Macdonald characters from a new formula for Macdonald polynomials

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Main results

- We establish a new creation formula for Macdonald polynomials.
- We introduce Macdonald characters and we prove that they can be characterized as shifted symmetric functions satisfying a vanishing condition.
- We conjecture that these characters satisfy some positivity properties.

Macdonald polynomials and Gamma Operator

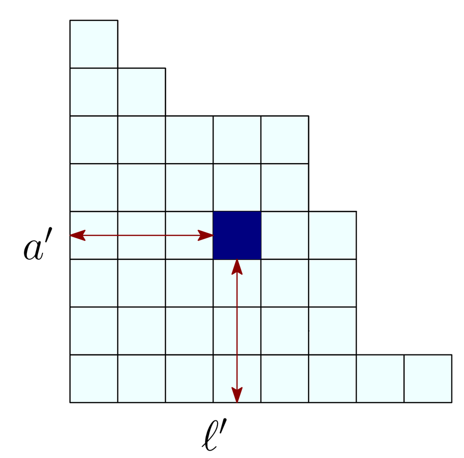
Let $J_\lambda^{(q,t)}$ denote the integral form of Macdonald polynomials (this normalization is directly related to Jack polynomials $J_\lambda^{(\alpha)}$).

We consider the integral version of the *Nabla* and the *Delta* operators of Bergeron–Garsia–Haiman–Tesler:

$$\nabla \cdot J_\lambda^{(q,t)} = (-1)^{|\lambda|}\left(\prod_{\square\in\lambda} q^{a'(\square)}t^{-\ell'(\square)}\right) J_\lambda^{(q,t)},$$

$$\Delta_v \cdot J_\lambda^{(q,t)} = \prod_{\square\in\lambda}\left(1 - v\cdot q^{a'(\square)}t^{-\ell'(\square)}\right) J_\lambda^{(q,t)},$$

where the products run over the cells of the Young diagram of λ, and where a' and ℓ' are respectively the co-arm and the co-length of the cell.



We introduce the following operator

$$\Gamma(u,v) := \Delta_{1/v}\mathcal{P}_{\frac{uv(1-t)}{1-q}}\Delta_{1/v}^{-1},$$

where $\mathcal{P}_{\frac{uv(1-t)}{1-q}}$ denotes the multiplication by the following plethystic exponential

$$\mathrm{Exp}\left[\frac{uv(1-t)}{1-q}\right] := \sum_{n\geq 0} h_n\left[\frac{uv(1-t)}{1-q}X\right] = \sum_{\mu \text{ partition}} \frac{(uv)^{|\mu|}}{z_\mu}\prod_{i\in\mu}\frac{1-t^i}{1-q^i}p_i[X]$$

This operator is a close relative of the Theta operator introduced in [3].

A new creation formula for Macdonald polynomials

Theorem (B.D–D'Adderio '24+). For any partition $\lambda = [\lambda_1,\lambda_2,\dots,\lambda_k]$, we have

$$\Gamma(u,q^{\lambda_1})\Gamma(t^{-1}u,q^{\lambda_2})\cdots\Gamma(t^{-(k-1)}u,q^{\lambda_k})\cdot 1 = t^{-n(\lambda)}\nabla J_\lambda^{(q,t)}\left[uX+\frac{1}{1-t}\right],$$

where $n(\lambda) = \sum_{\square\in\lambda}\ell'(\square)$. Moreover,

$$\Gamma^{(+)}_{\lambda_1}\Gamma^{(+)}_{\lambda_2}\cdots\Gamma^{(+)}_{\lambda_k}\cdot 1 = J_\lambda^{(q,t)} \quad \text{where} \quad \Gamma^{(+)}_m := [u^m]\nabla^{-1}\Gamma(u,q^m)\nabla.$$

Proof:

- five-term relation of Garsia–Mellit,
- Pieri rule.

Macdonald characters

Definition. Fix a partition μ. We define *the Macdonald character* associated to μ as the function $\widetilde{\boldsymbol\theta}_\mu^{(q,t)}(v_1,v_2,\dots)$ such that for each $k\geq 1$

$$\widetilde{\boldsymbol\theta}_\mu^{(q,t)}(v_1,v_2,\dots,v_k,1,1,\dots) := \left\langle p_\mu, \Gamma(1,v_1)\Gamma(t^{-1},v_2)\cdots\Gamma(t^{-k-1},v_k)\cdot 1\right\rangle_{q,t}$$

Macdonald characters can be thought of as functions on Young diagrams via

$$\widetilde{\boldsymbol\theta}_\mu^{(q,t)}(\lambda) := \widetilde{\boldsymbol\theta}_\mu^{(q,t)}(q^{\lambda_1},q^{\lambda_2},\dots,q^{\lambda_k},1,1,\dots).$$

The evaluation of these characters on partitions λ of size $|\mu|$ is closely related to the power-sum expansion of Macdonald polynomials:

$$(-1)^{|\lambda|}q^{n(\lambda')}t^{-2n(\lambda)}J_\lambda^{(q,t)} = \sum_{\mu\vdash|\lambda|}\frac{\widetilde{\boldsymbol\theta}_\mu^{(q,t)}(\lambda)}{z_\mu(q,t)}p_\mu.$$

Macdonald characters are two-parameter deformation of Jack characters $\theta_\mu^{(\alpha)}$ introduced by Lassalle, and which have been a useful tool to understand the asymptotic behavior of large Young diagrams under a Jack deformation of the Plancherel measure.

Characterization of $\widetilde{\theta}_\mu^{(q,t)}$ as shifted symmetric functions

We say that a function $f(v_1,v_2,v_3\dots)$ is shifted symmetric if it is symmetric in the variables $v_1, v_2t^{-1}, v_3t^{-2}\dots$.

Theorem (B.D–D'Adderio '24+). Let μ be a partition. The Macdonald character $\widetilde{\boldsymbol\theta}_\mu^{(q,t)}$ is the unique function which satisfies the following properties:

- $\widetilde{\boldsymbol\theta}_\mu^{(q,t)}$ is shifted symmetric of degree $|\mu|$,
- $\widetilde{\boldsymbol\theta}_\mu^{(q,t)}(\lambda) = 0$ for any partition $|\lambda| < |\mu|$,
- the top homogeneous part of $\widetilde{\boldsymbol\theta}_\mu^{(q,t)}$ is $p_\mu(v_1,t^{-1}v_2,t^{-2}v_3,\dots)$.

Jack characters have a similar characterization due to Féray.

Connection to shifted Macdonald polynomials

Theorem (Okounkov '98) Let μ be a partition. There exists a unique function $J_\mu^*(v_1,v_2,\dots)$ such that

- J_μ^* is shifted symmetric of degree $|\mu|$.
- (normalization property)
$$J_\mu^*(\mu) = (-1)^{|\mu|}q^{n(\mu')}t^{-2n(\mu)}j_\mu^{(q,t)}.$$
- (vanishing property) for any partition $\mu\not\subset\lambda$
$$J_\mu^*(\lambda) = 0.$$

Moreover, the top homogeneous part of J_μ^* is $J_\mu^{(q,t)}(v_1,t^{-1}v_2,t^{-2}v_3,\dots)$. The polynomials J_μ^* are called *shifted Macdonald polynomials*

We extend the map $J_\mu^{(q,t)}\longmapsto J_\mu^*$ into an isomorphism φ from the space of symmetric functions into the space of shifted symmetric functions. We then have

$$\varphi(p_\mu) = \widetilde{\boldsymbol\theta}_\mu^{(q,t)}.$$

We deduce that for any symmetric function f, we have

$$\varphi(f)(v_1,\dots,v_k) = \left\langle f, \Gamma(1,v_1)\Gamma(t^{-1},v_2)\cdots\Gamma(t^{-(k-1)},v_k)\cdot 1\right\rangle_{q,t}.$$

A Macdonald version of Lassalle's conjecture

We introduce the change of variables

$$\begin{cases}\alpha := \frac{1-q}{1-t}\\ \gamma := t-1.\end{cases} \longleftrightarrow \begin{cases} q = 1+\gamma\alpha\\ t = 1+\gamma\end{cases}$$

Conjectrure. The normalized character

$$\frac{t^{(k-1)|\mu|}}{(1-t)^{|\mu|}}\widetilde{\boldsymbol\theta}_\mu^{(q,t)}(s_1,s_2,\dots,s_k)$$

is a polynomial in the parameters $\gamma, \alpha-1, (1-s_1)/\gamma, (1-s_2)/\gamma,\dots,(1-s_k)/\gamma$ with non-negative integer coefficients.

This is a Macdonald version of Lassalle's conjecture on Jack characters, recently proved in [2] using a combinatorial expansion of Jack characters in terms of maps (graphs on surfaces).

Other positivity conjectures

- **(A Macdonald generalization of Stanley's conjecture on the structure coefficients of Jack polynomials).**

  Conjecture. For any partitions λ, μ and ν, the quantity
$$\frac{1}{(1-t)^{|\lambda|+|\mu|+|\nu|}}\left\langle J_\lambda^{(q,t)}J_\mu^{(q,t)}, J_\nu^{(q,t)}\right\rangle_{q,t}$$
  is a polynomial in γ and α with non-negative integer coefficients.

- **(A Macdonald version of Goulden–Jackson b-conjecture).** Define, for π,μ,ν of the same size the coefficient $\boldsymbol h^\pi_{\mu,\nu}$ by the expansion
$$\log\left(\sum_{\lambda \text{ partition}} u^{|\lambda|}t^{-2n(\lambda)}q^{n(\lambda')}\frac{J_\lambda^{(q,t)}[X]J_\lambda^{(q,t)}[Y]J_\lambda^{(q,t)}[Z]}{(1-t)^{|\lambda|}j_\lambda^{(q,t)}}\right) = \sum_{m\geq 0}\sum_{\pi,\mu,\nu\vdash m}\frac{u^m\boldsymbol h^\pi_{\mu,\nu}(\alpha,\gamma)}{\alpha[m]_q}p_\pi[X]p_\mu[Y]p_\nu[Z].$$
  The coefficients $\boldsymbol h^\pi_{\mu,\nu}$ are related to the structure coefficients of Macdonald characters $\widetilde{\boldsymbol\theta}_\mu^{(q,t)}$.

  Conjectrure. The quantity $t^{|\pi|(|\pi|-1)}z_\pi z_\mu z_\nu \boldsymbol h^\pi_{\mu,\nu}$ is a polynomial in γ and $\alpha-1$ with non-negative integer coefficients.

References

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