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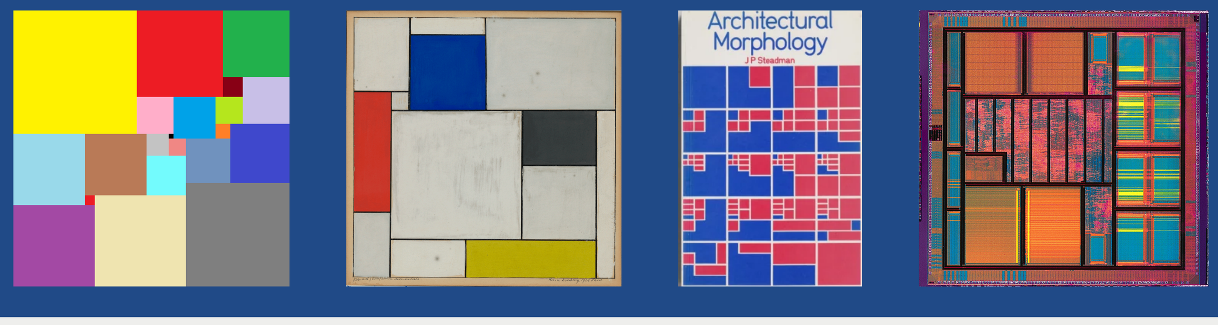

When geometry meets generating functions: rectangulations and permutations

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What is a rectangulation?

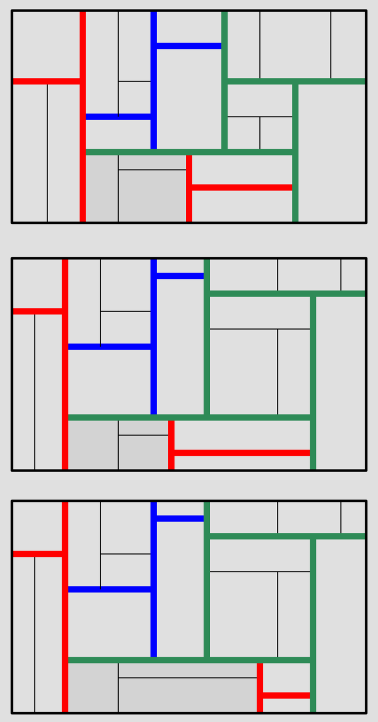
A rectangulation of size n is a tiling of a rectangle by n rectangles such that no four rectangles meet in a point.

Examples:

- Solutions to Squaring the square are rectangulations (Blanche Descartes, 1936).
- The artwork *Composition décentralisée*, by Theo van Doesburg (1883–1931).
- The cover of a book on the geometry of building plans. It is not a rectangulation, since it contains instances of 4 rectangles meeting in a point.
- VLSI (Very-Large-Scale Integration): a technology for integrated circuits.

Patterns in rectangulations

Two rectangulations are equivalent if they coincide after (horizontal or vertical) translations of some segments (without meeting an endpoint of any other segment).



- Only the first two rectangulations are equivalent.
- *Patterns* marked: ┼┤ (green), ┤├ (blue), ╫ (red).
- Goal: to enumerate non-equivalent rectangulations of size n that avoid a given set of patterns.
- Example: ┤├ ┬┴ -avoiding rectangulations are enumerated by Baxter numbers: $\sum_{k=1}^{n} \frac{\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}{\binom{n+1}{1}\binom{n+1}{2}}$.
- Example: Av(┬┴ ┴┬ ┤├ ┬┴) rectangulations are *guillotine diagonal rectangulations*
  ⇝ bijection with separable permutations
  ⇝ Schröder numbers: $\frac{1}{n}\sum_{k=0}^{n} 2^k \binom{n}{k}\binom{n}{k-1}$.

Summary of results: solving Merino & Mütze's conjectures

We solve all conjectures from [5] related to Av(┬┴ ┴┬ ┤├ ┬┴) that additionally avoid some patterns from the set { ┤ ┼ ╫ ╪ }:

Entry in [5]	Guil. diag. rect. avoiding. . .	Separable perm. avoiding. . .	G.f.	OEIS
1234	∅	∅	alg.	OEIS A006318
12345	┤	2143	alg.	OEIS A106228
12347	╫	21354	alg.	OEIS A363809
123456	┤ ┼	2143, 3412	alg.	OEIS A078482
123457	┤ ╫	2143	alg.	OEIS A033321
123458	┤ ╪	2143, 45312	alg.	OEIS A363810
123478	╫ ╪	21354, 45312	rat.	OEIS A363811
1234567	┤ ┼ ╫	2143, 3412	alg.	OEIS A363812
1234578	┤ ╫ ╪	2143, 45312	rat.	OEIS A363813
12345678	┤ ┼ ╫ ╪	2143, 3412	rat.	OEIS A006012

Main theorem: ℕ-algebraicity

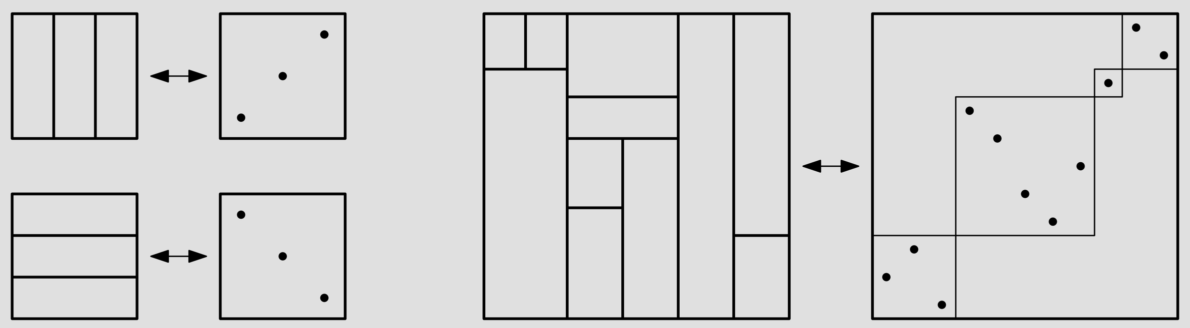
Each of these Av(┬┴ ┴┬ ┤├ ┬┴) classes of rectangulations has an ℕ-algebraic generating function.

1. For rectangulations avoiding ┬┴ ┴┬ ┤├ ┬┴ :
$$F(t) = \left(1 - t - \sqrt{1 - 6t + t^2}\right) / 2.$$
2. For rectangulations avoiding ┬┴ ┴┬ ┤├ ┬┴ ┤ :
$$tF^3 + 2tF^2 + (2t - 1)F + t = 0.$$
3. For rectangulations avoiding ┬┴ ┴┬ ┤├ ┬┴ ╫ : $F(t)$ satisfies
$$t^4(t-2)^2F^4+t(t-2)(4t^3-7t^2+6t-1)F^3+(2t^4-t^3-2t^2+5t-1)F^2-(4t^3-7t^2+6t-1)F+t^2=0.$$

(Etc. according to the table — 10 cases in total.)

Proof of Case 1: guillotine rectangulations

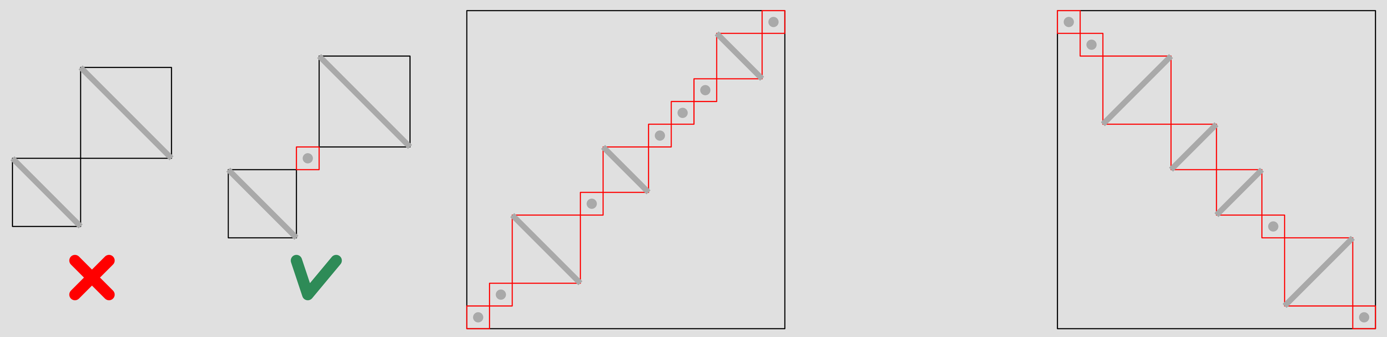
Av(┬┴ ┴┬ ┤├ ┬┴) ⟷ bijection with separable permutations



$$\left\{F = t + A + D,\ A = \frac{(t+D)^2}{1-(t+D)},\ D = \frac{(t+A)^2}{1-(t+A)}\right\} \Rightarrow F = \frac{1-t-\sqrt{1-6t+t^2}}{2}$$

Proof of Case 2: ┤ -avoiding guillotine rectangulations

Step 1: A guillotine rectangulation $\mathcal{R}$ avoids ┤ iff $\delta(\mathcal{R})$ avoids 2143.
Step 2: Enumeration of 2143-avoiding separable permutations.



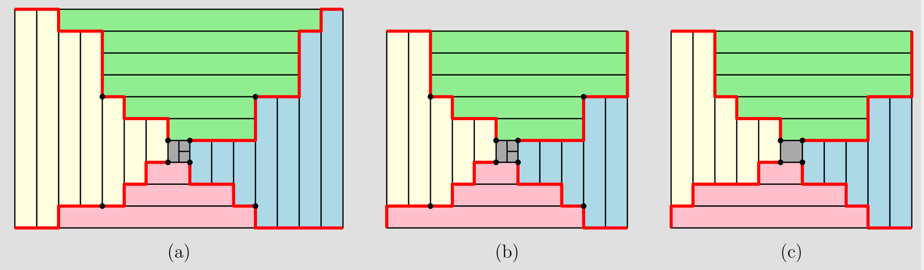
Step 3: $\left\{A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2}\frac{1}{1-\frac{tD}{1-t}} - 1\right)D,\quad D = \frac{(t+A)^2}{1-(t+A)}\right\} \Rightarrow$ the claim.

Proof of Case 3: ┼ -avoiding guillotine rectangulations

Step 1: A guillotine rectangulation $\mathcal{R}$ avoids ╫ iff $\delta(\mathcal{R})$ avoids 21354.
Step 2: Enumeration of 21354-avoiding separable permutations.
This is achieved via auxiliary families:
213-avoiding and 132-avoiding separable permutations.
Step 3: System of equations that leads to the result.

Beyond guillotines: whirls

{ ┬┴ ┤├ ┬┴ ┴┬ ┴┬ ╫ ╪ } avoiding rectangulations ⇝



(a): peelable whirl = (at least) one side is a rectangle
(b): non peelable whirl
(c): simple whirl = non-peelable whirl with precisely one windmill ┴┬ whose interior is not further partitioned.

Theorem (Algebraicity of simple whirls)

x_i counts the number of rectangles of colour i touching the border.

$$F(t, x_1, x_2, x_3, x_4) = t^5 \frac{1}{2\alpha}\left(\beta - \sqrt{\beta^2 - 4\alpha e_4^2}\right)$$

with $\alpha = \prod_{i=1}^4 (1-x_i+tx_i^2)$ *and* $\beta = (2e_4t^2-t(4e_4-3e_3+2e_2)+e_4-e_3+e_2-e_1+2)e_4$, *where* $e_m=[t^m]\prod_{i=1}^4(1+tx_i)$ *= elementary symmetric polynomial of total degree m.*

Proof: generating tree leads to a noteworthy functional equation:

$$F(t,x_1,x_2,x_3,x_4) = t^5x_1x_2x_3x_4 + tx_1x_2x_3x_4[x_3x_4]F(t,1,x_2,x_3,x_4) + tx_1x_2x_3x_4\frac{[x_1x_4]F(t,x_1,x_2,x_3,x_4) - [x_1x_4]F(t,x_1,1,x_3,x_4)}{x_2-1} + tx_1x_3x_4\frac{[x_1]F(t,x_1,x_2,x_3,x_4) - [x_1]F(t,x_1,x_2,1,x_4)}{x_3-1} + tx_1x_4\frac{F(t,x_1,x_2,x_3,x_4) - F(t,x_1,x_2,x_3,1)}{x_4-1}. \quad \square$$

In particular, g.f. of simple whirls is $F(t,1,1,1,1) = t^5C(t)^4$, where $C(t)$ = g.f. of Catalan numbers. (⇝ also bijection: Hugh Thomas.)

Conclusion and perspectives

- In progress: non-diagonal and non-guillotine models from [5].
- In progress: general treatment of patterns in rectangulations, links to inversion sequences [2], mesh patterns, etc.
- Nature of the generating function? Stanley–Wilf constant C? ($\simeq C^n$ rectangulations, for any given set of forbidden patterns.)

References

[1] Andrei Asinowski, Jean Cardinal, Stefan Felsner, and Éric Fusy. *Combinatorics of rectangulations: Old and new bijections*. arXiv, 2024.

[2] Andrei Asinowski and Michaela Polley. ┬-avoiding rectangulations, inversion sequences, and Dyck paths. *Permutation Patterns 2024.*

[3] Cyril Banderier, Mireille Bousquet-Mélou, Alain Denise, Philippe Flajolet, Danièle Gardy, and Dominique Gouyou-Beauchamps. *Generating functions for generating trees*. Disc. Math., 2002.

[4] Cyril Banderier and Michael Drmota. *Formulae and asymptotics for coefficients of algebraic functions*. Comb. Probab. Comput., 2015.

[5] Arturo Merino and Torsten Mütze. *Combinatorial generation via permutation languages. III. Rectangulations*. Disc. Comp. Geom., 2023.