Non symmetric Cauchy kernels, Demazure measures and LPP

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Introduction

We use non symmetric Cauchy kernel identities to get the laws of last passage percolation (LPP) models in terms of Demazure characters. The construction is based on the restrictions of the RSK correspondence to augmented stair (Young) shape matrices and rephrased in a unified way compatible with crystal bases.

Preliminaries

Cauchy kernel identity

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j}=\sum_{\lambda\in\mathcal{P}_{\min(m,n)}}s_\lambda(x)s_\lambda(y)$$

LHS rewritten in the basis of Schur polynomials. $\mathcal{P}_r$ the set of partitions with at most r parts.

Non-symmetric Cauchy kernel identity, Lascoux 2000.

$$\prod_{1\le j\le i\le n}\frac{1}{1-x_iy_j}=\sum_{\mu\in\mathbb{Z}^n_{\ge 0}}\overline{\kappa}^{\mu}(x)\kappa_\mu(y)$$

LHS rewritten in the bases of Demazure and Demazure atom polynomials: $\overline{\kappa}^{\mu}(x_1,\dots,x_n)=\overline{\kappa}_{\sigma_0\mu}(x_n,\dots,x_1)$ *opposite Demazure atom polynomial* of $\overline{\mathrm{B}}^{\mu}$ and $\kappa_\mu(y)$ *Demazure character* of B_μ.

Bicrystals and RSK correspondence

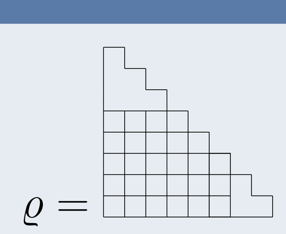
$$\psi:\begin{cases}\mathcal{M}_{m,n}\to\bigsqcup_{\lambda\in\mathcal{P}_{\min(m,n)}}\mathrm{B}_m(\lambda)\times\mathrm{B}_n(\lambda)\\ \quad A\longmapsto(P(A),Q(A))\end{cases}$$

$$\prod_{1\le i\le m,1\le j\le n}\frac{1}{1-x_iy_j}=\sum_{A\in\mathcal{M}_{m,n}}x^{\mathrm{wt}(P(A))}y^{\mathrm{wt}(Q(A))}=\sum_{\lambda\in\mathcal{P}_{\min(m,n)}}s_\lambda(x)s_\lambda(y).$$

Remark: $\mathrm{B}_m(\lambda)$ tableau crystal on the alphabet $[m]$ with highest weight element the key tableau $K(\lambda)$, $\lambda\in\mathcal{P}_{\min(m,n)}$.

Demazure crystals and restriction of RSK to Ferrers shape matrices

Stair RSK



The restriction of the RSK correspondence ψ to $\mathcal{M}^{\varrho}_{n,n}$, $n\times n$ *lower triangular matrices*, gives a one-to-one correspondence

$$\psi:\mathcal{M}^{\varrho}_{n,n}\to\bigsqcup_{\mu\in\mathbb{Z}^n_{\ge0}}\overline{\mathrm{B}}^{\mu}\times\mathrm{B}_\mu\qquad\text{Lascoux, 2000, A.-Emami,15, Choi-Kwon, 18}$$

$$A\mapsto(P,Q),\quad K_+(Q)\le K_-(P)=K(\mu)$$

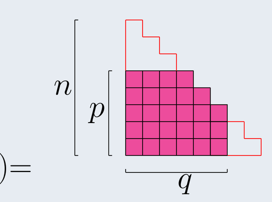
$$\prod_{1\le j\le i\le n}\frac{1}{1-x_iy_j}=\sum_{A\in\mathcal{M}^{\varrho}_{n,n}}x^{\mathrm{wt}(P(A))}y^{\mathrm{wt}(Q(A))}=\sum_{\mu\in\mathbb{Z}^n_{\ge0}}\sum_{\substack{(P,Q)\\K^+(Q)\le K^-(P)=K(\mu)}}x^{\mathrm{wt}(P)}y^{\mathrm{wt}(Q)}$$

$$=\sum_{\mu\in\mathbb{Z}^n_{\ge0}}\sum_{(P,Q)\in\overline{\mathrm{B}}^{\mu}\times\mathrm{B}_\mu}x^{\mathrm{wt}(P)}y^{\mathrm{wt}(Q)}$$

$$=\sum_{\mu\in\mathbb{Z}^n_{\ge0}}\overline{\kappa}^{\mu}(x)\kappa_\mu(y).$$

Remark: B_μ Demazure crystal consisting of all tableaux Q with right key $K_+(Q)\le K(\mu)$.
$\overline{\mathrm{B}}^{\mu}$ opposite Demazure atom crystal consisting of all tableaux P with left key $K^-(P)=K(\mu)$.

Truncated stair RSK, bubble sort operators and parabolic map

$n\ge q\ge p\ge 1$, $\Lambda(p,q)=$ 

$$\psi:\mathcal{M}^{\Lambda(p,q)}_{n,n}\to\bigsqcup_{\mu\in\mathbb{Z}^p_{\ge0}}\overline{\mathrm{B}}^{\mu}_p\times\mathrm{B}_{q,\widetilde{\mu}}$$

$$A\mapsto(P(A),Q(A))\quad K^-(P)=K(\mu),\ K^+(Q)\le K(\tilde\mu).$$

$$\prod_{(i,j)\in\Lambda(p,q)}\frac{1}{1-x_iy_j}=\sum_{\mu\in\mathbb{Z}^p_{\ge0}}\overline{\kappa}^{\mu}(x_{n-p+1},\dots,x_n)\kappa_{\widetilde{\mu}}(y_1,\dots,y_q).$$

- The *parabolic map*: for $\sigma\in\mathfrak{S}_n$, the set $\mathfrak{S}_q^{\le\sigma}=\{v\in\mathfrak{S}_q\mid v\le\sigma\}$ has **unique maximal element** σ^{I_q} for the (strong) Bruhat order $\le$; A. -Gobet-Lecouvey, 22.
- For $\mu=(\mu_1,\dots,\mu_p)\in\mathbb{Z}^p_{\ge0}$, let $\lambda\in\mathcal{P}_p$ and $\tau\in\mathfrak{S}^{\lambda}_p$ such that $\mu=\tau\lambda$: $\widetilde{\mu}=(\sigma_{\mathbf{0}}\tau)^{\mathbf{I_q}}(\lambda,\mathbf{0^{q-p}},\mathbf{0^{n-q}})$, $\sigma_0\in\mathfrak{S}_n$, A.-Emami, 14, A.-Gobet-Lecouvey, 22.

$n=5$, $p=3$, $q=4$, $\lambda=(4,3,1)$, $\mu=(1,3,4)$

$$A_{\Lambda(3,4)}=\begin{pmatrix}\mathbf{0}&0&0&0&0\\0&\mathbf{0}&0&0&0\\\mathbf{0}&\mathbf{1}&\mathbf{0}&0&0\\\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{0}&0\\\mathbf{1}&\mathbf{0}&\mathbf{2}&\mathbf{1}&\mathbf{0}\end{pmatrix}\quad 45\otimes34\otimes455\otimes5\otimes\emptyset\quad 5\to4\to5\to5\to3\to4\to45$$

$$P=\begin{array}{|c|c|c|c|}\hline3&4&4&5\\\hline4&5&5\\\cline{1-3}5\\\cline{1-1}\end{array}=K^-(P)=K(0,0,1,3,4)=K(0,0,\mu),\qquad Q=\begin{array}{|c|c|c|c|}\hline1&1&2&2\\\hline3&3&3\\\cline{1-3}4\\\cline{1-1}\end{array}$$

$$K^+(Q)=\begin{array}{|c|c|c|c|}\hline2&2&2&2\\\hline3&3&3\\\cline{1-3}4\\\cline{1-1}\end{array}=K(0,4,3,1,0)\ \le K(\tilde\mu)=\begin{array}{|c|c|c|c|}\hline2&3&3&3\\\hline3&4&4\\\cline{1-3}4\\\cline{1-1}\end{array}=K(0,1,4,3,0)$$

$\sigma_0\in\mathfrak{S}_5$, $\sigma_0(\mu,0,0)=(00431)=s_2s_3s_4s_1s_2s_3(43100)$ $(\mathbf{s_2s_3s_4s_1s_2s_3})^{I_4}:=\mathbf{s_2s_3\hat{s}_4s_1s_2s_3}=\mathbf{s_2s_3s_1s_2s_3}=\mathbf{s_2s_1s_3s_2s_3}$

$\tilde\mu=s_2s_1s_3s_2s_3(43100)=(01430)=\pi_2\pi_1\pi_3\pi_2\pi_3(43100)$ $\mathrm{B}_{q,\widetilde{\mu}}=\mathrm{B}_{\pi_2\pi_1\pi_3\pi_2\pi_3(43100)}$, π_i bubble sort operator (♣).

Last passage percolation in a Young diagram

For $A=[a_{i,j}]\in\mathcal{M}_{n,n}$, the **last passage percolation** (time) associated to A:

$$perc(A)=\max_{\pi\text{ in }A}\{\textstyle\sum\text{ entries along a path }\pi\text{ in }A\text{ with steps }\leftarrow,\downarrow\text{ starting in }(1,n)\text{ and ending in }(n,1)\}$$
$$=\ \text{maximal row lenght of }P(A)\text{ (or }Q(A)).$$

$$perc(A_{\Lambda(3,4)})=4,\quad perc(A_{(7,4,2,2,2)})=5$$

The random matrix $\mathcal{W}$ in $\mathcal{M}_{n,n}$ whose entry $w_{i,j}$ follows a geometric distribution of parameter u_iv_j

$\mathbb{P}(w_{i,j}=k)=(1-u_iv_j)(u_iv_j)^k$ for any $k\in\mathbb{Z}_{\ge0}$ gives $\mathbb{P}(\mathcal{W}=A)=\prod_{1\le i\le n,1\le j\le n}(1-u_iv_j)\prod_{1\le i\le n,1\le j\le n}(u_iv_j)^{a_{i,j}}.$

Schur and Demazure measures

The law of the random variable $G=perc(\mathcal{W})$:

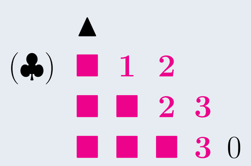
$$\mathcal{W}\in\mathcal{M}_{n,n},\qquad\mathbb{P}(G=k)=\prod_{1\le i\le n,1\le j\le n}(1-u_iv_j)\sum_{\lambda\in\mathcal{P}_n|\lambda_1=k}s_\lambda(u)s_\lambda(v)\quad\textbf{Schur measure}.$$

$$\mathcal{W}\in\mathcal{M}^{\Lambda(n,n)}_{n,n},\quad\mathbb{P}(G=k)=\prod_{1\le j\le i\le n}(1-u_iv_j)\sum_{\mu\in\mathbb{Z}^n_{\ge0}|\max(\mu)=k}\overline{\kappa}^{\mu}(u)\kappa_\mu(v).$$

$$\mathcal{W}\in\mathcal{M}^{\Lambda(p,q)}_{n,n},\quad\mathbb{P}(G=k)=\prod_{(i,j)\in\Lambda(p,q)}(1-u_iv_j)\sum_{(\mu_1,\dots,\mu_p)\in\mathbb{Z}^p_{\ge0}|\max(\mu)=k}\overline{\kappa}_{(\mu_p,\dots,\mu_1)}(u_n,\dots,u_{n-p+1})\kappa_{\widetilde{\mu}}(v_1,\dots,v_q).$$

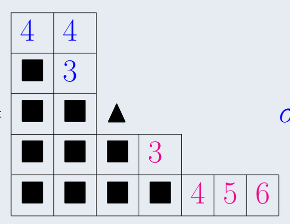
Augmented stair RSK

$n=5$, $\varrho_\Lambda=(3,2,1)$, $\Lambda=(3,4,4)$;



$\sigma(\Lambda,NW)=1$, $\sigma(\Lambda,SE)=s_2s_1s_3s_2s_3$

$n=8$, $\Lambda=(7,4,2,2,2)$, $m=4$, $\varrho_\Lambda=(4,3,2,1)$

$\Lambda=$ 

$\sigma(\Lambda,NW)=s_4s_3s_4$, $\sigma(\Lambda,SE)=s_3s_6s_5s_4$

$$\psi:\mathcal{M}^{D_\Lambda}_{n,n}\to\bigsqcup_{(\mu_1,\dots,\mu_m)\in\mathbb{Z}^m_{\ge0}}\iota\left(\dot\Delta_{\sigma(\Lambda,NW)}\overline{\mathrm{B}}_{(\mu_m,\dots,\mu_1)}\right)\times\Delta_{\sigma(\Lambda,SE)}\mathrm{B}_{(\mu_1,\dots,\mu_m)}$$

where $\Delta_{\sigma(\Lambda,SE)}=\Delta_{j_1}\cdots\Delta_{j_b}$, $\dot\Delta_{\sigma(\Lambda,NW)}=\dot\Delta_{i_1}\cdots\dot\Delta_{i_a}$, ι Schützenberger involution,

$$\prod_{(i,j)\in\Lambda}\frac{1}{1-x_iy_j}=\qquad\text{Lascoux 2000}$$
$$\sum_{(\mu_1,\dots,\mu_m)\in\mathbb{Z}^m}D_{\sigma(\Lambda,NW)}\overline{\kappa}_{(\mu_m,\dots,\mu_1)}(x_n,\dots,x_{n-m+1})D_{\sigma(\Lambda,SE)}\kappa_{(\mu_1,\dots,\mu_m)}(y_1,\dots,y_m)$$

where $D_{\sigma(\Lambda,NW)}=D_{i_1}\cdots D_{i_a}$ and $D_{\sigma(\Lambda,SE)}=D_{j_1}\cdots D_{j_b}$ are Demazure operators.

$$A_{(7,4,2,2,2)}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\\mathbf{0}&\mathit{1}&0&0&0&0&0&0\\\mathit{1}&\mathit{1}&0&0&0&0&0&0\\\mathbf{0}&\mathbf{0}&0&0&0&0&0&0\\\mathit{2}&\mathbf{0}&\mathbf{1}&\mathbf{1}&0&0&0&0\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{2}&0\end{pmatrix}\quad 577\otimes45\otimes7\otimes7\otimes8\otimes\emptyset\otimes88\otimes\emptyset$$

$$P=\begin{array}{|c|c|c|c|c|}\hline4&5&5&7&7\\\hline7&7&8\\\cline{1-3}8&8\\\cline{1-2}\end{array}\qquad K^-(P)=\begin{array}{|c|c|c|c|c|}\hline4&4&4&4&4\\\hline7&7&8\\\cline{1-3}8&8\\\cline{1-2}\end{array}=K(0^3,5,0^2,2,3)$$

$$Q=\begin{array}{|c|c|c|c|c|}\hline1&1&1&2&2\\\hline3&4&7\\\cline{1-3}5&7\\\cline{1-2}\end{array}\qquad K^+(Q)=\begin{array}{|c|c|c|c|c|}\hline2&2&2&2&2\\\hline4&4&7\\\cline{1-3}7&7\\\cline{1-2}\end{array}=K(0,5,0,2,0^2,3,0)$$

$$\iota\dot\Delta_4\dot\Delta_3\dot\Delta_4\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,\mu_1,0^4)}=$$
$$=\begin{cases}\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,0,0^4)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_2,0^4)}\bigsqcup\iota\overline{\mathbf{B}}_{(\mu_4,\mu_3,\mathbf{0^2},\mu_2,\mathbf{0^3})}\ \text{if }\mu_2>\mu_1=0\quad(\spadesuit)\\ \iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,0,0^4)}\ \text{if }\mu_1=\mu_2=0\\ \iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,\mu_1,0^4)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,0,\mu_1,0^3)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_2,\mu_1,0^3)}\text{if }\mu_1=\mu_2>0\\ \emptyset,\ \text{if }\mu_1>\mu_2\ge0\\ \iota\overline{\mathrm{B}}_{(\mu_4,\dots,\mu_1,0^4)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_1,\mu_2,0^4)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,0,\mu_1,0^3)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_2,\mu_1,0^3)}\bigsqcup\\ \bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_1,\mu_2,0^3)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_1,0,\mu_2,0^3)},\ \text{if }\mu_2>\mu_1>0.\end{cases}$$

$$K^-(P)=K(0^3,5,0^2,2,3)\Leftrightarrow P\in\iota\overline{\mathbf{B}}_{(\mathbf{3,2,0^2,5,0^3})}=\overline{\mathrm{B}}^{(0^3,5,0^2,2,3)}\ (\spadesuit)\Rightarrow\mu=(0,5,2,3)$$
$$K^+(Q)=K(0,5,0,2,0^2,3,0)\le K(\pi_3\pi_6\pi_5\pi_4(\mu,0^4))=K(0,5,0,2,0^2,3,0)\Rightarrow(P,Q)\in\iota\overline{\mathrm{B}}_{(3,2,0^2,5,0^3)}\times\mathrm{B}_{(0,5,0,2,0^2,3,0)}$$

Demazure measure

$$\mathcal{W}\in\mathcal{M}^{\Lambda}_{n,n},\quad\mathbb{P}(A_\Lambda=k)=$$
$$\prod_{(i,j)\in D_\Lambda}(1-u_iv_j)\sum_{(\mu_1,\dots,\mu_m)\in\mathbb{Z}^m|\max(\mu)=k}D_{\sigma(\Lambda,NW)}\overline{\kappa}_{(\mu_m,\dots,\mu_1)}(u_n,\dots,u_{n-m+1})D_{\sigma(\Lambda,SE)}\kappa_{(\mu_1,\dots,\mu_m)}(v_1,\dots,v_m).$$