Character factorisations, z-asymmetric partitions and plethysm

Seamus Albion, Universität Wien

Cores and quotients of partitions

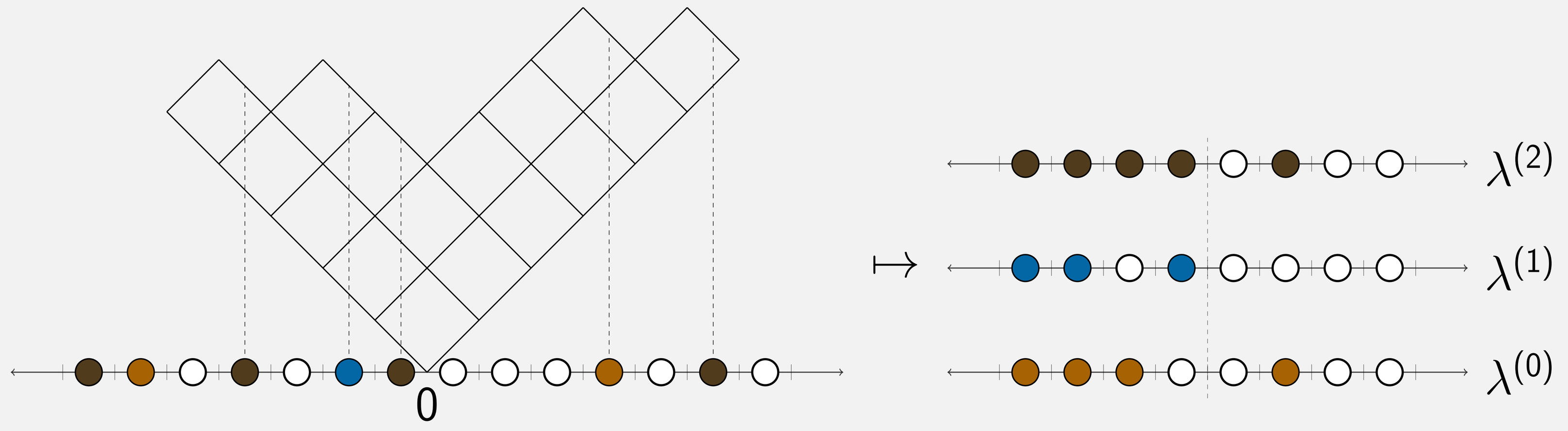
For each integer $t \geqslant 2$ the *Littlewood decomposition* of a partition is a bijection

$$\lambda \longmapsto \big(t\text{-core}(\lambda), (\lambda^{(0)}, \dots, \lambda^{(t-1)})\big)$$

where

- t-core(λ) is partition with no hook of length t and
- $(\lambda^{(0)}, \dots, \lambda^{(t-1)})$ is a t-tuple of arbitrary partitions.

The quotient may be read off the *Maya diagram*:



To find t-core(λ) push all beads to the left in the picture on the right. Alternatively, define the *charge* $\mathbf{c}(\lambda) = (c_0, \dots, c_{t-1})$ by

$$c_r := \#\text{beads at positive positions} - \#\text{spaces at negative positions}.$$

This restricts to a bijection

$$\{t\text{-cores}\} \longleftrightarrow \{(c_0, \dots, c_{t-1}) \in \mathbb{Z}^t : c_0 + \cdots c_{t-1} = 0\}.$$

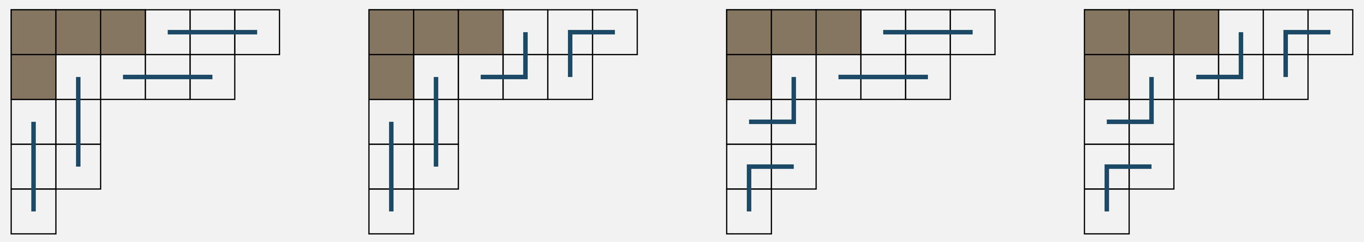
Our example is

$$(6,5,2,2,1) \longleftrightarrow ((3,1),((2),(1),(1)))$$

and $\mathbf{c}((3,1)) = (0,-1,1)$.

Ribbon tilings of skew shapes and signs

Recall a *t-ribbon* is an edge-connected skew shape with no 2×2 square and t-cells. Below are the four 3-ribbon tilings of the skew shape $(6,5,2,2,1)/(3,1)$.



In general for partitions $\mu \subseteq \lambda$

$$\lambda/\mu \text{ tileable by } t\text{-ribbons} \iff t\text{-core}(\lambda) = t\text{-core}(\mu) \text{ and } \mu^{(r)} \subseteq \lambda^{(r)}$$

for each $0 \leqslant r \leqslant t-1$.

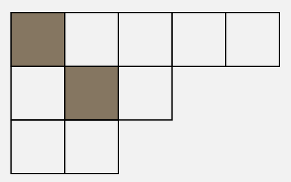
The *height* of a ribbon, $\mathrm{ht}(R)$ is one less than the number of rows. Given a tiling of λ/μ by ribbons $R_1, \dots, R_m$ we define the sign

$$\mathrm{sgn}_t(\lambda/\mu) := (-1)^{\sum_{i=1}^m \mathrm{ht}(R_i)}.$$

It is a non-trivial fact that his sign depends only on the shape and not the tiling.

Frobenius notation

We may write partitions in their Frobenius coordinates as $\lambda = (\lambda_1 - 1, \dots \lambda_d - d \mid \lambda'_1 - 1, \dots, \lambda'_d - d)$ where $d := \mathrm{rk}(\lambda)$ is the side-length of the Durfee square. Further let $\mathrm{rk}_k(\lambda)$ be the side length of the Durfee square of λ with its first k rows removed.



$$\lambda = (5,3,2) = (4,1 \mid 2,1),\ \mathrm{rk}(\lambda) = 2,\ \mathrm{rk}_2(\lambda) = 1$$

Verschiebung operators and a particular plethysm

Fix an integer $t \geqslant 2$. The *t-th Verschiebung operator* is defined by

$$\varphi_t h_r = \begin{cases} h_{r/t} & \text{if } t \mid r, \\ 0 & \text{otherwise.} \end{cases}$$

This is the adjoint of plethysm by a power sum p_t with respect to the Hall inner product:

$$\langle f \circ p_t, g \rangle = \langle f, \varphi_t g \rangle \quad \text{for all } f, g \in \Lambda. \tag{1}$$

Littlewood (and in some special cases with Richardson) computed the action of φ_t on the Schur functions. This was extended to skew Schur functions by Farahat (when $\mu = t$-core(λ)) and Macdonald (general μ).

Theorem (Littlewood, Richardson, Farahat, Macdonald): If the skew shape λ/μ is tileable by t-ribbons then

$$\varphi_t s_{\lambda/\mu} = \mathrm{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}},$$

otherwise $\varphi_t s_{\lambda/\mu} = 0$.

z-asymmetric partitions and their Littlewood decomposition

Following Ayyer and Kumari we call a partition *z-asymmetric* if it has the form

$$(a_1 + z, \dots, a_d + z \mid a_1, \dots, a_d)$$

for some $z \in \mathbb{Z}$. Note $(6,5,2,2,1) = (5,4 \mid 4,3)$ to the left is 1-asymmetric (also variously called *doubled distinct* or *threshold*).

Using the well-known symmetry of the Littlewood decomposition

$$\lambda' \longleftrightarrow \big(t\text{-core}(\lambda)', ((\lambda^{(t-1)})', \dots, (\lambda^{(0)})')\big),$$

it follows that a partition λ is *self-conjugate* (0-asymmetric) if and only if

$$c_r + c_{t-r-1} = 0 \quad \text{and} \quad \lambda^{(r)} = (\lambda^{(t-r-1)})' \quad \text{for } 0 \leqslant r \leqslant t-1,$$

where $\mathbf{c}(t\text{-core}(\lambda)) = (c_0, \dots, c_{t-1})$.

This generalises to z-asymmetric partitions in a natural way.

Theorem: For $0 \leqslant z \leqslant t-1$ a partition is z-asymmetric if and only if the following hold: (i) $\mathbf{c}(t\text{-core}(\lambda)) = (c_0, \dots, c_{t-1})$ satisfies

$$c_r + c_{z-r-1} = 0 \quad \text{for } 0 \leqslant r \leqslant z-1 \quad \text{and} \quad c_s + c_{t+z-s-1} = 0 \quad \text{for } z \leqslant s \leqslant t-1;$$

(ii) there exist partitions $\nu^{(0)}, \dots, \nu^{(z-1)}$ such that for $0 \leqslant z \leqslant t-1$ and $c_r \geqslant 0$,

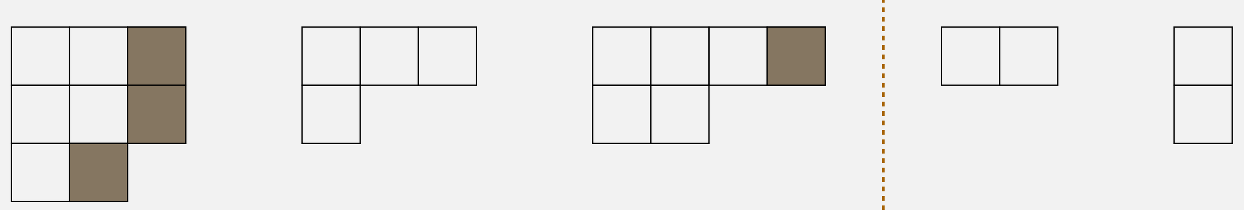
$$\lambda^{(r)} = \nu^{(r)} + (1^{c_r + \mathrm{rk}_{c_r}(\nu^{(r)})}) \quad \text{and} \quad \lambda^{(z-r-1)} = (\nu^{(r)})' + (1^{\mathrm{rk}_{c_r}(\nu^{(r)})}),$$

moreover, for $z \leqslant s \leqslant t-1$.

$$\lambda^{(s)} = (\lambda^{(t+z-s-1)})'.$$

Example of the theorem

The 5-quotient of $\lambda = (20,15,13,11,7,5 \mid 17,12,10,8,4,2)$, which is 3-asymmetric and the charge of its 5-core:



2 0 −2 1 −1

Universal characters

The *universal characters* sp_λ, o_λ and $\mathrm{so}^{\pm}_\lambda$ are symmetric function lifts of the characters of the symplectic and orthogonal groups, usually defined by Jacobi–Trudi-type determinants. The even orthogonal universal character is given by

$$\mathrm{o}_\lambda := \det_{1 \leqslant i,j \leqslant \ell(\lambda)} \big(h_{\lambda_i - i + j} - h_{\lambda_i - i - j}\big) = \sum_{\mu \text{ 1-asymmetric}} (-1)^{|\mu|/2} s_{\lambda/\mu}.$$

As one may expect, the factorisation of o_λ under φ_t is intimately connected with the $z = 1$ case of the previous theorem (due to Garvan, Kim and Stanton).

Theorem (Lecouvey, Ayyer and Kumari, SA): If t-core(λ) is 1-asymmetric then

$$\varphi_t \mathrm{o}_\lambda = (-1)^{|t\text{-core}(\lambda)|/2} \mathrm{sgn}_t(\lambda/t\text{-core}(\lambda))\, \mathrm{o}_{\lambda^{(0)}} \prod_{r=1}^{\lfloor (t-1/2) \rfloor} \mathrm{rs}_{\lambda^{(r)}, \lambda^{(t-r)}} \times \begin{cases} \mathrm{so}^-_{\lambda^{(t/2)}} & t \text{ even,} \\ 1 & t \text{ odd.} \end{cases}$$

Otherwise $\varphi_t \mathrm{o}_\lambda = 0$.

Here $\mathrm{rs}_{\lambda,\mu}$ is the universal character associated to the rational representation of $\mathrm{GL}_n(\mathbb{C})$ indexed by the pair of partitions (λ, μ).

A uniform extension

Bressoud and Wei introduced a uniform generalisation of the universal characters which may be defined by

$$\chi_\lambda(z) := \sum_{\mu \text{ } z\text{-asymmetric}} (-1)^{(|\mu| + (z-1)\mathrm{rk}(\mu))/2} s_{\lambda/\mu}.$$

This also has a Jacobi–Trudi-type determinantal expression.

Theorem: For $0 \leqslant z \leqslant t-1$ we have $\varphi_t \chi_\lambda(z) = 0$ unless $\mathbf{c}(t\text{-core}(\lambda))$ satisfies (i) from the column to the left and $\lambda \supseteq \nu_{\mathbf{c}}$, in which case

$$\varphi_t \chi_\lambda(z) = \varepsilon \prod_{r=0}^{\lfloor (z-2/2) \rfloor} \mathrm{rs}_{\lambda^{(r)}, \lambda^{(t-r)}; c_r} \prod_{s=z}^{\lfloor (t+z-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(s)}, \lambda^{(t+z-s-1)}} \times \begin{cases} 1 & z \text{ \& } t \text{ even,} \\ \mathrm{o}_{\lambda((z-1)/2)} & z \text{ \& } t \text{ odd,} \\ \mathrm{so}_{\lambda((t+z-1)/2)} & z \text{ even, } t \text{ odd,} \\ \mathrm{o}_{\lambda((z-1)/2)} \mathrm{so}^-_{\lambda((t+z-1)/2)} & z \text{ odd, } t \text{ even,} \end{cases}$$

where

$$\varepsilon := (-1)^{(|\nu_{\mathbf{c}}| + (z-1)\mathrm{rk}(\nu_{\mathbf{c}}))/2} \mathrm{sgn}_t(\lambda/\nu_{\mathbf{c}}),$$

with $\nu_{\mathbf{c}}$ defined below.

To recover the case of o_λ one sets $z = 1$.

Small z-asymmetric partitions

Given an integer vector $\mathbf{c}$ satisfying (i) to the left the minimal z-asymmetric partition with $\mathbf{c}(t\text{-core}(\lambda)) = \mathbf{c}$ has quotient

$$\lambda^{(r)} = (1^{c_r}) \quad \text{for } 0 \leqslant r \leqslant z-1 \text{ with } c_r > 0.$$

Denote this partition by $\nu_{\mathbf{c}}$. For example the 3-core with $\mathbf{c} = (-2,2,0)$ has minimal 2-symmetric partition $(8,6,2,2,1,1)$:

$(-2,2,0) \longrightarrow$