Shards for the affine symmetric group

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Abstract

The poset of biclosed sets in a root system is a natural extension of the weak Bruhat order on the associated Coxeter group [3]. In the case of the affine symmetric group $\widetilde{S}_n$, there is a combinatorial model for this poset, introduced in [1], in terms of total orders of $\mathbb{Z}$. It was shown in [1, 2] that this poset is a complete lattice. We show that completely join-irreducible elements of the extended weak order biject with shards in the affine braid arrangement, which are certain cones. We also give a parametrization of both objects by "type-$\widetilde{A}$ arc diagrams".

Cyclic arc diagrams

Shard arcs

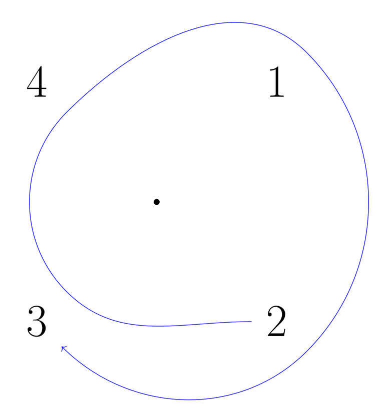
A **shard arc** for $\widetilde{S}_n$ is the data of:

- an initial value i and a terminal value j, such that $1 \le i \le n$ and $i < j$ and $i \not\equiv j \mod n$, and
- for each intermediate value k with $i < k < j$, a choice of "inside" or "outside".

These data must satisfy a non-crossing condition: it must be possible to draw the arc in a cyclic arc diagram without self-crossing. (See below.) Two shard arcs are equivalent if they differ by translating by n.

To draw a cyclic arc diagram, start with the numbers $1, \dots, n$ arranged in a circle, and draw an arc moving clockwise, starting at i, passing $j - i - 1$ intermediate values which are on either the inside or the outside, and ending at the residue of j mod n.

$i = 2$
$3:$ outside
$4:$ outside
$5:$ inside
$6:$ inside
$j = 7$



From extended weak order to arcs

Lower walls

Given a TITO $(\prec)$, we say that (i, j) is a **lower wall** if $i < j$ and $j \prec i$ is a cover relation in the TITO.

We can upgrade a lower wall (i, j) of $(\prec)$ to a shard arc with initial value i and terminal value j. For each intermediate value k, we choose "inside" if $k \prec j$, and we choose "outside" if $i \prec k$.

Any completely join-irreducible TITO has a unique lower wall (up to translation by n); hence, there is a map

$$\text{(complete join-irreducibles)} \Rightarrow \text{(shard arcs)}.$$

Extended weak order

The **affine symmetric group** $\widetilde{S}_n$ is the group of bijections $\widetilde{\pi} : \mathbb{Z} \to \mathbb{Z}$ satisfying:

- $\widetilde{\pi}(i + n) = \widetilde{\pi}(i) + n$ for all $i \in \mathbb{Z}$, and
- $\sum_{i=1}^{n} \widetilde{\pi}(i) = \sum_{i=1}^{n} i$.

Extended weak order

A **translationally invariant total order** (**TITO**) is a total ordering $(\prec)$ of $\mathbb{Z}$ so that:

- For all $i, j \in \mathbb{Z}$, we have $i \prec j$ if and only if $i + n \prec j + n$, and
- For all $i \in \mathbb{Z}$, if $i + n \prec i$ then there exists a k with $i + n \prec k \prec i$.

An **inversion** of a TITO $(\prec)$ is a pair (i, j) so that $i < j$ and $j \prec i$.

The **extended weak order** for $\widetilde{S}_n$ is the poset of TITOs, where $(\prec_1) \le (\prec_2)$ if every inversion of $(\prec_1)$ is an inversion of $(\prec_2)$.

Some examples of TITOs for $n = 4$:

$$\cdots \prec -1 \prec 0 \prec 2 \prec 1 \prec 3 \prec 5 \prec 4 \prec 6 \prec \cdots$$

$$\cdots \prec 5 \prec 4 \prec 3 \prec 2 \prec 1 \prec 0 \prec -1 \prec -2 \prec \cdots$$

$$\cdots \prec -1 \prec 1 \prec 3 \prec 5 \prec 7 \prec \cdots \prec 6 \prec 4 \prec 2 \prec 0 \prec -2 \prec \cdots$$

$$\cdots \prec 1 \prec 5 \prec \cdots \prec 6 \prec 7 \prec 2 \prec 3 \prec -2 \prec \cdots \prec 0 \prec 4 \prec \cdots.$$

Affine braid arrangements

Affine braid arrangements

The **affine braid arrangement** $\widetilde{\mathcal{B}}_n$ consists of hyperplanes $\widetilde{H}_{ab}$ in $\mathbb{R}^{n+1}$, where

$$\widetilde{H}_{ab} \coloneqq \{(y, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_a = x_b\}.$$

Convention: $x_{a+kn} = x_a + ky$ for any $k \in \mathbb{Z}$.

Each hyperplane defines two half-spaces:

$$\widetilde{H}^+_{ab} \coloneqq \{(y, x_1, \dots, x_n) \mid x_a \le x_b\}$$
$$\widetilde{H}^-_{ab} \coloneqq \{(y, x_1, \dots, x_n) \mid x_a \ge x_b\}.$$

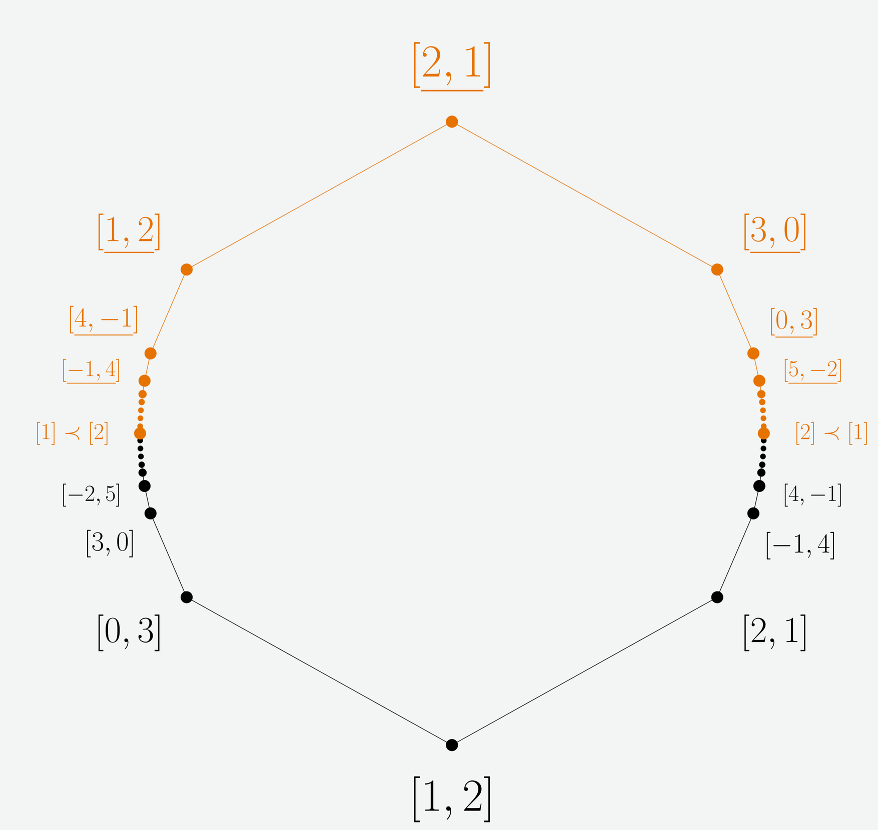
Window notation

Any TITO breaks up into subintervals isomorphic to the ordering on $\mathbb{Z}$, called **blocks**. We represent a block using a **window**; if the block contains k residue classes mod n, then a window consists of k consecutive elements of the block. We underline the window if it contains an element i so that $i + n \prec i$.

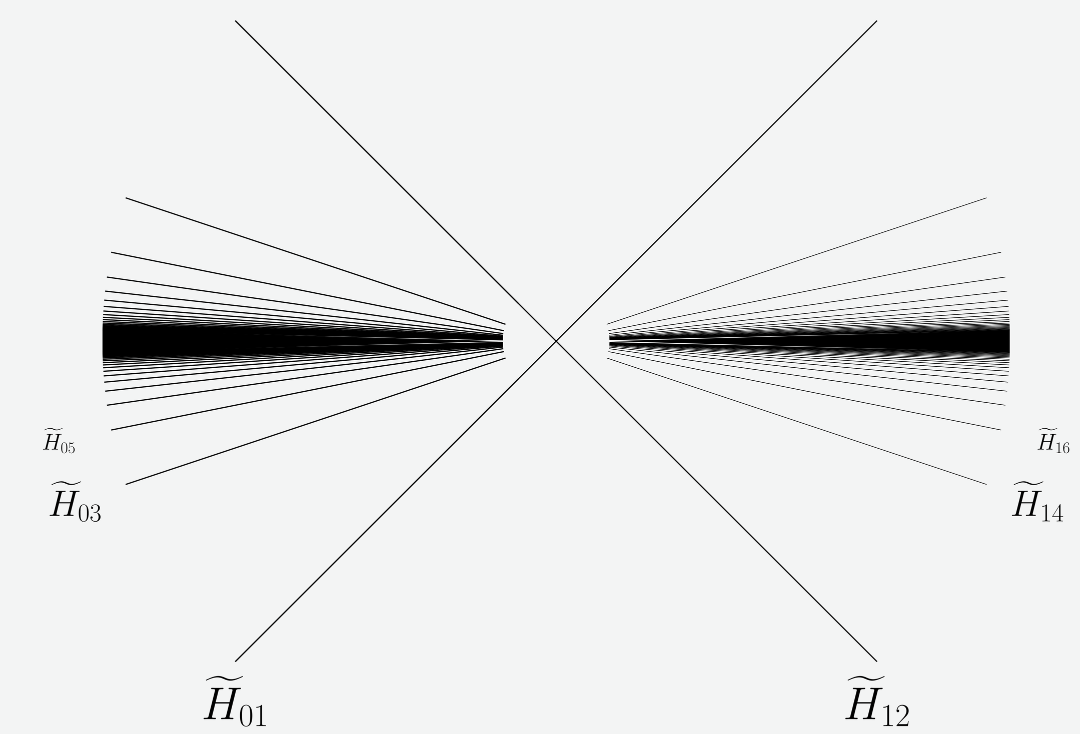
These are the window notations for the TITOs to the left:

$$[0, 2, 1, 3]$$
$$\underline{[3, 2, 1, 0]}$$
$$[1, 3] \prec \underline{[2, 0]}$$
$$[1] \prec \underline{[2, 3]} \prec [0].$$

Extended weak order for $\widetilde{S}_2$



The Hasse diagram for extended weak order. Elements of weak Bruhat order are shown in black, and new elements from extended weak order are in orange.



The intersection of the affine braid arrangement $\widetilde{\mathcal{B}}_2$ with a two-dimensional subspace of $\mathbb{R}^3$.

Shards

Shards

A **shard** of $\widetilde{\mathcal{B}}_n$ is a maximum-dimensional cone of the form

$$\widetilde{H}_{ij} \cap \bigcap_{i<k<j} \widetilde{H}^{\pm}_{ik}$$

for some $i < j$ with $i \not\equiv j \mod n$ and choice of sign for each intermediate k.

Each shard arc encodes a shard in the following way: we pick i and j to be the initial and terminal value of the arc, and for each intermediate k, we use $\widetilde{H}^+_{ik}$ if we chose "inside", and we use $\widetilde{H}^-_{ik}$ if we chose "outside".

Main Theorem

The maps

$$\text{(complete join-irreducibles)} \Rightarrow \text{(shard arcs)} \Rightarrow \text{(shards)}$$

are both bijections.

Conjecture

For any affine Coxeter group W, there is a bijection between the complete join-irreducibles of its extended weak order and the shards in the W-Coxeter arrangement, which is compatible with lower walls.

The conjecture is known for finite Coxeter groups [4].

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