From coloured permutations to Hadamard products and zeta functions

Angela Carnevale*, Vassilis Dionyssis Moustakas†, Tobias Rossmann*

*University of Galway, Ireland †Università di Pisa, Italy

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Abstract

We devise a constructive method for computing explicit combinatorial formulae for Hadamard products of certain rational generating functions. The latter arise naturally when studying so-called ask zeta functions of direct sums of modules of matrices or class- and orbit-counting zeta functions of direct products of groups. Our method relies on shuffle compatibility of coloured permutation statistics and coloured quasisymmetric functions, extending recent work of Gessel and Zhuang [2].

Coloured permutations

For $r \gg 0$, consider the poset $\Gamma = \{0 > 1 > 2 > \cdots > r\}$ of colours. On the set of coloured positive integers $\Gamma \times \mathbb{Z}_{>0}$, consider the colour order $\cdots < 1^1 < 2^1 < \cdots < 1^0 < 2^0 < \cdots$. A coloured permutation is a string $\boldsymbol{a} = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$ for distinct $\sigma_1, \dots, \sigma_n \in \mathbb{Z}_{>0}$, and arbitrary $\gamma_1, \dots, \gamma_n \in \Gamma$. The descent set of $\boldsymbol{a}$ is

$$\mathrm{Des}(\boldsymbol{a}) = \{i \in [n-1] : \sigma_i^{\gamma_i} > \sigma_{i+1}^{\gamma_{i+1}}\} \cup \{0 : \gamma_1 \neq 0\}.$$

The descent number and comajor index are $\mathrm{des}(\boldsymbol{a}) = |\mathrm{Des}(\boldsymbol{a})|$ and $\mathrm{comaj}(\boldsymbol{a}) = \sum_{i \in \mathrm{Des}(\boldsymbol{a})} (n-i)$, respectively. Let $\mathrm{col}_j(\boldsymbol{a}) := |\{i \in [n] : \gamma_i = j\}|$. The colour vector of a $\boldsymbol{a}$ is $\mathbf{col}(\boldsymbol{a}) = (\mathrm{col}_0(\boldsymbol{a}), \dots, \mathrm{col}_{r-1}(\boldsymbol{a}))$. The coloured descent set of $\boldsymbol{a}$ is

$$\mathrm{sDes}(\boldsymbol{a}) = \{i^{\gamma_i} : i \in [n-1],\ \gamma_i \neq \gamma_{i+1} \text{ or } (\gamma_i = \gamma_{i+1} \text{ and } \sigma_i > \sigma_{i+1})\} \cup \{n^{\gamma_n}\}.$$

Coloured configurations

A labelled coloured configuration is a pair (f, α), where f is an element of the form $f = \sum_{\boldsymbol{a} \in \mathcal{A}} f_{\boldsymbol{a}} \boldsymbol{a} \in \mathbb{N}_0 \mathcal{A}$, where almost all $f_{\boldsymbol{a}} \in \mathbb{N}_0$ are zero and $\alpha \colon \Gamma \to \{\pm X^k : k \in \mathbb{Z}\}$ such that $\mathrm{supp}(\alpha) \subseteq \mathrm{pal}^*(f)$, where $\mathrm{supp}(\alpha) = \{c \in \Gamma : \alpha(c) \neq 1\}$ and $\mathrm{pal}^*(f)$ denotes the set of nonzero colours appearing in $\boldsymbol{a}$. By taking shuffles of elements in $\mathbb{N}_0\mathcal{A}$, we consider labelled coloured configurations of the form $(f \sqcup g, \alpha \cup \beta)$, where f and g have disjoint set of symbols and $\alpha(c) = \beta(c)$ for $c \in \mathrm{pal}^*(f) \cap \mathrm{pal}^*(g)$. We call (f, α) and (g, β) coherent.

Coloured quasisymmetric functions

Write $\boldsymbol{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \dots)$ and $\mathrm{Des}^*(\boldsymbol{a}) = \mathrm{Des}(\boldsymbol{a}) \setminus \{0\}$. The coloured quasisymmetric function attached to $\boldsymbol{a} = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$ is

$$F_{\boldsymbol{a}} = F_{\boldsymbol{a}}(\boldsymbol{x}^{(0)}, \dots, \boldsymbol{x}^{(r-1)}) = \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \\ j \in \mathrm{Des}^*(\boldsymbol{a}) \Rightarrow i_j < i_{j+1}}} x_{i_1}^{(\gamma_1)} x_{i_2}^{(\gamma_2)} \cdots x_{i_n}^{(\gamma_n)}.$$

The space $\mathrm{QSym}^{(r)}$ spanned by all $F_{\boldsymbol{a}}$ forms a $\mathbb{Q}$-algebra.

Coloured shuffle algebras

We say that st is shuffle compatible if for all disjoint coloured permutations $\boldsymbol{a}$ and $\boldsymbol{b}$, the multiset $\{\{\mathrm{st}(\boldsymbol{c}) : \boldsymbol{c} \in \boldsymbol{a} \sqcup \boldsymbol{b}\}\}$ only depends on $\mathrm{st}(\boldsymbol{a}), \mathrm{st}(\boldsymbol{b})$ and the lengths of $\boldsymbol{a}$ and $\boldsymbol{b}$. Write $\boldsymbol{p} = (p_0, \dots, p_{r-1})$. Let $\mathbb{Q}[\![t*]\!]$ denote $\mathbb{Q}[\![t]\!]$ with multiplication given by the Hadamard product in t. By extending the concept of shuffle algebras from [2] to the coloured setting, we obtain the following. Full details are provided in [1]; cf. [3, §4].

Theorem

- The tuple of statistics $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$ is shuffle compatible.
- An injective algebra homomorphism $\mathcal{A}^{(r)}_{(\mathrm{des},\mathrm{comaj},\mathbf{col})} \to \mathbb{Q}[\boldsymbol{p}, x][\![t*]\!]$ is given by

$$[\boldsymbol{a}]_{(\mathrm{des},\mathrm{comaj},\mathbf{col})} \mapsto \frac{\boldsymbol{p}^{\mathbf{col}(\boldsymbol{a})} x^{\mathrm{comaj}(\boldsymbol{a})} t^{\mathrm{des}(\boldsymbol{a})}}{(1-t)(1-xt)\cdots(1-x^{|\boldsymbol{a}|}t)}.$$

- sDes is shuffle compatible and the linear map $\mathcal{A}^{(r)}_{\mathrm{sDes}} \to \mathrm{QSym}^{(r)}$ defined by $[\boldsymbol{a}]_{\mathrm{sDes}} \mapsto F_{\boldsymbol{a}}$ is a $\mathbb{Q}$-algebra isomorphism.

Main result

Given a labelled coloured configuration (f, α) and $\varepsilon \in \mathbb{Z}$, we define a rational formal power series

$$W^{\varepsilon}_{f,\alpha} = W^{\varepsilon}_{f,\alpha}(X, Y) = \sum_{\boldsymbol{a} \in \mathrm{supp}(f)} f_{\boldsymbol{a}} \frac{\alpha(\boldsymbol{a}) X^{\varepsilon\, \mathrm{comaj}(\boldsymbol{a})} Y^{\mathrm{des}(\boldsymbol{a})}}{(1-Y)(1-X^{\varepsilon}Y)\cdots(1-X^{\varepsilon|\boldsymbol{a}|}Y)} \in \mathbb{Q}(X)[\![Y]\!].$$

Theorem

Let (f, α) and (g, β) be coherent labelled coloured configurations. Then

$$W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta} = W^{\varepsilon}_{f \sqcup g,\, \alpha \cup \beta}$$

for each $\varepsilon \in \mathbb{Z}$. In particular, for fixed ε, the set $\{W^{\varepsilon}_{f,\alpha} : (f, \alpha) \text{ is a labelled coloured configuration}\}$ is closed under $*_Y$.

Applications to zeta functions

In the following table, $\underline{n} = \sum_{\nu_i \in \{0,i\}} 1^{\nu_1} \cdots n^{\nu_n}$ and "%" indicates that an entry coincides with the one immediately above it. In each case, $\mathsf{Z}(Y) = W^{\varepsilon}_{f,\alpha}(q, u(q)Y)$ (subject perhaps to mild restrictions on q). This can be proved using formulae from [4, 5].

Zeta function $\mathsf{Z}(Y)$	f	α	ε	$u(X)$	$W^{\varepsilon}_{f,\alpha}(X,Y)$
$\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(Y)$	$\underline{1}$	$1 \leftarrow -X^{-d}$	$d-e$	1	$\frac{1-X^{-e}Y}{(1-Y)(1-X^{d-e}Y)}$
$\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(Y)$, $\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times(d-1)}(\mathfrak{O})}(Y)$	%	%	1	%	$\frac{1-X^{1-d}Y}{(1-Y)(1-XY)}$
$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y)$	%	%	%	$X^{\binom{d}{2}}$	%
$\mathsf{Z}^{\mathrm{ask}}_{\Delta_n \vee \mathsf{K}_{n+1}}(Y)$	$\underline{2}$	$1, 2 \leftarrow -X^{-n-1}$	1	X^{-1}	$\frac{(1-X^{1-n}Y)(1-X^{-n}Y)}{(1-Y)(1-XY)(1-X^2Y)}$
$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}_{\Delta_n \vee \mathsf{K}_{n+1}}}(Y)$	%	%	%	$X^{3\binom{n+1}{2}-1}$	%
$\mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_{d+1}}(Y)$	$\underline{d}$	$1, \dots, d \leftarrow -X^{-1}$	0	X	$\frac{(1-X^{-1}Y)^d}{(1-Y)^{d+1}}$

Zeta functions

Let $\mathfrak{O}$ be a compact discrete valuation ring with maximal ideal $\mathfrak{P}$, and residue field size q. The (local) ask zeta function associated with $M \subseteq \mathrm{M}_{d\times e}(\mathfrak{O})$ is the formal power series $\mathsf{Z}^{\mathrm{ask}}_M(Y) = \sum_{k=0}^{\infty} \alpha_k(M) Y^k$, where $\alpha_k(M)$ denotes the average size of the kernels within the reduction of M modulo $\mathfrak{P}^k$. Let G be a linear group scheme over $\mathfrak{O}$, embedded into $d \times d$ matrices. For example, consider:

- U_d, upper unitriangular $d \times d$ matrices.
- $\mathsf{F}_{2,d}$, derived from the free class-2-nilpotent group on d generators.
- G_Γ, the graphical group scheme associated with a graph Γ as in [5, §3.4].

The orbit-counting zeta function of G is $\mathsf{Z}^{\mathrm{oc}}_{\mathsf{G}}(Y) = \sum_{k=0}^{\infty} b_k(\mathsf{G}) Y^k$, where $b_k(\mathsf{G})$ denotes the number of orbits of $\mathsf{G}(\mathfrak{O}/\mathfrak{P}^k)$ acting on $(\mathfrak{O}/\mathfrak{P}^k)^d$. The class-counting zeta function of G is $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}}(Y) = \sum_{k=0}^{\infty} c_k(\mathsf{G}) Y^k$, where $c_k(\mathsf{G})$ denotes the number of conjugacy classes of $\mathsf{G}(\mathfrak{O}/\mathfrak{P}^k)$.

Hadamard products and zeta functions

Modules of matrices, linear group schemes, and graphs all admit natural operations which correspond to taking Hadamard products of associated zeta functions. Hence, our main result and the table provide us with an effective method for computing zeta functions of suitable direct products of groups. Given a set of (uncoloured) permutations P, write $\Pi(P) = \{\sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n} : \sigma_1 \cdots \sigma_n \in P,\ \gamma_i \in \{0, \sigma_i\}\}$. The following is an example of a group-theoretic application of our main result.

Corollary

$$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d_1} \times \cdots \times \mathsf{F}_{2,d_n}}\left(q^{-\sum_{i=1}^{n} \binom{d_i}{2}} Y\right) = \frac{\sum_{\boldsymbol{a} \in \Pi(\mathrm{S}_n)} \alpha_q(\boldsymbol{a}) q^{\mathrm{comaj}(\boldsymbol{a})} Y^{\mathrm{des}(\boldsymbol{a})}}{(1-Y)(1-qY)\cdots(1-q^nY)}.$$

References

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