Eulerian Polynomials for Digraphs

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Overview

We study a generalization of the Eulerian polynomial to digraphs.

Background

For a permutation $\sigma \in \mathfrak{S}_n$, let $\mathrm{des}(\sigma)$ be the number of descents of σ. The *Eulerian polynomial* is defined to be

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)}.$$

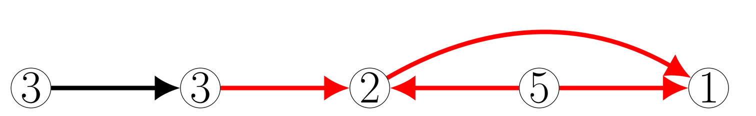
This has the well-known property:

$$A_n(-1) = \begin{cases} 0 & n \text{ even} \\ (-1)^{(n-1)/2}|Alt_n| & n \text{ odd} \end{cases}$$

where Alt_n is the set of *alternating* permutations $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$

Extending to digraphs

Let D be a digraph on n vertices. For a bijection $\sigma : V \to [n]$, a *descent* is an edge $u \to v$ such that $\sigma(u) > \sigma(v)$. Let $\mathrm{des}_D(\sigma)$ be the number of descents of σ.

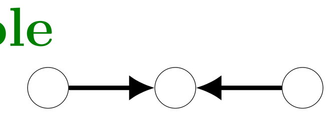


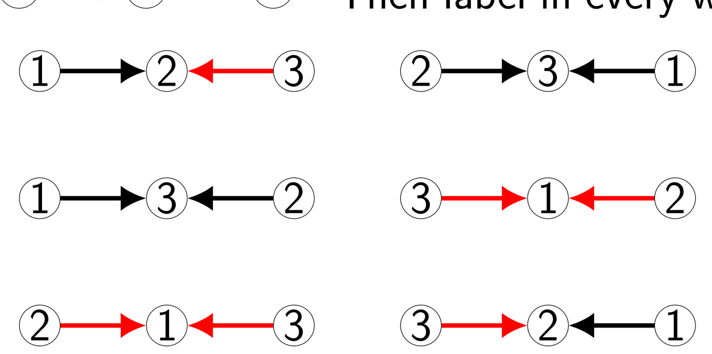
The **Eulerian polynomial of a digraph** D is defined as

$$A_D(t) = \sum_{\sigma \in \mathfrak{S}_D} t^{\mathrm{des}_D(\sigma)}$$

where $\mathfrak{S}_D$ is the set of bijections $\sigma : V \to [n]$. $A_D(t)$ was first studied by Foata and Zeilberger in 1996.

Example

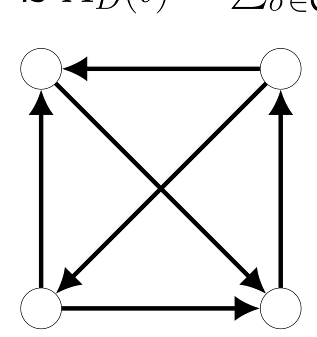
Let D be  Then label in every way possible



$$A_D(t) = 2 + 2t + 2t^2$$

Example (Audience question)

For what digraph(s) D is $A_D(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}(\sigma)}$?



$$A_D(t) = 1 + 3t + 5t^2 + 6t^3 + 5t^4 + 3t^5 + t^6$$

What is an "alternating permutation" of a directed graph?

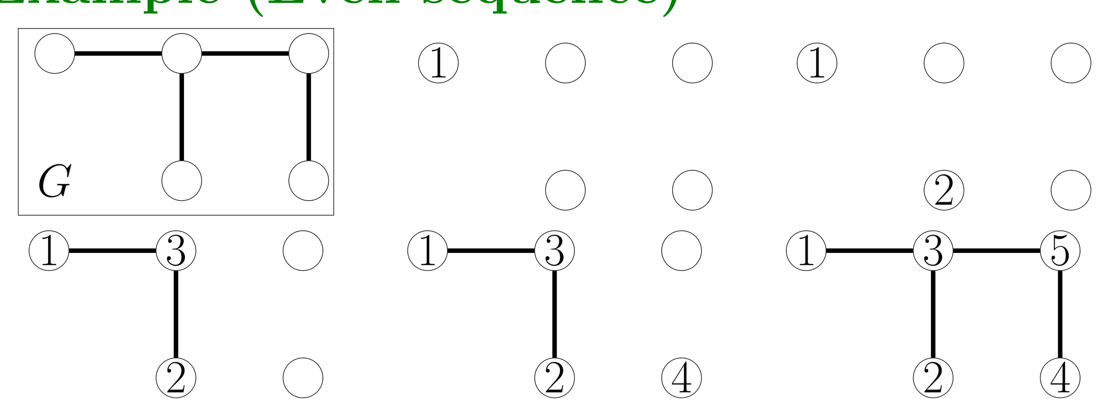
If $A_n(-1)$ is the number of *alternating permutations*, what is $A_D(-1)$ for a digraph D?

Lemma (Kalai 2002)

If D, D' are two orientations of the same graph G, then $|A_D(-1)| = |A_{D'}(-1)|$

An **even sequence** of an n-vertex graph G is an ordering $\pi = (\pi_1, \ldots, \pi_n)$ of the vertex set $V(G)$ such that if each of the subgraphs $G[\pi_1, \ldots, \pi_i]$ induced by the first i vertices of π have an *even* number of edges for all $1 \le i \le n$.

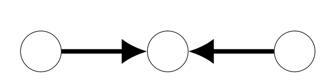
Example (Even sequence)

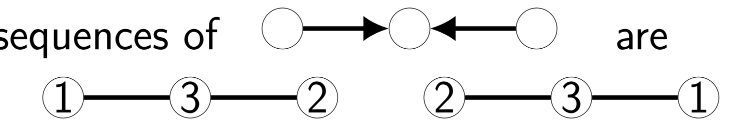


Theorem (Celano–Sieger–Spiro 2023)

If D is a digraph which is either bipartite, complete multipartite, or a blowup of a cycle, then $|A_D(-1)|$ is the number of even sequences of its underlying graph.

Example

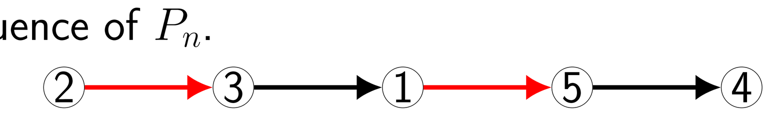
The even sequences of  are



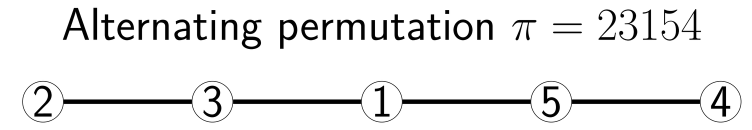
And $|A_D(-1)| = |2 + 2(-1) + 2(-2)^2| = 2$

Example

$\pi \in \mathfrak{S}_n$ is an alternating permutation if and only if π^{-1} is an even sequence of P_n.



Alternating permutation $\pi = 23154$



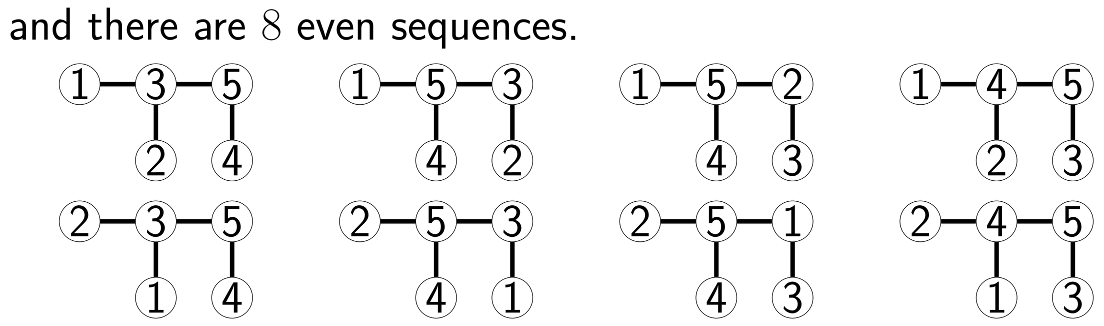
Even sequence $\pi^{-1} = 31254$

Example

If D directs the edges of G down and to the right, then

$$A_D(x) = 3 + 28x + 58x^2 + 28x^3 + 3x^4 \qquad |A_D(-1)| = 8$$

and there are 8 even sequences.

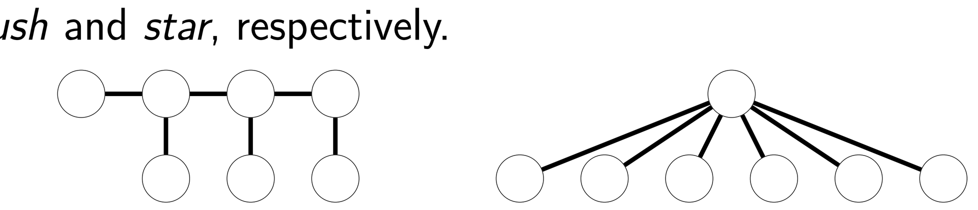


Example (Extreme values for trees)

For a given directed tree T on $2n+1$ vertices,

$$2^n n! \le |A_T(t)| \le (2n)!$$

where lower and upper bounds are achieved only by the *hairbrush* and *star*, respectively.



The *hairbrush* (lower bound) The *star* (upper bound)

Multiplicity of -1

When $A_D(-1) = 0$, we know that

$$A_D(t) = (1+t)^m B(t)$$

for some m and some $B(t)$ with $B(-1) \neq 0$. What is m? What is the largest it can be?

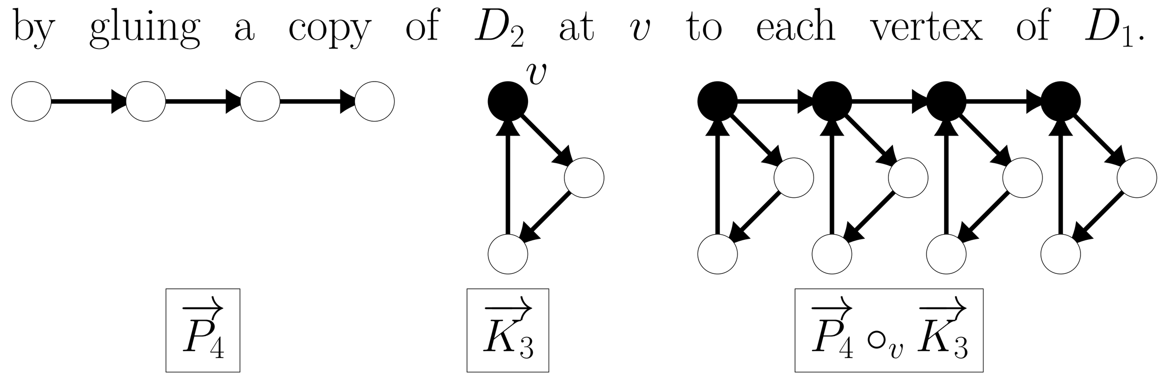
Example

If D is an <Audience Answer>, then

$$A_D(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}(\sigma)} = [n]_t! = (1+t)^{\lfloor n/2 \rfloor} B(t).$$

It turns out that $m = \lfloor n/2 \rfloor$ is the highest you'll ever see. We can study the multiplicity in more detail with the following construction:

Given digraphs D_1, D_2 and a root vertex $v \in D_2$, the *rooted product digraph*, denoted $D_1 \circ_v D_2$, is obtained by gluing a copy of D_2 at v to each vertex of D_1.



Lemma (Celano–Sieger–Spiro 2023)

If D_1 has d_1 vertices and D_2 has d_2 vertices then

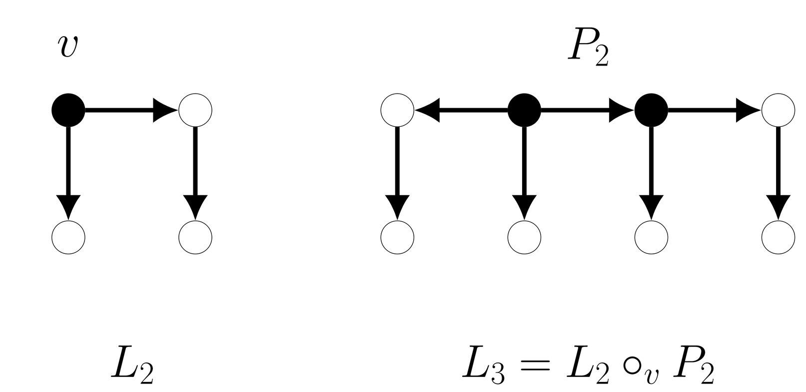
$$A_{D_1 \circ_v D_2}(t) = \frac{1}{d_2!}\binom{d_1 \cdot d_2}{d_1, \ldots, d_1} \cdot A_{D_1}(t) A_{D_2}(t)^{d_2}.$$

Define a family of graphs recursively by

$$L_1 = P_2 \quad \text{and} \quad L_{n+1} = L_n \circ P_2.$$

By the formula and induction, we get

$$A_{L_n}(t) = (2^m)! \left(\frac{1+t}{2}\right)^{2^n - 1}.$$



Disjoint products of these graphs bound m.

Theorem (Celano–Sieger–Spiro 2023)

Suppose m is multiplicity of -1 of $A_D(t)$ for a digraph D. Then

1. $m \le n - s_2(n)$ where $s_2(n)$ is the number of 1's in the binary expansion of n.
2. If D is any tournament, then $m = \lfloor n/2 \rfloor$

Open questions and conjectures

In general, there are more than $|A_D(-1)|$ even sequences.

Question 1: Can one give a combinatorial interpretation for $|A_D(-1)|$ for arbitrary digraphs D as some *special* even sequences?

The following is a very natural generalization of our multiplicity results for tournaments.

Conjecture 2: If D is the orientation of a complete multipartite graph which has r parts of odd size, then $m = \lfloor \frac{r}{2} \rfloor$.

We know when 0 and -1 are roots of $A_D(t)$. We have not found any other *integral* roots so far.

Question 3: Does there exist a digraph D such that $A_D(t)$ has an *integral* root which is not equal to either 0 or -1?

Another well-known property of $A_n(t)$ is that it is *unimodal* i.e. $a_0 \le a_1 \le \cdots a_{\lfloor n-1 \rfloor/2} \ge \cdots \ge a_{n-1}$. Due to their connection with Hessenberg varieties, naturally oriented *unit interval graphs* D have unimodal $A_D(t)$.

Question 4: For which digraphs D is $A_D(t)$ unimodal?

For Further Information

- K. Celano, N. Sieger, and S. Spiro. *Eulerian polynomials for digraphs.* 2023. arXiv:2309.07240.
- D. Foata and D. Zeilberger, *Graphical major indices*, Journal of Computational and Applied Mathematics **68** (1996), no. 1, 79–101.
- G. Kalai, *A fourier-theoretic perspective on the condorcet paradox and arrow's theorem*, Advances in Applied Mathematics **29** (2002), no. 3, 412–426.