Positroids and Boundary Charts

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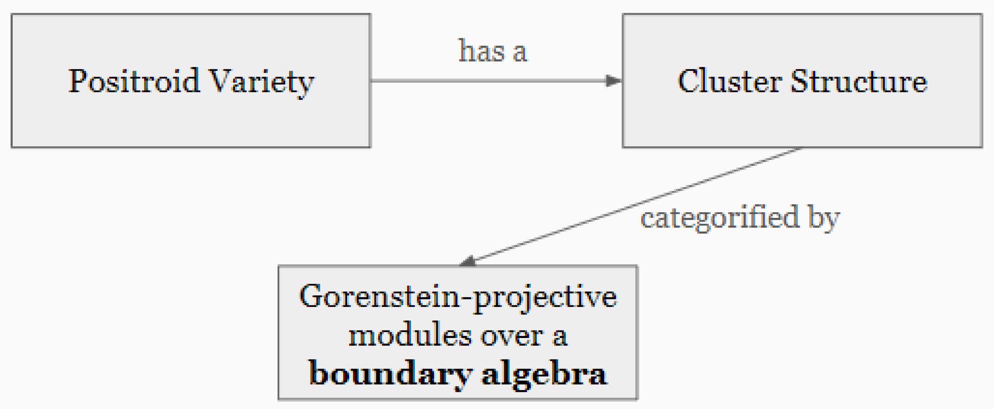
Introduction



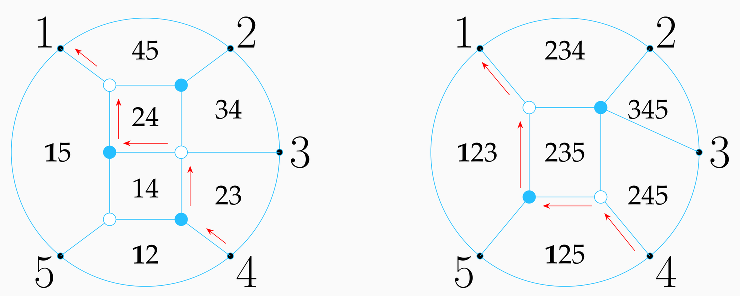
This categorification was first studied in the full Grassmannian [Sco06, JKS16, BKM16] and then generalized to all positroid varieties [Pre22,GL23]. It has proven useful, for example, in proving that two cluster structures on positroid varieties quasi-coincide [Pre23].

The construction of the boundary algebra of a connected positroid in the literature has a number of involved steps. We give a **direct combinatorial description of the boundary algebra and its generating relations**. We will view this as a **cryptomorphism** for connected positroids.

Positroids

Positroids are realizable matroids reflecting the combinatorial structure of the *totally nonnegative Grassmannian*.

A (reduced) **plabic graph** (planar bi-colored) is a planar graph embedded in a disc. Interior vertices are assigned empty (○) or filled (●). Boundary vertices must be incident to exactly one edge.



$\pi = 3\,4\,5\,1\,2$ $\qquad$ $\pi = 4\,5\,2\,1\,3$

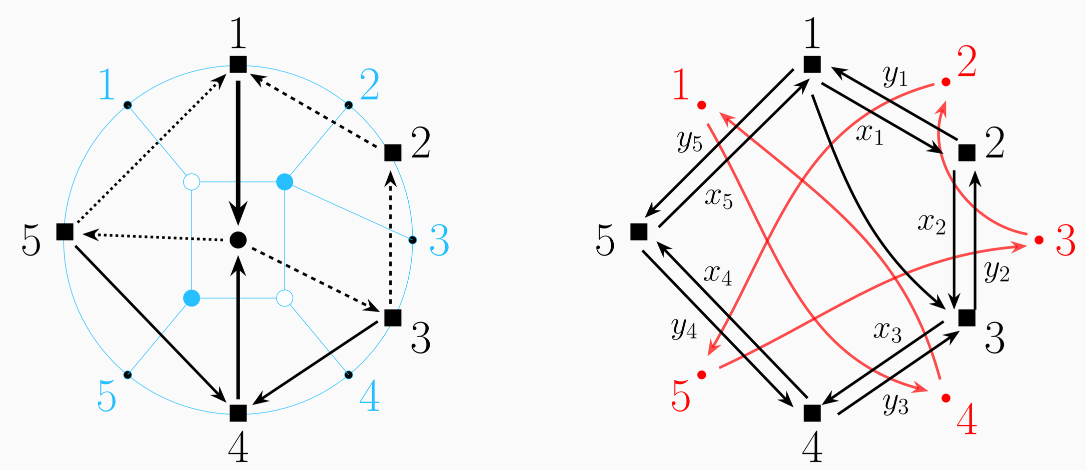
Positroids are indexed by **decorated permutations**. Obtain these from plabic graphs using the "*rules of the road*" from each vertex: Turn right at filled vertices and left at empty vertices. This yields the **trip permutation**. One must *decorate*, i.e. distinguish, different types of fixed points, but we only consider connected positroids, whose permutations have no fixed points.

For any bipartite plabic graph G, the **quiver** $Q(G)$ is the dual to G with the exterior vertex removed and with edges oriented so that ○ is on the left. This is illustrated in the next column. $Q(G)$ defines a cluster algebra which depends only on the trip permutation and is isomorphic to a cluster structure on a subset Π°_π of the Grassmannian called an **open positroid variety** [KLS13].

Boundary Algebras

Boundary algebras are an important tool in the categorification of the cluster structure on positroid varieties. Let G be a plabic graph with trip permutation π.

- Path algebra $\mathbb{C}Q(G)$: Algebra spanned by finite paths in quiver $Q(G)$, with operation of concatenation.
- Dimer algebra $A_{Q(G)}$: For each internal edge, identify two paths in $\mathbb{C}Q(G)$ as illustrated by the dashed and dotted edges in the left subfigure below.
- Boundary algebra B_π: In $A_{Q(G)}$, let e_i be the empty path at boundary vertex i, and $e = \sum_i e_i$. Then $B_\pi := eA_{Q(G)}e$ consists of paths in the dimer algebra starting and ending at the boundary.



$Q(G)$ $\qquad$ Q°_π

$\pi = 4\,5\,2\,1\,3$

Paths between nonadjacent boundary vertices in $Q(G)$ which do not factor as products of other paths between boundary vertices are called nonadjacent arrows (like $1 \to 3$ above). Let Q°_π be the quiver with vertices $1, 2, \ldots, n$, arrows between adjacent vertices, and nonadjacent arrows.

Theorem [BS24]: The boundary algebra is a quotient of $\mathbb{C}Q^\circ_\pi(G)$ which can be computed using only the data of the nonadjacent arrows of Q°_π along with the number X_p of permutation strands which are to the left of and antiparallel each nonadjacent arrow p.

One of the most difficult steps of the construction involves taking the **cancellative closure** of an ideal I_π. The cancellative closure, denoted $\mathrm{C}(I_\pi)$, is the smallest ideal containing I_π such that for any arrow p of Q°_π and element q of $\mathbb{C}Q^\circ_\pi$, we have $pq \in \mathrm{C}(I_\pi) \iff q \in \mathrm{C}(I_\pi)$. Our work allows us to give explicit generators for $\mathrm{C}(I_\pi)$.

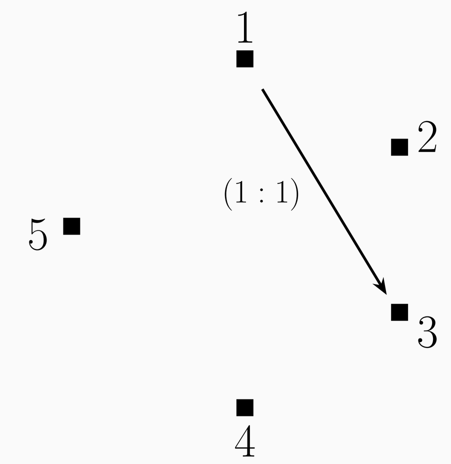
We will now describe how to recover the permutation of a connected positroid from its boundary algebra and develop a combinatorial structure describing these algebras. We thus offer a new combinatorial object in bijection with connected positroids, that is, a **cryptomorphism** of connected positroids.

Boundary Charts

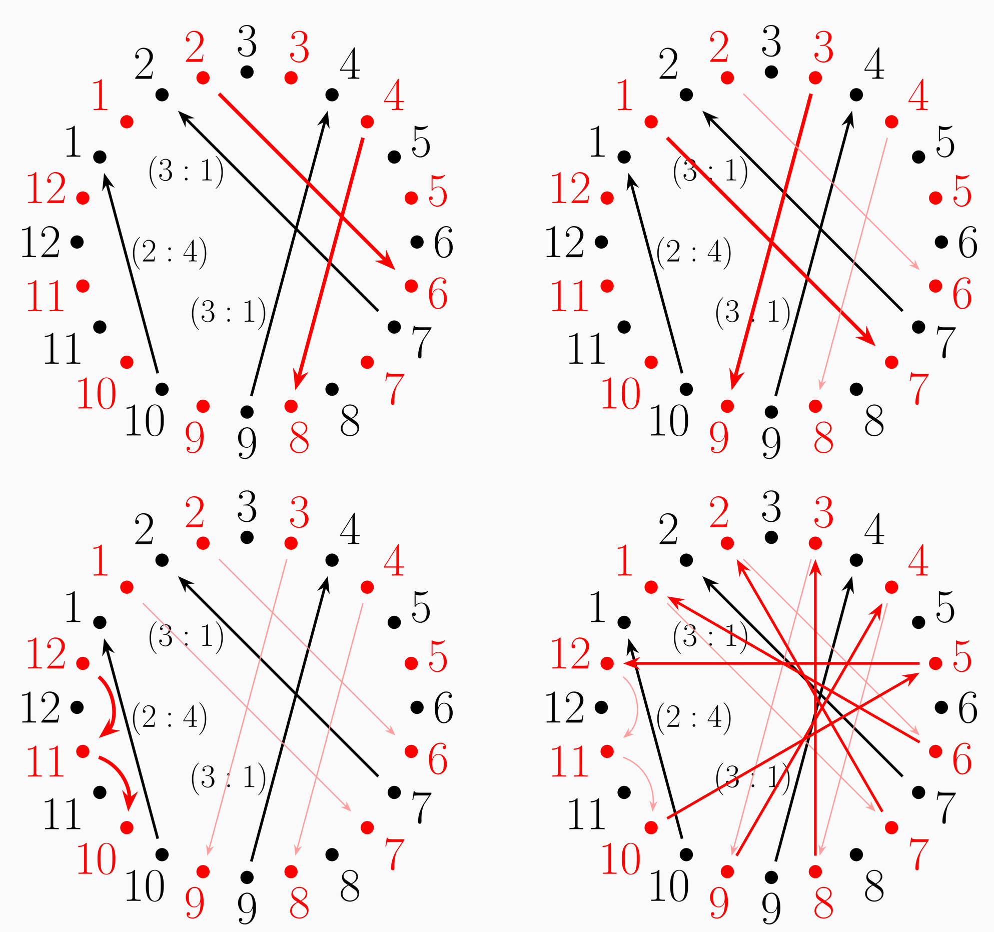
Goals: Characterize the quivers Q°_π and determine π from its boundary algebra.

A **boundary chart** encapsulates the important data from the quiver Q°_π. It has n vertices, nonadjacent arrows between them, and for each arrow p a pair of integers $(X_p : Y_p)$ satisfying $Y_p - X_p + \mathrm{reach}_p = k$ for some fixed k.

The boundary algebra of a connected positroid gives rise to a boundary chart. We call such a boundary chart **realizable**. For example, for $\pi = 4\,5\,2\,1\,3$:



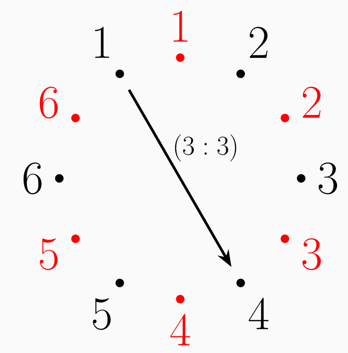
To recover the permutation π (and the corresponding positroid) from a realizable boundary chart: Inducting on the arrows, from right to left, let Y'_p be the difference between Y_p, and the number of permutation strands antiparallel and to the right of p. We connect the i^{th} closest permutation-vertex from the head of p to the $(Y'_p+1-i)^{\text{th}}$ closest permutation-vertex from the tail of p, skipping over vertices j which already have strands starting or ending there (See figure below, top row). This totals Y_p strands antiparallel and right of p. Similarly draw X_p antiparallel strands to the left of p (bottom left). Connect all other permutation-vertices i to $i-k$ (bottom right).



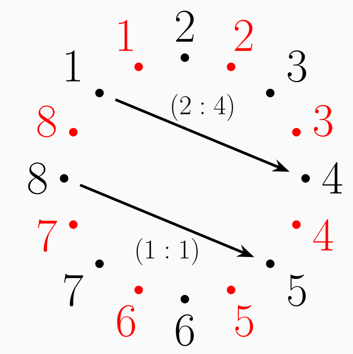
$k = 5, \quad n = 12, \quad \pi = 7\,6\,9\,8\,12\,1\,2\,3\,4\,5\,10\,11$

Boundary Charts (cont.)

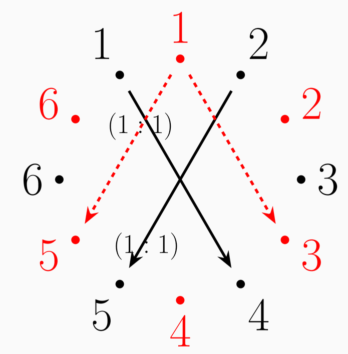
Not all boundary charts are realizable. The following would render a boundary chart unrealizable:



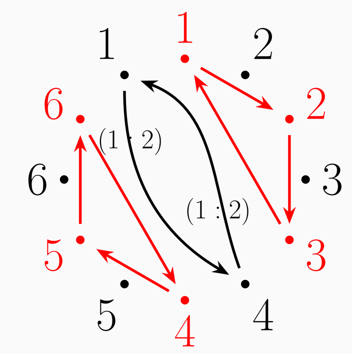
There is no space for three permutation strands to the left of the arrow $1 \to 4$.



The two strands left and antiparallel to $1 \to 4$ will overwhelm the arrow $8 \to 5$.



The crossing arrows create two different strands originating at 1.



The digon with X_p and Y_p values summing to $n = 6$ causes a disconnected permutation.

Theorem: These are the only obstacles to realizability.

Applications of Boundary Charts

1. Boundary charts allow us to compute explicit generators for $C(I_\pi)$. To do so, we need to make use of information that is readily available from the boundary chart of π, but not from π, as well as information readily available from π but not from its boundary chart. Thus, a permutation and its boundary chart offer different insight into Π°_π.
2. Realizable boundary charts are a new cryptomorphic description of connected positroids.

References

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