbf{M}_L$ has state space L. For each $j \in \mathcal{J}_L$, fix a probability $p_j \in [0,1]$. Starting at a state $u \in L$, we choose a random subset S of $\mathcal{D}_L(u)$ by adding each element $j \in \mathcal{D}_L(u)$ into S with probability p_j and then transition to the new state $u' = \mathrm{Row}(\bigvee S)$.

Theorem ([1]). *Let L be a semidistrim lattice, and fix a probability $p_j \in (0,1)$ for each join-irreducible element $j \in \mathcal{J}_L$. The rowmotion Markov chain $\mathbf{M}_L$ is irreducible.*

When L is obtained by adding a minimum element and a maximum element to a disjoint union of two chains, we compute the stationary distribution of $\mathbf{M}_L$ explicitly.



Suggestions for Future Work

It would be interesting to prove that other families of toggle Markov chains exhibit cutoff. Some particularly interesting toggle Markov chains $\mathbf{T}(\mathcal{K}, \mathbf{x})$ are as follows:

- Let P be the set of vertices of a graph G, let $\mathcal{K}$ be the collection of independent sets of G, and let $\mathbf{x}$ be some special ordering of P. For example, if G is a cycle graph, then $\mathbf{x}$ could be the ordering obtained by reading the vertices of G clockwise.
- Let P be an n-element set, and let $\mathbf{x}$ be an arbitrary ordering of the elements of P. For $0 \leq k \leq n$, let $\mathcal{K} = \{I \subseteq P : |I| \leq k\}$.