# On the scaling of random Tamari intervals and Schnyder woods of random triangulations (with an asymptotic D-finite trick)

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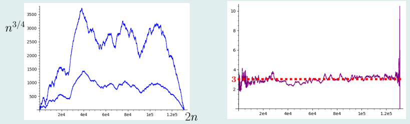


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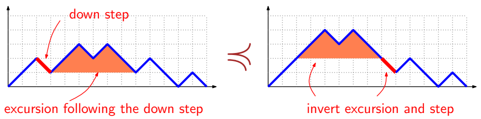
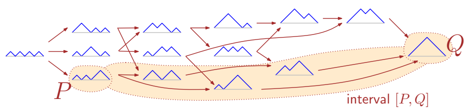
## Dyck paths and Tamari lattice

•Dyck path: steps $\pm 1$, from 0 to 0, stays $\geq 0$.

• Tamari cover relation:




•We get the famous Tamari partial order

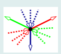


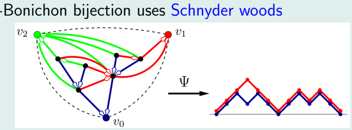

## Counting intervals

• [Chapoton 06] The number of pairs $[P,Q]$ with $P \preccurlyeq Q$ is:

$$I_n = \frac{2}{n(n+1)}\binom{4n+1}{n-1}.$$

• ... nice!. Also the number of plane triangulations with $n+2$ vertices [Tutte 62, Bernardi-Bonichon 09]. The Bernardi-Bonichon bijection uses Schnyder woods






• Only the beginning of many analogies between Tamari invervals and planar maps! See e.g. Wenjie Fang's great works. Or this:

$$(n+1)^{l-2}\prod_{i=1}^{\ell(\lambda)}\binom{2\lambda_i}{\lambda_i} \overset{?}{\leftrightarrow} 2(n-1)_{\ell(\lambda)-2}\prod_{i=1}^{\ell(\lambda)}\binom{2\lambda_i-1}{\lambda_i}$$

bipartite maps of profile $\lambda \vdash n$ — labelled Tamari intervals of profile $\lambda \vdash$

## New results: Random Tamari intervals

• **Theorem [C'24].** Let $(P_n, Q_n)$ a uniformly random Tamari interval of size $n$. Let $I \in [0, 2n]$ uniformly random. Then

$$\frac{Q_n(I)}{n^{3/4}} \longrightarrow Z \quad , \quad \mathbf{E}[Z^k] = \frac{\sqrt{3}\cdot 2^{-\frac{k}{4}-1}}{\sqrt{\pi}}\frac{\Gamma(\frac14 k+\frac13)\Gamma(\frac14 k+\frac23)}{\Gamma(\frac14 k+\frac12)}.$$

Note: $Z = (XY)^{1/4}$ where $X \sim \beta(\frac13, \frac16)$ and $Y \sim \Gamma(\frac23, \frac12)$.

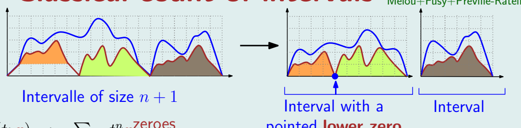
Moreover $\frac{P_n(I)}{n^{3/4}} \longrightarrow \frac{Z}{3}$.

• **Theorem [C'24].** $\frac{\tilde{Q}_n(J) - 3\tilde{P}_n(J)}{\sqrt{n}} = O_p(1)$ Therefore $\frac{\tilde{P}_n(J)}{\tilde{Q}_n(J)} \to \frac13$.

See Wenjie Fang's simulations for the paths and their ratio (title box, to the left ←).

## Classical count of intervals

[Chapoton, Bousquet-Mélou+Fusy+Préville-Ratelle]




$$F(t;x) =: \sum_{intervals} t^n x^{\text{zeroes}}$$

$$F(t;x) = x + t\sum_{i\geq 1}\Big([x^i]F(t;x)\Big)\Big(x + x^2 + \cdots + x^i\Big)F(t,x)$$

$$= x + tx\frac{F(t,x) - F(t,1)}{x-1}F(t,x)$$

• Polynomial equation with one catalytic variable. The theory of Bousquet-Mélou–Jehanne covers this effectively → Chapoton's theorem. Explicitly:

$$F(x) = \frac{1+u}{(1+zu)(1-z)^3}(1-2z-z^2u) \quad , \quad F(1) = \frac{1-2z}{(1-z)^3}$$

$$t = z(1-z)^3, \quad x = \frac{1+u}{(1+zu)^2}.$$

## Modest addition: a marked point

$H(x;t,s)$ gen. fun. of intervals with a marked point on upper path:

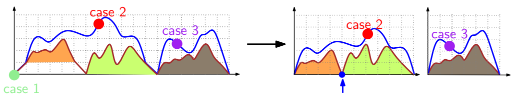
$$H(x) \equiv H(t,x,s) := \sum_{n\geq 0} t^n \sum_{(P,Q)\in\mathcal{I}_n} x^{\text{zeroes}(P)} \sum_{i=0}^{2n} s^{Q(i)}.$$

$t$: size
$x$: nb. of zeroes of $P$
$s$: height of point

We (trivially) get an equation for $H$ by pointing the previous decomposition

$$H(x) = F(x) + sxt\frac{H(x)-H(1)}{x-1}F(x) + xt\frac{F(x)-F(1)}{x-1}H(x).$$

case 1 — case 2 — case 3




Since we know $F(x)$, this is only a linear equation with one catalytic variable $(x)$, where $s$ is considered a parameter. This is directly solved by the kernel method!

## Kernel method

Write: $K(x)H(x) = F(x) - \frac{sxtF(x)H(1)}{(x-1)}$

$$K(x) = \left(-1 + sxt\frac{F(x)}{x-1} + xt\frac{F(x)-F(1)}{x-1}\right).$$

Find $x = X(t,s)$ that cancels $K(x)$, substitute and find $H(1)$.

• **Theorem [C'24].** The series $H(1) \equiv H(1;t,s)$ is algebraic, with an explicit rational parametrization,

$$H(1) = \frac{(1-2z-Uz^2)^2(1+U)}{(1-z)^6} \qquad t = z(1-z)^3, \quad s = \frac{U(1-z)^3}{z(1+U)^2(1-Uz^2-2z)}$$

⟶ This results contains in principle all the distribution of the random variable $Q_n(I)$... But how to deduce the wanted asymptotics???

$$H(1) \equiv H(1,t,s) := \textstyle\sum_{n\geq 0} t^n \sum_{(P,Q)\in\mathcal{I}_n}\sum_{i=0}^{2n} s^{Q(i)}.$$

## Transfer theorem and D-finiteness...

• **Transfer Theorem** [Flajolet-Odlyzko]. Let $f(t)$ algebraic with unique dominant singularity at $\rho > 0$. If $f(t) \sim c(1-t/\rho)^\alpha$ when $t \to \rho$, then $[t^n]f(t) \sim c\Gamma(-\alpha)n^{-\alpha-1}\rho^{-n}$ when $n \to \infty$. $(\alpha \notin \mathbb{N})$.

• **Application to asymptotics of moments (classical)**

Let $h_k = \left(\frac{\partial}{\partial s}\right)^k H(1)\Big|_{s=1}$, then $\frac{[t^n]h_k}{[t^n]h_0} = \mathbf{E}[(Q_n(I))_k]$

$(m)_k := m(m-1)\dots(m-k+1)$

→ to perform the asymptotics of moments, it is enough to know the main singularity of $h_k$ for each $k \geq 0$.

→ Given an algebraic equation for $H(1)$, I can do this (in principle) for any given $k \geq 0$.

• **The killing trick – this is great!**

Any algebraic function is D-finite (solution of a linear ODE with polynomial coefficients). Our series in $t, s$ is algebraic, hence it is algebraic in the variable $(s-1)$, over $\mathbb{Q}(t)$

Hence it is D-finite: its coefficients, the $h_k$, satisfy a polynomial recursion!

## The D-finite trick

Let $h =\equiv h(t,s)$ an algebraic function.

Let $h_k \equiv h_k(t) = \left(\frac{\partial}{\partial s}\right)^k h\Big|_{s=1}$.

As a series in $(s-1)$ it is algebraic over $\mathbb{Q}(t)$
so it is D-finite: the $h_k$ satisfy a recurrence relation

$$h_k(t) = \sum_{d=1}^{L} Rat_d(t,k)h_{k-d}(t)$$

$Rat_d =$ explicit rational function of $k$ (algebraic in $t$)

Under reasonable hypotheses one can determine the main singularity of $h_k$ easily by induction on $k$!!!
In our case $L = 6$ but only two terms contribute to the asymptotics, we get immediately

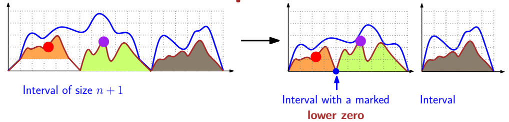
$$h_k(t) \sim c_k(1 - t/(27/256))^{1-\frac34 k}$$

where $c_k = \frac{\sqrt{6}(3k-4)(3k-8)}{96}c_{k-2}$. (order two recurrence).

We immediately get the formula for $\mathbf{E}[Z^k]$.

This trick seems very powerful!

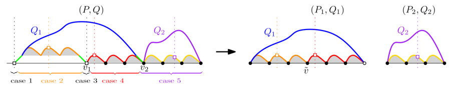
## About the lower path...



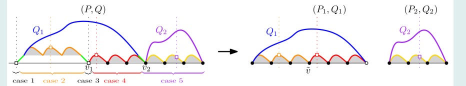

● ● → to track the lower height, we need to know if the marked point comes before or after the marked zero...

→ we need two catalytic variables (!!!)

$$G(x,y) \equiv G(t,x,y,w) := \sum_{n\geq 0} t^n \sum_{(P,Q)\in\mathcal{I}_n}\sum_{i=0}^{2n} w^{P(i)} x^{\text{contact}_{<i}(P)} y^{\text{contact}_{\geq i}(P)}.$$




## Two catalytic variables (but not really)




$$G(x,y) = F(y) + txw\frac{G(1,y)-G(1,1)}{y-1}F(y) + tx\frac{F(y)-yF(1)}{y-1}F(y) + t\frac{x^2}{y}\frac{G(x,y) - \frac{y}{x}F(x) - G(1,y) + yF(1)}{x-1}F(y) + tx\frac{F(x)-F(1)}{x-1}G(x,y).$$

→ Contains $G(x,y), G(1,y), G(1,1)$... but no $G(x,1)$.

→ therefore it is, in fact, not hard to solve!

... first solve the catalytic equation in $x$ (with $y$ a parameter)

... then solve the catalytic equation in $y$

→ To deduce asymptotics, the "D-finite trick" works again!

(needs computer algebra! order 9 recurrence with polynomial coefficients)

## Conclusion

• Bertoin-Curien-Riera (book to come) can do the full scaling limit for the upper path (but maybe not the explicit limit law)

• This limit law "should" be universal for nonnegative Bousquet-Mélou–Jehanne equations, and one should try to prove it.

• The "asymptotic D-finite trick" is great and I'd like to have other applications of it!

## References

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