Murnaghan-Type Representations for the Elliptic Hall Algebra

Milo Bechtloff Weising, arXiv: 2405.00756

University of California, Davis

Introduction

The ring of **symmetric functions** Λ over the two-parameter field $\mathbb{Q}(q,t)$ is a fundamental object in modern combinatorics relating to representation theory and geometry. There are many special operators which act on Λ occurring in modern algebraic combinatorics including the **Macdonald operator** Δ which acts diagonally on the **symmetric Macdonald functions** $P_\lambda[X;q,t]$. One may understand the operator Δ using the **Cherednik operators** and the **double affine Hecke algebras** of Cherednik which act on finite variable non-symmetric polynomials $\mathbb{Q}(q,t)[x_1,\ldots,x_n]$. In order to relate these two notions one must perform a symmetrization and stable-limit process which the authors Schiffmann-Vasserot show gives more than just an understanding of Δ. They show that the **elliptic Hall algebra** $\mathcal{E}^+$ of Burban-Schiffmann acts on Λ and this action contains many of the important operators in Macdonald theory including Δ. We will take this approach to construct a family of representations for $\mathcal{E}^+$ generalizing Λ.

Background

Elliptic Hall Algebra

The (positive) elliptic Hall algebra $\mathcal{E}^+$ of Burban-Schiffmann is the stable-limit of the (positive) spherical double affine Hecke algebras in type GL_n. The algebra $\mathcal{E}^+$ is generated by the special elements $P_{0,\ell}, P_{r,0}$ for $\ell \in \mathbb{Z}\setminus\{0\}$ and $r>0$ which are limits of the special elements of the spherical double affine Hecke algebras

- $P_{0,\ell} := \lim_n P_{0,\ell}^{(n)} = \lim_n \epsilon^{(n)}\left(\sum_{i=1}^n \theta_i^\ell\right)\epsilon^{(n)}$
- $P_{r,0} := \lim_n P_{r,0}^{(n)} = \lim_n q^r\epsilon^{(n)}\left(\sum_{i=1}^n X_i^r\right)\epsilon^{(n)}$.

Here $\epsilon^{(n)}$ is the **trivial Hecke idempotent** and θ_i are the **Cherednik elements**.

Vector-Valued Polynomials

For a partition λ with size $|\lambda|=n$, Dunkl-Luque define the space of **vector-valued** (vv.) polynomials V_λ as

$$V_\lambda := \mathbb{Q}(q,t)[x_1,\ldots,x_n]\otimes S_\lambda$$

where S_λ is the Specht-module corresponding to λ for the finite Hecke algebra with quadratic relation $(T_i-1)(T_i+t)=0$. The space S_λ has a distinguished basis $\{e_\tau|\tau\in \mathrm{SYT}(\lambda)\}$ of weight vectors for the Jucys-Murphy elements. We define the operators

$$T_i(x^\alpha\otimes v) := x^{s_i(\alpha)}\otimes T_i(v) + (1-t)\frac{x^\alpha - x^{s_i(\alpha)}}{x_i - x_{i+1}}\otimes v$$

and

$$\pi_n(x^\alpha\otimes v) := (qx_1)^{\alpha_n}x_2^{\alpha_1}\cdots x_n^{\alpha_{n-1}}\otimes t^{n-1}T_1^{-1}\cdots T_{n-1}^{-1}(v).$$

We also define X_i as the obvious multiplication operator and using $\theta_i := t^{-(n-i)}T_{i-1}^{-1}\cdots T_1^{-1}\pi_n T_{n-1}\cdots T_i$, we define the action of θ_i. These operators generate a representation of the (positive) double affine Hecke algebra of type GL_n. This representation admits a basis of θ-weight vectors $\{F_\tau|\tau\in \mathrm{PSYT}_{\geq 0}(\lambda)\}$ called the **non-symmetric vv. Macdonald polynomials** indexed by certain labelled diagrams $\tau\in\mathrm{PSYT}_{\geq 0}(\lambda)$ called (non-negative) **periodic standard Young tableaux**. By Hecke-symmetrizing using $\epsilon^{(n)}$ one obtains the **symmetric vv. Macdonald polynomials** P_T indexed by (non-negative) **reverse semi-standard Young tableaux** of shape λ. These are a basis for $W_\lambda := \epsilon^{(n)}V_\lambda$.

We give a non-symmetric example here for the diagram $\begin{smallmatrix}1q & 2q\\ 3 & \end{smallmatrix}\in \mathrm{PSYT}_{\geq 0}(2,1)$:

$$\begin{aligned}F_{\begin{smallmatrix}1q & 2q\\ 3 & \end{smallmatrix}} &= t^{-2}X_1X_2\otimes e_{\begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix}} + t^{-2}\left(\frac{1-t}{1-qt^2}\right)X_2X_3\otimes e_{\begin{smallmatrix}1 & 3\\ 2 & \end{smallmatrix}}\\ &+\frac{t^{-2}}{1+t}\left(\frac{1-t}{1-qt^2}\right)X_2X_3\otimes e_{\begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix}} - t^{-3}\left(\frac{1-t}{1-qt^2}\right)X_1X_3\otimes e_{\begin{smallmatrix}1 & 3\\ 2 & \end{smallmatrix}}\\ &+\frac{t^{-1}}{1+t}\left(\frac{1-t}{1-qt^2}\right)X_1X_3\otimes e_{\begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix}}\end{aligned}$$

Murnaghan-Type Representations

In the following result we see that there exist representations of $\mathcal{E}^+$ which may be built from the spaces V_λ along certain sequences of partitions λ directly generalizing the way one may obtain Λ from the polynomial spaces $\mathbb{Q}(q,t)[x_1,\ldots,x_n]$. The proof of the following theorem involves carefully defining quotient maps $V_\lambda\to V_{\lambda'}$ with exceptional stability properties.

For a partition λ and $n\geq n_\lambda := |\lambda|+\lambda_1$ set $\lambda^{(n)} := (n-|\lambda|,\lambda)$. Define $\Omega(\lambda)$ to be the set of (non-negative) reverse semi-standard fillings of the infinite diagram $\lambda^{(\infty)}$ which are 0 almost everywhere.

Theorem 1 [BW 24]

For any partition λ there exists a stable-limit construction for the family of spherical double affine Hecke algebra modules $(W_{\lambda^{(n)}})_{n\geq n_\lambda}$ generating a module $\widetilde{W}_\lambda$ for the positive elliptic Hall algebra $\mathcal{E}^+$. This space has a distinguished basis of weight vectors $\mathfrak{P}_T$ for the action of $P_{0,1}$ called the **generalized Macdonald functions** indexed by $T\in\Omega(\lambda)$.

Examples

For $\lambda=(3,2,1)$ we have the diagram

$$T = \begin{array}{|c|c|c|c|c|c|c|c|l}\hline 5 & 3 & 3 & 2 & 1 & 0 & 0 & 0 & \cdots\\ \hline 3 & 2 & 0 & \multicolumn{6}{l}{}\\ \cline{1-3} 1 & 1 & \multicolumn{7}{l}{}\\ \cline{1-2} 0 & \multicolumn{8}{l}{}\\ \cline{1-1}\end{array} \in \Omega(3,2,1).$$

The corresponding generalized Macdonald function $\mathfrak{P}_T$ has X-degree $5+3+3+2+1+3+2+1+1+1=21$ and is an eigenvector for the action of $P_{0,1}$ with eigenvalue

$$(q^5-1)+(q^3-1)(t^{-1}+t^1+t^2)+(q^2-1)(t^0+t^3)+(q-1)(t^{-2}+t^{-1}+t^4).$$

Here $c(\square)$ is the **content** of the box $\square$.

Main Properties

- For $\lambda=\emptyset$, $\widetilde{W}_\emptyset=\Lambda$ and $\Omega(\emptyset)$ is naturally in bijection with the set of partitions. Further, in this case $\mathfrak{P}_T = P_T[X;q,t]$ up to a scalar.
- For distinct partitions $\lambda\neq\mu$, $\widetilde{W}_\lambda\ncong\widetilde{W}_\mu$ as $\mathcal{E}^+$ modules.
- Each $\mathfrak{P}_T$ is the limit of the symmetric vv. Macdonald polynomials $P_{T|_{\lambda^{(n)}}}$.
- $\mathfrak{P}_T$ is X-homogeneous of degree $\sum_{\square\in\lambda^{(\infty)}}T(\square)$
- The spectrum of $P_{0,1}$ on $\widetilde{W}_\lambda$ is simple and given by
$$\{\sum_{\square\in\lambda^{(\infty)}}(q^{T(\square)}-1)t^{c(\square)}\mid T\in\Omega(\lambda)\}.$$
- Each of the representations $\widetilde{W}_\lambda$ has a unique 1-dimensional subspace of minimal X-degree spanned by $\mathfrak{P}_{T_\lambda^{\min}}$ where $T_\lambda^{\min}$ is the unique element of $\Omega(\lambda)$ with minimal sum of entries given by $T_\lambda^{\min}(\square) := \#\{\square'\in\lambda^{(\infty)}|\square' \text{ strictly below } \square\}$.
- Each $\widetilde{W}_\lambda$ is a cyclic $\mathcal{E}^+$ module generated by $\mathfrak{P}_{T_\lambda^{\min}}$.
- The element $P_{r,0}$ acts on $\widetilde{W}_\lambda$ by multiplication by $q^rp_r[X]$.
- For all $\ell\in\mathbb{Z}\setminus\{0\}$
$$P_{0,\ell}(\mathfrak{P}_T) = \left(\sum_{\square\in\lambda^{(\infty)}}(q^{\ell T(\square)}-1)t^{\ell c(\square)}\right)\mathfrak{P}_T.$$

Pieri Rule

We find the following Pieri rule for the multiplication by $e_r^\bullet$ map on $\widetilde{W}_\lambda$ generalizing the ordinary Pieri rule for the symmetric Macdonald functions $P_\lambda[X;q,t]$. We achieve this formula by studying the combinatorics of the expansion of the P_T into the F_τ basis coming from inversion sets and using a few techniques coming from double affine Hecke algebras. We give here only the finite rank Pieri rule for the P_T from which the full Pieri rule for the $\mathfrak{P}_T$ may be directly inferred.

Theorem 2 [BW 24]

For $T\in\mathrm{RSSYT}_{\geq 0}(\lambda)$ and $1\leq r\leq n$ we have the expansion

$$e_r[X_1+\ldots+X_n]P_T = \sum_S d_{S,T}^{(r)}P_S$$

where $d_{S,T}^{(r)}$ is the product of

$$t^{\binom{r}{2}}e_r\left[\frac{1-t^n}{1-t}\right]\frac{[\mu(S)]_t!}{[n]_t!}\prod_{(\square_1,\square_2)\in\mathrm{Inv}(\min(S))}\left(\frac{q^{S(\square_1)}t^{c(\square_1)}-q^{S(\square_2)}t^{c(\square_2)+1}}{q^{S(\square_1)}t^{c(\square_1)}-q^{S(\square_2)}t^{c(\square_2)}}\right)$$

and the sum

$$\sum_{\substack{\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi^r(\tau)\in\mathrm{PSYT}_{\geq 0}(\lambda;S)}} t^{c_\tau(1)+\ldots+c_\tau(r)}\prod_{(\square_1,\square_2)\in\mathrm{Inv}(\tau)}\left(\frac{q^{T(\square_1)}t^{c(\square_1)+1}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)}}\right)\times$$
$$\prod_{(\square_1,\square_2)\in\mathrm{Inv}(\Psi^r(\tau))}\left(\frac{q^{S(\square_1)}t^{c(\square_1)}-q^{S(\square_2)}t^{c(\square_2)}}{q^{S(\square_1)}t^{c(\square_1)}-q^{S(\square_2)}t^{c(\square_2)+1}}\right)$$

and S ranges over all $S\in\mathrm{RSSYT}_{\geq 0}(\lambda)$ one can obtain from T by adding r to the boxes of T with at most one 1 being added to each box. Further, $d_{S,T}^{(1)}\neq 0$ whenever S may be obtained from T by adding the value of 1 to a single box of T.

q,t Product-Series Identities

By understanding the expansions of the P_T into the F_τ and applying some elementary non-archimedean analysis techniques we find an interesting and nontrivial family of product-series identities in $\mathbb{Q}(q)((t))$. Here $\mathrm{APSYT}_{\geq 0}(\lambda)$ is the set of **asymptotic periodic standard Young tableaux** with basement shape λ which are certain labellings of $\lambda^{(\infty)}$.

Theorem 3 [BW 24]

For $T\in\Omega(\lambda)$ we have the following equality in $\mathbb{Q}(q)((t))$:

$$\frac{\prod_{\square\in\lambda^{(\mathrm{rk}(T))}}\left(1-q^{-T(\square)}t^{\mathrm{rk}(T)-|\lambda|-c(\square)}\right)}{(1-t)^{\mathrm{rk}(T)}[\mu(T|_{\lambda^{(\mathrm{rk}(T))}})]_t!}\prod_{(\square_1,\square_2)\in\mathrm{I}(\lambda^{(\mathrm{rk}(T))})}\left(\frac{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)}}{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)+1}}\right)$$
$$=\sum_{\tau\in\mathrm{APSYT}_{\geq 0}(\lambda;T)}t^{\mathrm{inv}(\tau)}\prod_{(\square_1,\square_2)\in\mathrm{Inv}(\tau)}\left(\frac{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)-1}}{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)+1}}\right).$$

In particular, the above **infinite** series is actually **rational** i.e. in $\mathbb{Q}(q,t)$.

References

[1] Milo Bechtloff Weising, *Murnaghan-type representations of the positive elliptic Hall algebra*, arXiv.2405.00756 (2023).

[2] C. F. Dunkl and J.-G. Luque, *Vector valued Macdonald polynomials*, Sém. Lothar. Combin. **66** (2011/12), Art. B66b, 68. MR2971011

[3] Olivier Schiffmann and Eric Vasserot, *The elliptic Hall algebra and the K-theory of the Hilbert scheme of $\mathbb{A}^2$*, Duke Math. J. **162** (2013), no. 2, 279–366. MR3018956

mjbechtloffweising@ucdavis.edu