Dyck combinatorics in Kazhdan–Lusztig theory

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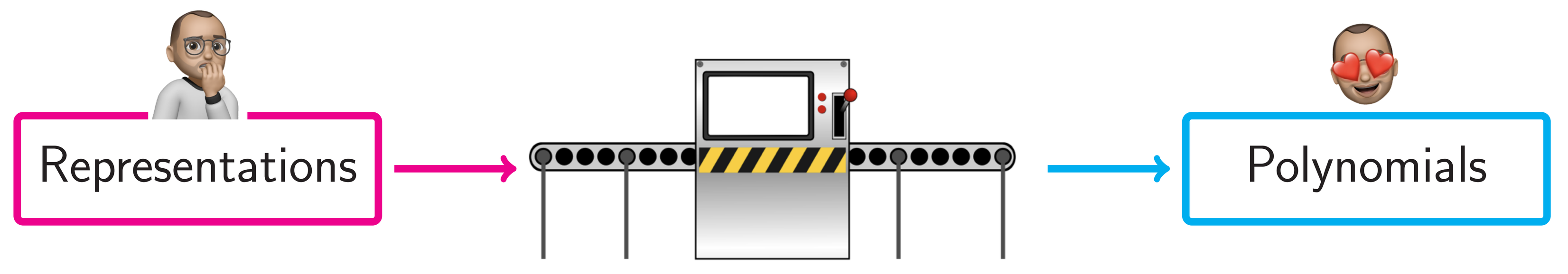
Kazhdan–Lusztig theory

Kazhdan–Lusztig theory is a machine that encodes (graded) representation theoretic information in terms of polynomials. This makes scary abstract structures more into something more concrete.

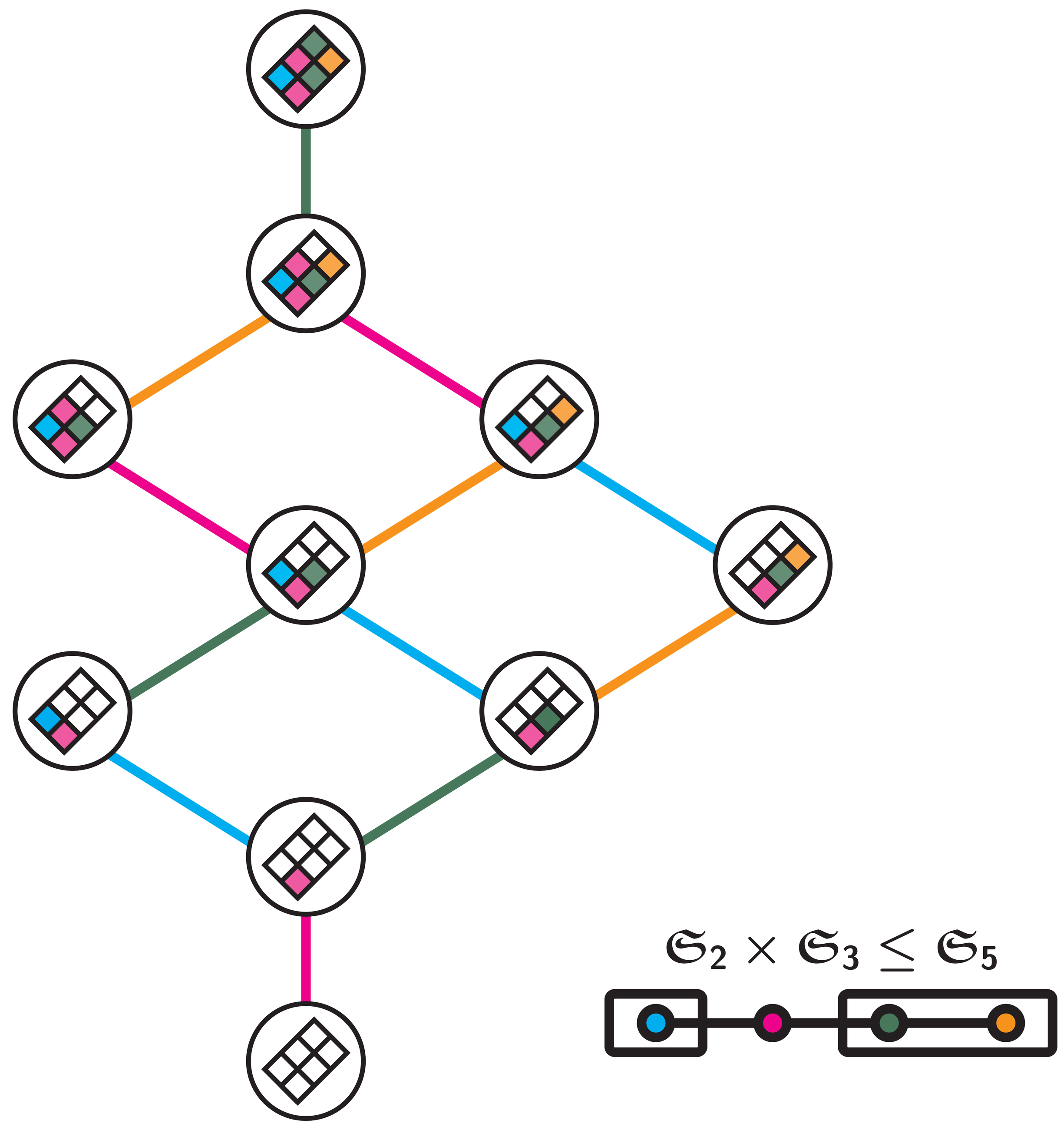


Kazhdan–Lusztig theory

We ask *"what are the limits to what Kazhdan–Lusztig combinatorics can tell us about the representation theoretic structures?"*

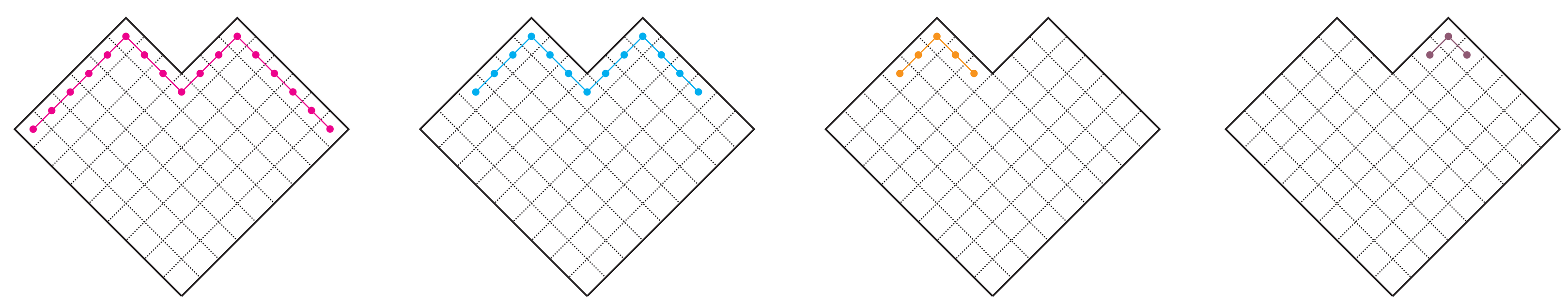
Let's start at the very beginning

What a very good place to start! The nicest Kazhdan–Lusztig polynomials are those of maximal parabolics of symmetric groups. The cosets for these parabolics are labelled by partitions in a rectangle and are partially ordered by inclusion. For example the cosets of $\mathfrak{S}_2 \times \mathfrak{S}_3 \leq \mathfrak{S}_5$ are as follows,

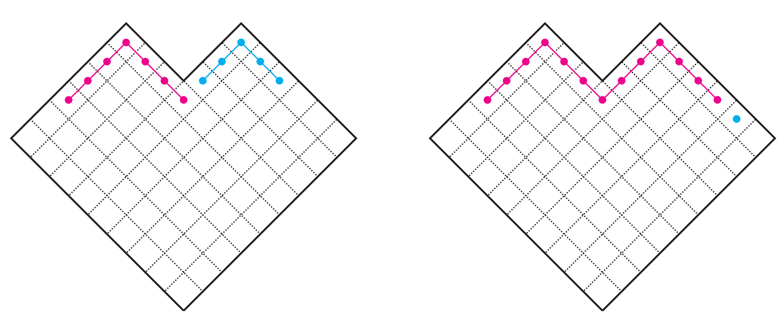


Kazhdan–Lusztig polynomials as Dyck tilings

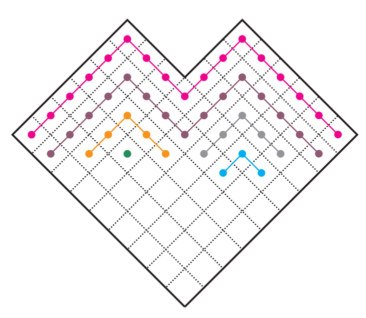
We draw Dyck paths on our partitions as follows



and we say a pair of Dyck paths is bad if their union forms an interval, for example



are both bad. We say that $\mu \setminus \lambda$ has a Dyck pair of degree k if it is a disjoint union of k Dyck paths which are pairwise good. For example



is Dyck a pair of degree $\mathbf{6}$. The Kazhdan–Lusztig polynomials of maximal parabolics of symmetric groups are enumerated by Dyck tilings as follows

$$n_{\lambda,\mu}(q) = \begin{cases} q^{\deg(\mu\setminus\lambda)} & \text{if } \mu \setminus \lambda \text{ is a Dyck pair;} \\ 0 & \text{otherwise.} \end{cases}$$

We let $\mathbf{Dyck}_{m,n}(\lambda) = \{\mu \mid \mu \setminus \lambda \text{ is a Dyck pair}\}$.

Character formulas

Our love of these polynomials comes from the fact that they control the structure of parabolic Verma modules for Lie algebras, general linear supergroups, the Khovanov arc algebras (of categorical knot theory), Brauer and walled Brauer algebras. . .

All these categories possess "Verma modules" $V_{m,n}(\lambda)$ (easily constructed via explicit bases) and simple modules $L_{m,n}(\mu)$ (which are more mysterious). The composition factor multiplicity of the former in the latter is given by the associated Kazhdan–Lusztig polynomial

$$n_{\lambda,\mu}(q)|_{q=1} = [V_{m,n}(\lambda) : L_{m,n}(\mu)]$$

and so we obtain a combinatorial understanding of the simple modules of these categories in terms of Dyck tilings.

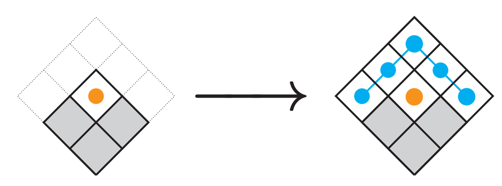
Categorification

The categorification programme seeks to use monoidal (or if you prefer, *diagrammatic*) presentations in order find new structural information. . . can we use this to get more structural information about Verma modules?

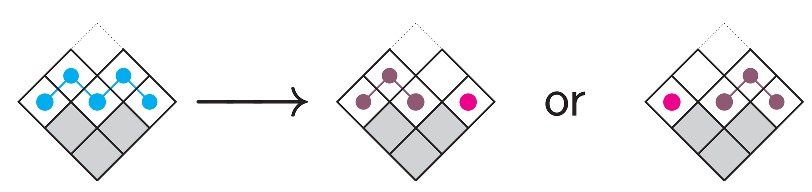
A new partial ordering on Dyck tilings

Let $\mu \setminus \lambda$ and $\nu \setminus \lambda$ be Dyck pairs with $\deg(\mu \setminus \lambda) = k$ and $\deg(\nu \setminus \lambda) = k + 1$. We write $\mu \to \nu$ if either:

- $\nu = \mu + P$ for P an addable Dyck path.



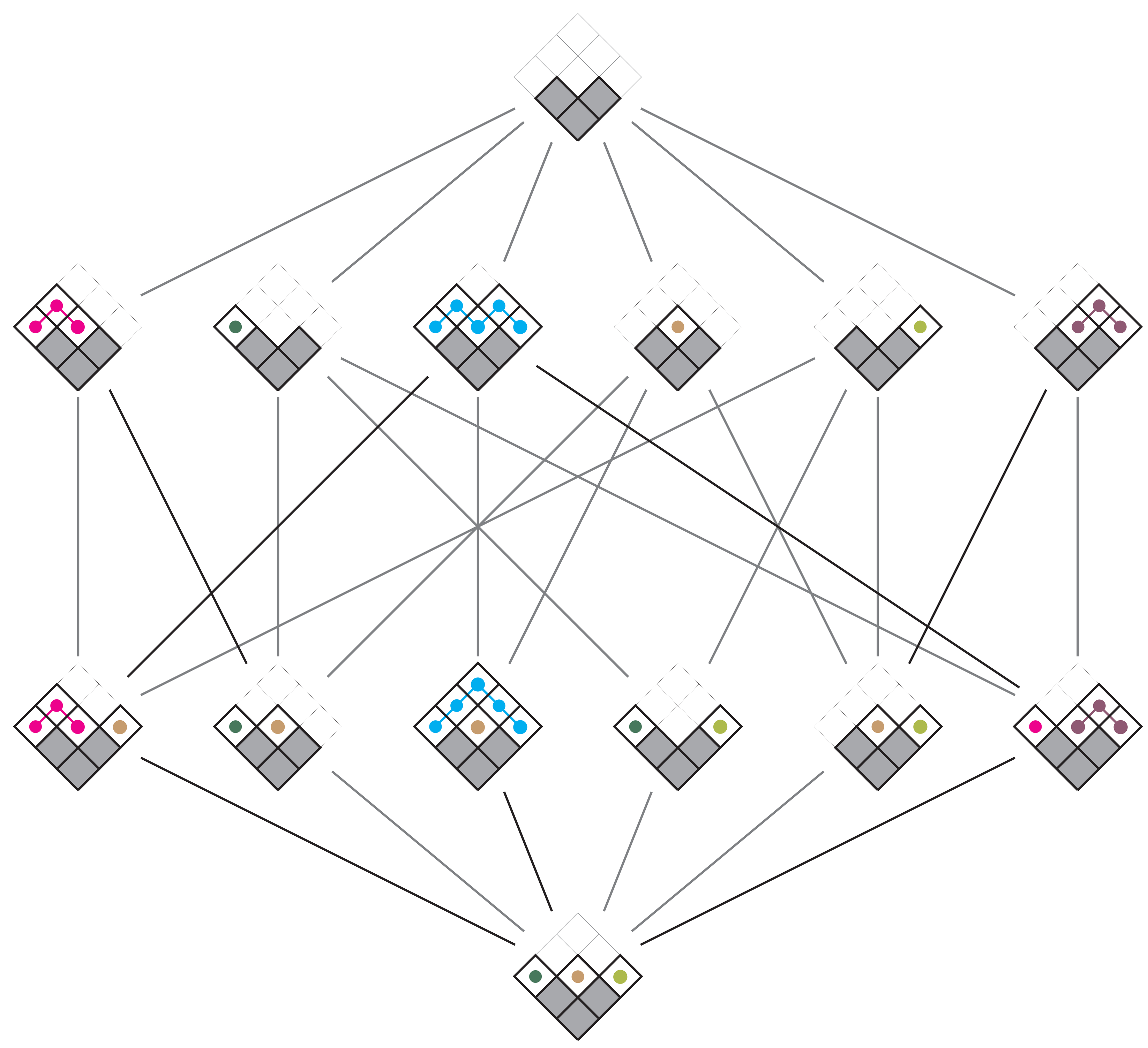
- $\nu = \mu - P$ for P a removable Dyck path



- We extend this to a partial ordering on $\mathbf{Dyck}_{m,n}(\lambda)$ by transitivity.
- We are no longer counting the sizes of sets (the remit of classical Kazhdan–Lusztig theory). Rather, we are considering the richer structures offered by considering maps between these sets — we call this "meta Kazhdan–Lusztig theory".

An example of the new partial ordering

The partial ordering on $\mathbf{Dyck}_{3,3}(2, 1)$ is as follows



Theorem: Submodule lattices of Verma modules

And now for our "ta-da!" moment: The submodule lattice of $V_{m,n}(\lambda)$ is given by the partial ordering on Dyck tilings!

An example of a submodule lattice of a Verma module

For example, the Verma module $V_{3,3}(2, 1)$ has 14 simple composition factors and the submodule lattice is depicted above. Notice that $L_{3,3}(2, 1)$ is at the top of the Verma and $L_{3,3}(3, 2, 1)$ at the bottom of the module.

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