CONFIGURATION SPACES AND PEAK REPRESENTATIONS

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GOAL: Use results relating Solomon's **Descent algebra** in Types A and B with **configuration spaces** to build an analogous story for the **Peak algebra**

COMBINATORIAL ALGEBRAS → KEY CONNECTION: SYMMETRY ← CONFIGURATION SPACES

TYPE A

ALGEBRA: Solomon's **Descent Algebra** [12]: $\mathrm{Sol}(\mathfrak{S}_n) \subset \mathbb{Q}[\mathfrak{S}_n]$

generated by sums of elements with the same **descent** set:

$$Y_J := \sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{Des}(w) = J \subset [n-1]}} w \quad \text{where} \quad \mathrm{Des}(w) = \{i \in [n-1] : w_i > w_{i+1}\}.$$

$\mathrm{Sol}(\mathfrak{S}_n)$ contains the **Type A Eulerian idempotents** due to Garsia-Reutenauer [7]:

$$\sum_{j=0}^{n-1} t^j \mathfrak{e}_A^{(j)} = \sum_{w \in \mathfrak{S}_n} \binom{t-1+n-|\mathrm{Des}(w)|}{n} w$$

THEOREM (Sundaram-Welker [9] + Hanlon [6]) For $d \geq 3$ and odd, the following decompositions of $\mathbb{Q}[\mathfrak{S}_n]$ coincide:

$$H^{2k}\mathrm{Conf}_n(\mathbb{R}^d) \cong \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = n-k}} \mathrm{Lie}_\lambda \cong \mathfrak{e}_A^{n-1-k}\mathbb{Q}[\mathfrak{S}_n] \text{ for } 0 \leq k \leq n-1$$

where $\{\mathrm{Lie}_\lambda\}_{\lambda \vdash n}$ are Thrall's **Higher Lie characters** [11].

Lie_λ has image under the Frobenius characteristic map

$$\mathrm{ch}(\mathrm{Lie}_\lambda) = h_{m_1}[L_1]h_{m_2}[L_2]\cdots h_{m_n}[L_n] \quad \text{for} \quad \lambda = (1^{m_1}, 2^{m_2}, \cdots, n^{m_n})$$

TOPOLOGICAL SPACE: the configuration space

$$\mathrm{Conf}_n(\mathbb{R}^d) := \{(p_1, \cdots, p_n) \in \mathbb{R}^{dn} : p_i \neq p_j\}$$

Cohomology presentation due to Arnol'd [3] and Cohen [6], and

$$H^*\mathrm{Conf}_n(\mathbb{R}^d) \cong \begin{cases} \text{Orlik Solomon algebra } \mathscr{OS}(\mathfrak{S}_n) & d \geq 2 \text{ and even} \\ \text{associated graded Varchenko-Gelfand ring } \mathscr{VG}(\mathfrak{S}_n) & d \geq 3 \text{ and odd [9]} \end{cases}$$

The Poincare polynomial of $H^*\mathrm{Conf}_n(\mathbb{R}^d)$ for $d \geq 2$ is :

$$\sum_{k=0}^{n-1} t^k \dim\left(H^{k(d-1)}\mathrm{Conf}_n(\mathbb{R}^d)\right) = (1+t)(1+2t)\cdots(1+(n-1)t)$$

$\mathfrak{P}_n$ is a subalgebra of $\mathrm{Sol}(\mathfrak{S}_n)$

$\mathfrak{gr}\left(H^*\mathrm{Conf}_n(\mathbb{RP}^2 \times \mathbb{R})\right)$ refines $H^*\mathrm{Conf}_n(\mathbb{R}^3)$

NEW: PEAK

ALGEBRA: Nyman's **Peak algebra** [10]: $\mathfrak{P}_n \subset \mathrm{Sol}(\mathfrak{S}_n) \subset \mathbb{Q}[\mathfrak{S}_n]$

generated by sums of elements with the same **peak set**:

$$Z_J := \sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{Peak}(w) = J \subset [n-1]}} w \quad \text{where} \quad \mathrm{Peak}(w) = \{i \in [n-1] : w_{i-1} < w_i > w_{i+1}\}$$

e.g. $Z_{1,3} = (\mathbf{2},1,\mathbf{4},3) + (\mathbf{3},1,\mathbf{4},2) + (\mathbf{3},2,\mathbf{4},1) + (\mathbf{4},1,\mathbf{3},2) + (\mathbf{4},2,\mathbf{3},1) \in \mathfrak{P}_4$

Aguiar-Bergeron-Nyman [1] show that $\mathfrak{P}_n$ is the image of $\mathrm{Sol}(B_n)$ under the sign-forgetting map $\varphi : B_n \twoheadrightarrow \mathfrak{S}_n$, e.g $\varphi(-3,2,-1) = (3,2,1)$

Define the **Peak idempotents** $\mathfrak{e}_{\mathfrak{P}}^{(j)} := \varphi(\mathfrak{e}_B^{(j)}) \in \mathfrak{P}_n$ for $0 \leq j \leq n$

MAIN RESULTS:

THEOREM (Aguiar, B, Reiner [2])

I. $H^{2k}\mathrm{Conf}_n(\mathbb{RP}^2 \times \mathbb{R})$ vanishes unless $k \equiv 0 \mod 2$

II. The peak idempotent $\mathfrak{e}_{\mathfrak{P}}^{(j)}$ vanishes unless $j \equiv n \mod 2$

III. As $\mathfrak{S}_n$ representations, $H^*\mathrm{Conf}_n(\mathbb{RP}^2 \times \mathbb{R}) \cong \mathbb{Q}[\mathfrak{S}_n]$

IV. The following decompositions of $\mathbb{Q}[\mathfrak{S}_n]$ coincide:

$$H^{2k}\mathrm{Conf}_n(\mathbb{RP}^2 \times \mathbb{R}) \cong \sum_{\substack{\lambda \vdash n \\ \mathrm{odd}(\lambda) = n-k}} \mathrm{Lie}_\lambda \cong \mathfrak{e}_{\mathfrak{P}}^{n-k}\mathbb{Q}[\mathfrak{S}_n] \text{ for } 0 \leq k \leq n$$

where $\mathrm{odd}(\lambda)$ is the number of odd parts of λ.

TOPOLOGICAL SPACE: the configuration space

$$\mathrm{Conf}_n(\mathbb{RP}^2 \times \mathbb{R}) \cong \left(\mathrm{Conf}_n^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}^3)\right)^{\mathbb{Z}/2\mathbb{Z}}$$

(assuming coefficients in a field with characteristic not dividing 2)

THEOREM (Aguiar, B, Reiner [2])

Letting $d_n^{(k)} := \dim\left(H^{k(d-1)}\mathrm{Conf}_n(\mathbb{RP}^2 \times \mathbb{R})\right)$ we have that

$$d_n^{(k)} = \#\{w \in \mathfrak{S}_n : n-k \text{ odd cycles}\} = d_{n-1}^{(k-1)} + (n-1)^2 d_{n-2}^{(k)}$$

We also give a **presentation** and **basis** for $H^*\mathrm{Conf}_n(\mathbb{RP}^2 \times \mathbb{R})$ and define a **filtration** whose associated graded ring $\mathfrak{gr}\left(H^*\mathrm{Conf}_n(\mathbb{RP}^2 \times \mathbb{R})\right)$ is a bi-graded ring refining $H^*\mathrm{Conf}_n(\mathbb{R}^3)$

$\mathrm{Sol}(B_n)$ projects onto $\mathfrak{P}_n$

$(\mathbb{Z}/2\mathbb{Z})^n$ invariant ring

TYPE B

ALGEBRA: Solomon's **Descent Algebra** [12] : $\mathrm{Sol}(B_n) \subset \mathbb{Q}[B_n]$

generated by sums of elements with the same (Coxeter) **descent** set:

$$Y_J := \sum_{\substack{w \in B_n \\ \mathrm{Des}(w) = J \subset [n]}} w \quad \text{where} \quad \mathrm{Des}(w) = \{i \in [n] : \ell(w) > \ell(ws_i)\}.$$

$\mathrm{Sol}(B_n)$ contains the **Type B Eulerian idempotents** due to Bergeron-Bergreron [4]:

$$\sum_{j=0}^{n} t^j \mathfrak{e}_B^{(j)} = \sum_{w \in B_n} \binom{\frac{t-1}{2} - 1 + n - |\mathrm{Des}(w)|}{n} w$$

THEOREM (B [5]) For $d \geq 3$ and odd,

The following decompositions of $\mathbb{Q}[B_n]$ coincide:

$$H^{2k}\mathrm{Conf}_n^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}^d) \cong \mathfrak{e}_B^{n-k}\mathbb{Q}[B_n] \text{ for } 0 \leq k \leq n$$

Bonus for experts: in fact for *any* finite Coxeter group of rank r, the following decompositions of $\mathbb{Q}[W]$ coincide:

$$\mathscr{VG}(W)_k \cong \mathfrak{e}_W^{r-k}\mathbb{Q}[W] \text{ for } 0 \leq k \leq r$$

where $\mathscr{VG}(W)_k$ is the degree k piece of the associated graded Varchenko-Gelfand ring $\mathscr{VG}(W)$ and $\mathfrak{e}_W^{r-k} \in \mathrm{Sol}(W)$

TOPOLOGICAL SPACE: the orbit configuration space

$$\mathrm{Conf}_n^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}^d) := \{(p_1, \cdots, p_n) \in \mathbb{R}^{dn} : p_i \neq \pm p_j \neq 0\}$$

Cohomology presentation due to Xicoténcatl [14], and

$$H^*\mathrm{Conf}_n^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}^d) \cong \begin{cases} \text{Orlik Solomon algebra } \mathscr{OS}(B_n) & d \geq 2 \text{ and even} \\ \text{associated graded Varchenko-Gelfand ring } \mathscr{VG}(B_n) & d \geq 3 \text{ and odd [9]} \end{cases}$$

The Poincare polynomial of $H^*\mathrm{Conf}_n^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}^d)$ for $d \geq 2$ is :

$$\sum_{k=0}^{n-1} t^k \dim\left(H^{k(d-1)}\mathrm{Conf}_n^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}^d)\right) = (1+t)(1+3t)\cdots(1+(2n-1)t)$$

EXAMPLE:

When $n = 6$, the regular representation $\mathbb{Q}[\mathfrak{S}_6]$ decomposes as follows:

k=6: $\mathrm{odd}(\lambda) = 0$	k=5: $\mathrm{odd}(\lambda) = 1$	k=4: $\mathrm{odd}(\lambda) = 2$	k=3: $\mathrm{odd}(\lambda) = 3$	k=2: $\mathrm{odd}(\lambda) = 4$	k=1: $\mathrm{odd}(\lambda) = 5$	k=0: $\mathrm{odd}(\lambda) = 6$
$\mathrm{Lie}_{(2,2,2)} + \mathrm{Lie}_{(6)} \cong$	No such $\lambda \vdash 6$	$\mathrm{Lie}_{(3,3)} + \mathrm{Lie}_{(4,1,1)} + \mathrm{Lie}_{(2,2,1,1)} \cong$	No such $\lambda \vdash 6$	$\mathrm{Lie}_{(2,1,1,1,1)} \cong$	No such $\lambda \vdash 6$	$\mathrm{Lie}_{(1,1,1,1,1,1)} \cong$
$\mathfrak{e}_{\mathfrak{P}}^0\mathbb{Q}[\mathfrak{S}_6] \cong$	$0 = \mathfrak{e}_{\mathfrak{P}}^1\mathbb{Q}[\mathfrak{S}_6]$	$\mathfrak{e}_{\mathfrak{P}}^2\mathbb{Q}[\mathfrak{S}_6] \cong$	$0 = \mathfrak{e}_{\mathfrak{P}}^3\mathbb{Q}[\mathfrak{S}_6]$	$\mathfrak{e}_{\mathfrak{P}}^4\mathbb{Q}[\mathfrak{S}_6] \cong$	$0 = \mathfrak{e}_{\mathfrak{P}}^5\mathbb{Q}[\mathfrak{S}_6]$	$\mathfrak{e}_{\mathfrak{P}}^6\mathbb{Q}[\mathfrak{S}_6] \cong$
$H^{12}\mathrm{Conf}_6(\mathbb{RP}^2 \times \mathbb{R})$	$= H^{10}\mathrm{Conf}_6(\mathbb{RP}^2 \times \mathbb{R})$	$H^8\mathrm{Conf}_6(\mathbb{RP}^2 \times \mathbb{R})$	$= H^6\mathrm{Conf}_6(\mathbb{RP}^2 \times \mathbb{R})$	$H^4\mathrm{Conf}_6(\mathbb{RP}^2 \times \mathbb{R})$	$= H^2\mathrm{Conf}_6(\mathbb{RP}^2 \times \mathbb{R})$	$H^0\mathrm{Conf}_6(\mathbb{RP}^2 \times \mathbb{R})$

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