Real Tropical Geometry, Determinants, and Matroidal Structures

Josephine Yu

FPSAC '24, Bochum

Outline

I. Introduction to Tropical Geometry.

II. Tropicalizing Principal Minors of Positive Definite Matrices **NEW**

stable polynoms, M-convex

(joint work with Abeer Al Ahmadieh, Felipe Rincón, Cynthia Vinzant)

Part I : Tropical Geometry

Classical		Tropical
hypersurfaces	$\rightsquigarrow$	polytopes, subdivisions
curves, divisors	$\rightsquigarrow$	graphs, chip-firing
linear spaces	$\rightsquigarrow$	(valuated) matroids
Grassmannians	$\rightsquigarrow$	moduli spaces of matroids & valuations
$Gr(2,n)$	$\rightsquigarrow$	phylogenetic trees
principal minors of positive def. matrices	$\rightsquigarrow$	???

TRANSACTIONS OF THE
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Volume 157, June 1971

THE LOGARITHMIC LIMIT-SET OF AN ALGEBRAIC VARIETY

BY

GEORGE M. BERGMAN[1]

Abstract. Let C be the field of complex numbers and V a subvariety of $(C-\{0\})^n$. To study the "exponential behavior of V at infinity", we define $V_\infty^{(a)}$ as the set of limit-points on the unit sphere S^{n-1} of the set of real n-tuples $(u_x \log|x_1|, \ldots, u_x \log|x_n|)$, where $x \in V$ and $u_x = (1+\sum(\log|x_i|)^2)^{-1/2}$. More algebraically, in the case of arbitrary base-field k we can look at places "at infinity" on V and use the values of the associated valuations on $X_1, \ldots, X_n$ to construct an analogous set $V_\infty^{(b)}$. Thirdly, simply by studying the terms occurring in elements of the ideal I defining V, we define another closely related set, $V_\infty^{(c)}$.

These concepts are introduced to prove a conjecture of A. E. Zalessky on the action of $GL(n, Z)$ on $k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, then studied further.

DEQUANTIZATION OF REAL ALGEBRAIC GEOMETRY ON LOGARITHMIC PAPER

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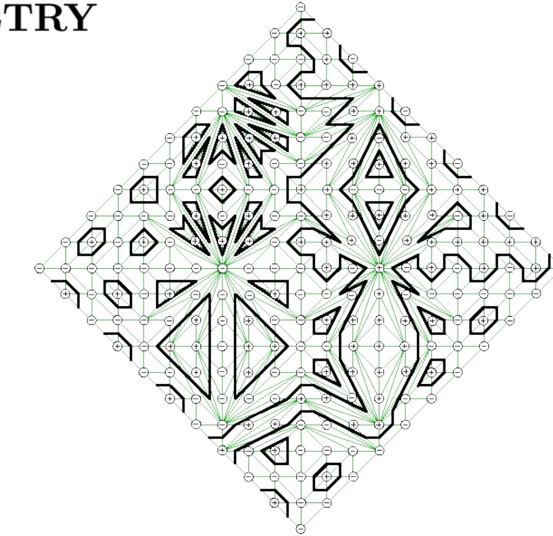
POMI, St. Petersburg, Russia

ABSTRACT. On logarithmic paper some real algebraic curves look like smoothed broken lines. Moreover, the broken lines can be obtained as limits of those curves. The corresponding deformation can be viewed as a quantization, in which the broken line is a classical object and the curves are quantum. This generalizes to a new connection between algebraic geometry and the geometry of polyhedra, which is more straightforward than the other known connections and gives a new insight into constructions used in the topology of real algebraic varieties.

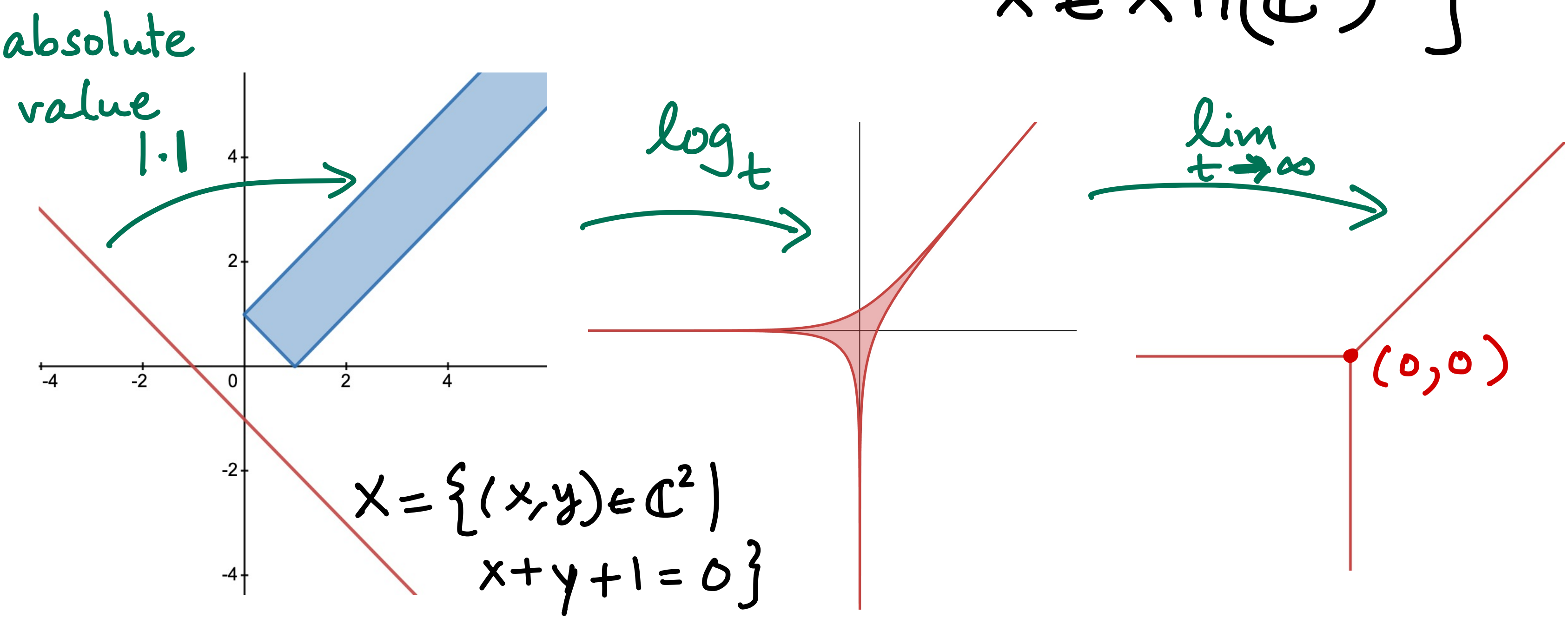


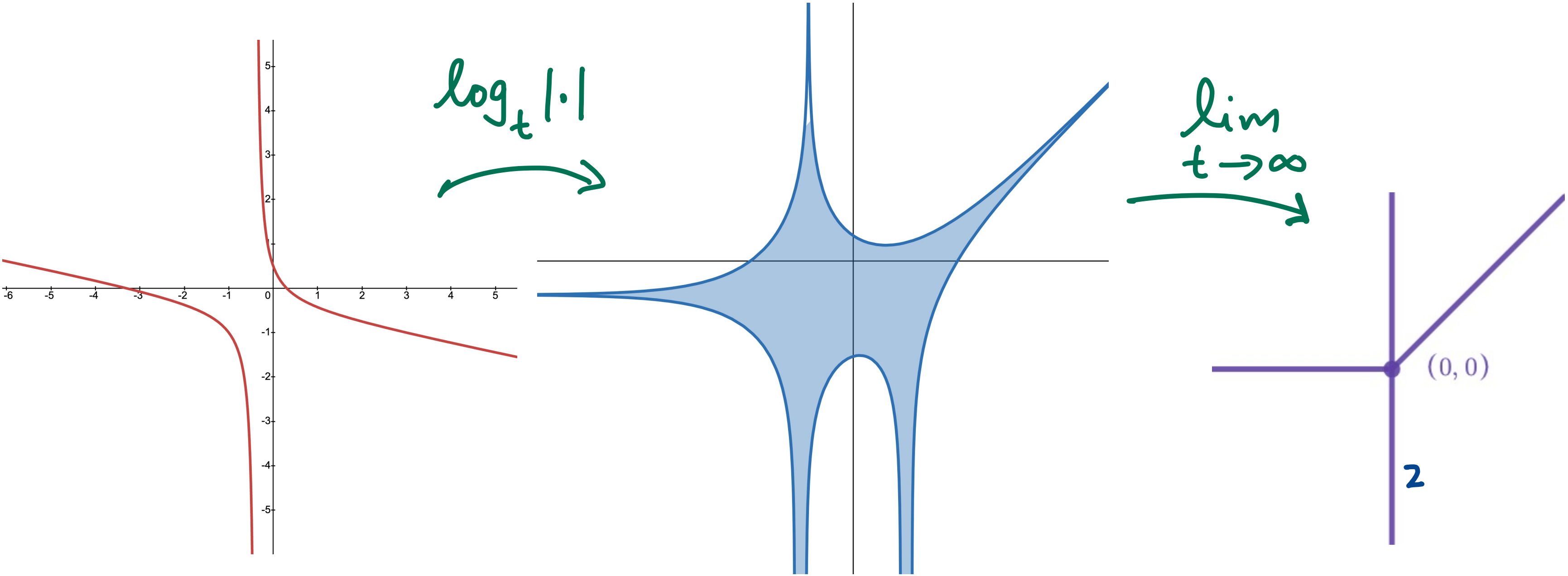
Viro's patchworking (starting 1980s)

⇝ Shaw's talk

Tropicalization = logarithmit limits

$$X \subseteq \mathbb{C}^n, \quad \mathrm{trop}(X) := \lim_{t\to\infty} \left\{ \left(\log_t|x_1|, \dots, \log_t|x_n|\right) : x \in X \cap (\mathbb{C}^*)^n \right\}$$





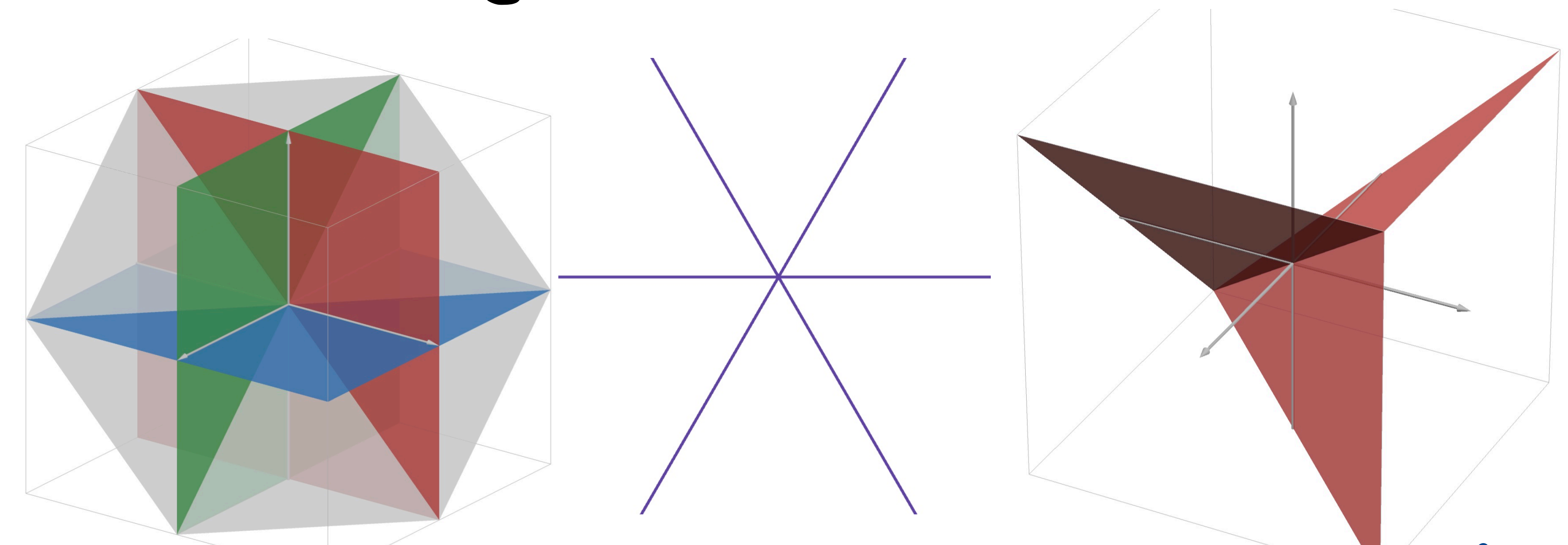
$$1 - 3x - x^2 - 2y - 5xy = 0$$

What does trop X know about X ?

It sees X near 0 or ∞.

Linear Space $L \subseteq \mathbb{C}^n$

trop L sees how coordinate planes $x_i = 0$ meet along L, i.e. the matroid of L.



Sturmfels, Ardila-Klivans (early 2000s)

Grassmannians

$\mathrm{Gr}_K(d,n) = \{ d\text{-dim. linear subspaces of } K^n \}$

Plücker embedding : $\mathrm{Gr}_K(d,n) \hookrightarrow \mathbb{P}_K^{\binom{n}{d}-1}$

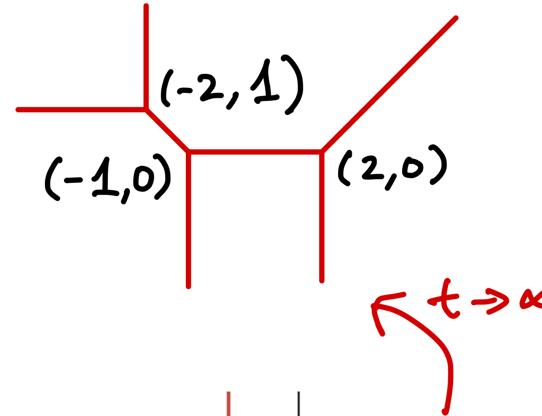
For a $d \times n$ rank d matrix A, $\mathrm{row}(A) \mapsto \left(d \times d \text{ minors of } A \right)$

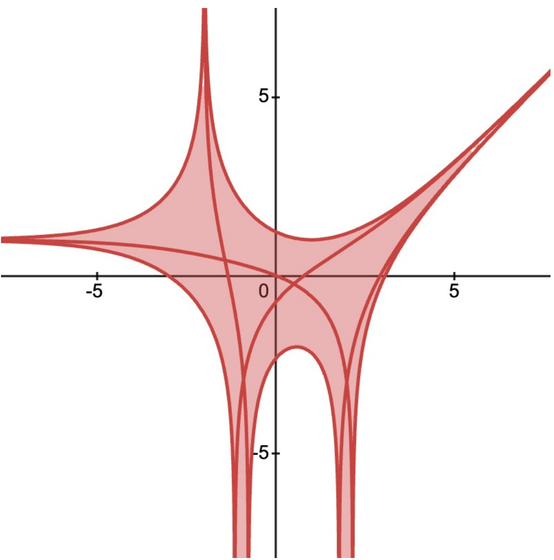
The zero pattern of the Plücker coordinates the matroid of linear independence among columns of A.

$\mathrm{trop}(\mathrm{Gr}_{\mathbb{C}}(d,n))$ sees matroids representable over $\mathbb{C}$.

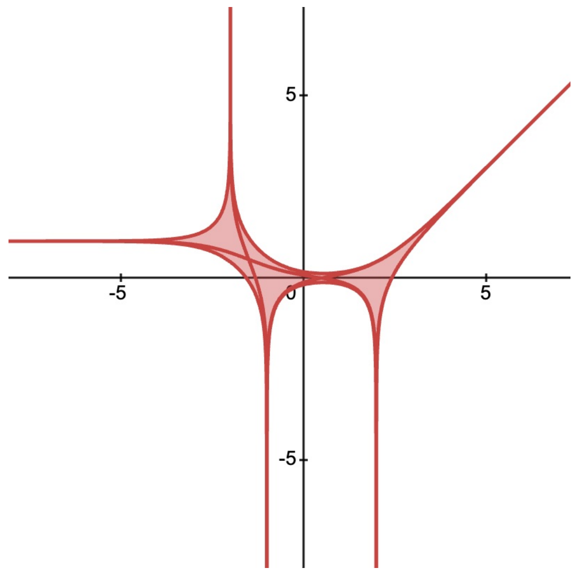
Log limits of families

$$t + t^2 x + x^2 + y + t^2 x y = 0$$

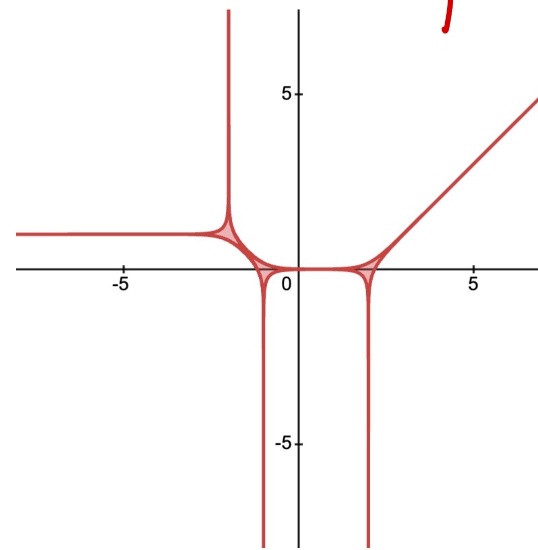




$t = 2$



$t = 5$



$t = 50$

Tropicalizing without limits

Let $K = \mathbb{C}\{\{t^{-1}\}\} = \bigcup_{k \geq 0} \mathbb{C}((t^{-\frac{1}{k}}))$ field of Puiseux series.

valuation $\nu : K \longrightarrow \mathbb{Q} \cup \{-\infty\}$

$0 \longmapsto -\infty$

series $a \longmapsto$ largest exponent.

Think: $t \to \infty$

$\nu \approx \log_t$

e.g. $\nu(2024\, t^{\frac{1}{3}} - 7 + 22\, t^{-4}) = \frac{1}{3}$

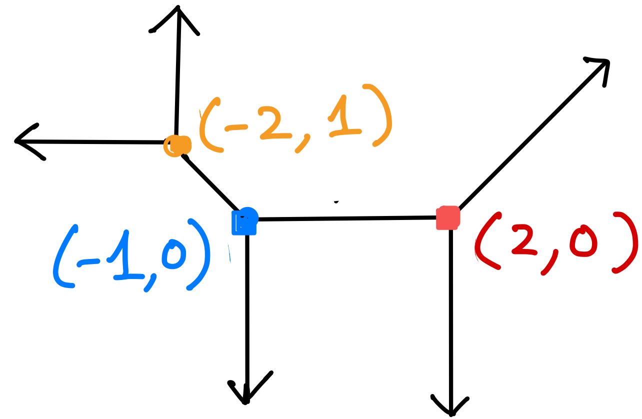
For $X \subseteq K^n$, $\operatorname{trop} X := \{(\nu(x_1), \dots, \nu(x_n)) \mid x \in X \cap (K^*)^n\}$

K can be any valued field, of arbitrary char.

For $(x,y) \in (K^*)^2$ to be a solution of

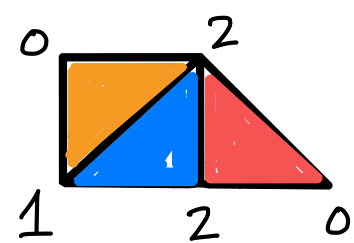
$$t + t^2 x + x^2 + y + t^2 x y = 0,$$

the highest degree terms must cancel, so maximum of



$$\left.\begin{array}{l} 1 \\ 2+\nu(x) \\ 2\nu(x) \\ \nu(y) \\ 2+\nu(x)+\nu(y) \end{array}\right\}$$

these must be attained at least twice.



<u>Ex.</u> Grassmannian $Gr_K(2,4) \hookrightarrow \mathbb{P}^5_K$

coordinates $P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}$ are 2×2 minors of 2×4 matrices $\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$

Plücker relation: $P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0$.

Their valuations satisfy:

$$\max\big(\nu(P_{12})+\nu(P_{34}),\ \nu(P_{13})+\nu(P_{24}),\ \nu(P_{14})+\nu(P_{23})\big)$$

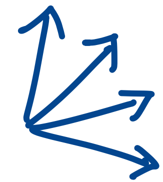
attained twice.

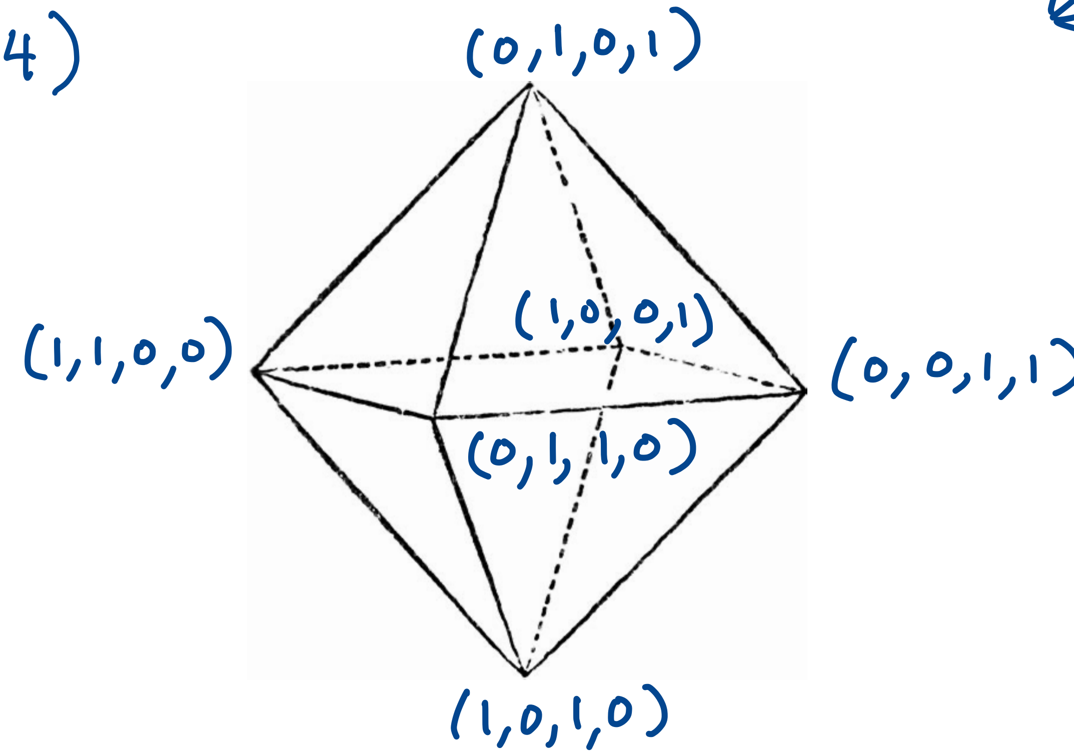
$\Downarrow$

Valuated matroids!

The matroid polytope of $U_{2,4}$

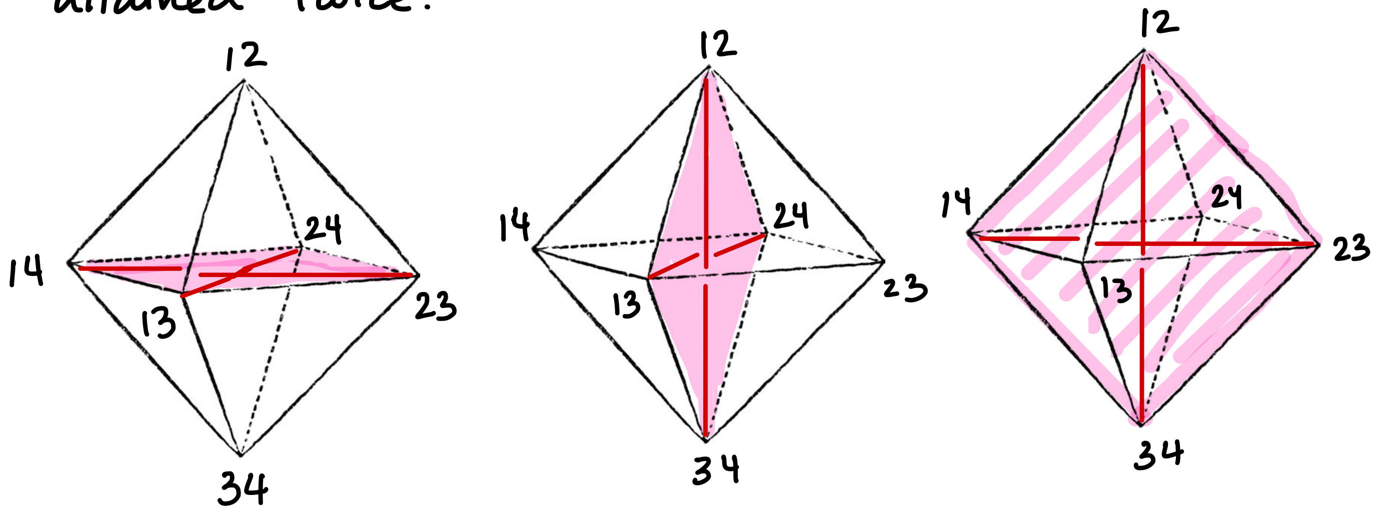
$\Delta(2,4)$





$$\max\left(\nu(P_{12})+\nu(P_{34}),\ \nu(P_{13})+\nu(P_{24}),\ \nu(P_{14})+\nu(P_{23})\right)$$

attained twice.



The upper convex hull gives a matroid subdivision

↳ talks of Joswig, Proudfoot

$\Longleftrightarrow (P_{ij})$ forms an M-concave function on vertices of $\Delta(2,4)$

("M" for matroids)

Valuated matroid representability problem over a field K:

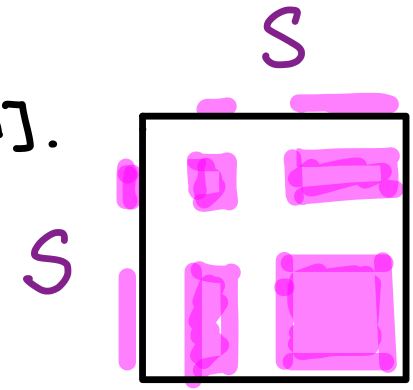
which M-concave functions on $\Delta(r,n)$ are in $\operatorname{trop} Gr_K(r,n)$?

Part II

Principal Minors (PM)

Let A be an $n \times n$ matrix, $S \subseteq [n]$.

$$A_S := \det(\text{principal submatrix indexed by } S)$$



(tropical)

PM map: $K^{n\times n} \longrightarrow K^{2^n} \xrightarrow{\text{trop}} \mathbb{R}^{2^n}$

$$A \longmapsto (A_S)_{S\subseteq[n]} \overset{\text{val}}{\longmapsto} (\nu(A_S))_{S\subseteq[n]}$$

Principal Minor Assignment Problem:

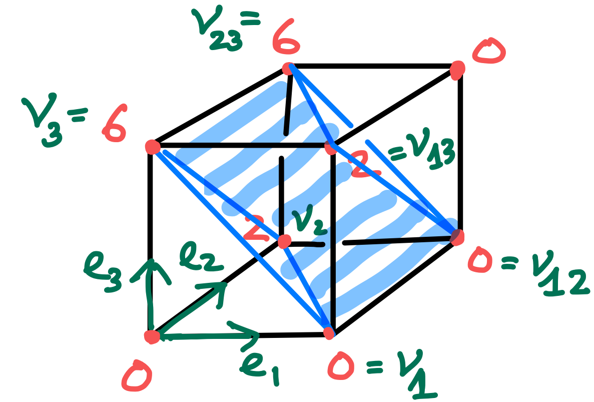
Characterize the image of the PM map. Find equations and inequalities among PMs.

For symmetric matrices over $\mathbb{R}$ or $\mathbb{C}$, this was solved by Oeding (2011), based on a conjecture of Holtz and Sturmfels (2007). Extended to arbitrary unique factorization domains by Al Ahmadieh & Vinzant (2021).

Example: $B = \begin{bmatrix} 1 & t & t^3 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$, $A = B^T B = \begin{bmatrix} 1 & t & t^2 \\ t & t^2+1 & t^3+t \\ t^2 & t^3+t & t^4+t^2+1 \end{bmatrix}$

$A = B^T B$ is positive definite

S	A_S	$\nu(A_S)$
ϕ	1	0
1	1	0
2	t^2+1	2
3	t^6+t^2+1	6
12	1	0
13	t^2+1	2
23	$t^6-2t^5+t^4+t^2+1$	6
123	1	0



$$\max(\nu_1 + \nu_{23},\ \nu_2 + \nu_{13},\ \nu_3 + \nu_{12})$$
$$0+6 \qquad 2+2 \qquad 6+0$$

is attained twice.

Our goal: Describe the tropicalization of the image of positive definite (PD) matrices under the PM map.

A symmetric matrix over $\mathbb{R}$ is called positive (semi)definite if all its principal minors are positive.
(nonnegative)

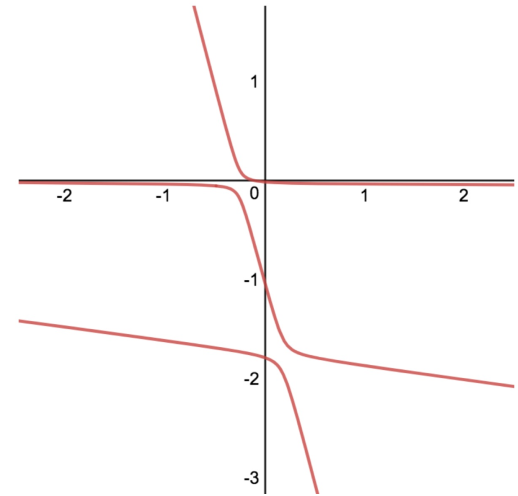
Stable Polynomials

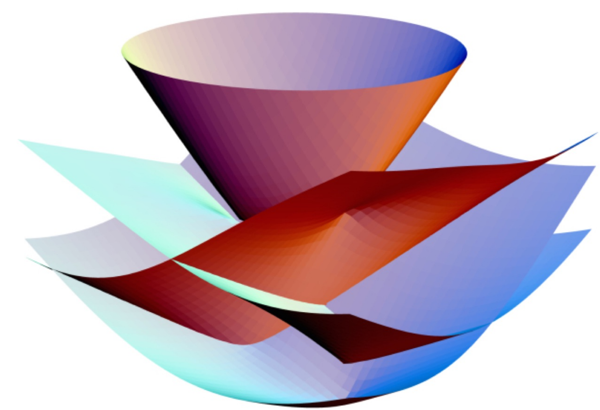
Def. A polynomial $f(x_1, x_2, \dots, x_n)$ with complex coefficients is called stable if for any $z \in \mathbb{C}^n$

$$f(z) = 0 \implies \operatorname{Im}(z) \notin (\mathbb{R}_{>0})^n.$$

- For univariate polynomials with $\mathbb{R}$ coeff, stable = real rooted.
- stable $\Rightarrow$ Lorentzian / strongly log concave

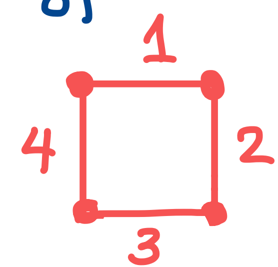
Geometrically, a real homogeneous polynomial f is stable iff every line in positive direction meets the hypersurface $f=0$ only at real points.





Lemma: If A_0 is Hermitian and $A_1, \dots, A_n$ are PSD, then $\det(A_0 + x_1 A_1 + \dots + x_n A_n)$ is stable.

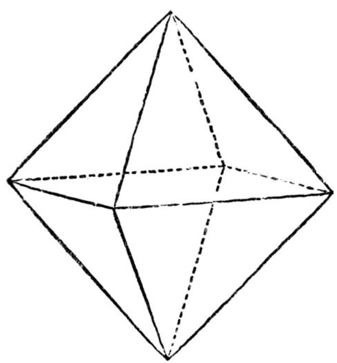
Cor (follows from Matrix Tree Theorem)
The spanning tree generating function of a graph is stable.

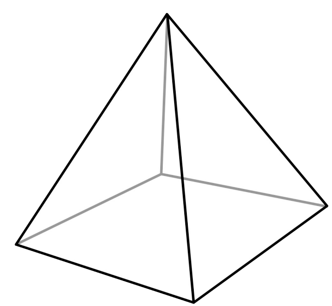


Ex. $x_1x_2x_3 + x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4$.
Its Newton polytope is the matroid polytope of the graphic matroid.

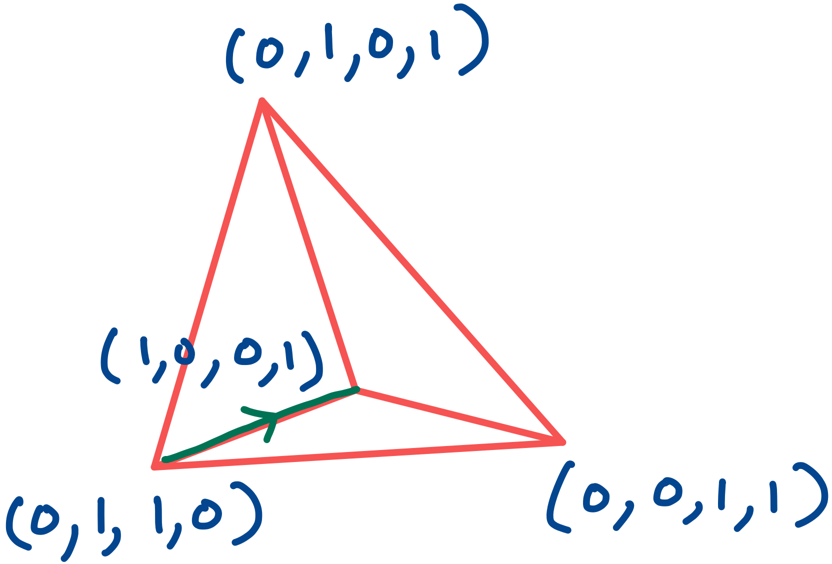
Theorem (Choe, Oxley, Sokal, Wagner, 2004)

The Newton polytope of a homogeneous multiaffine stable polynomial is a matroid base polytope, i.e., a 0/1 polytope with edges in directions $e_i - e_j$ only.





Not a matroid polytope

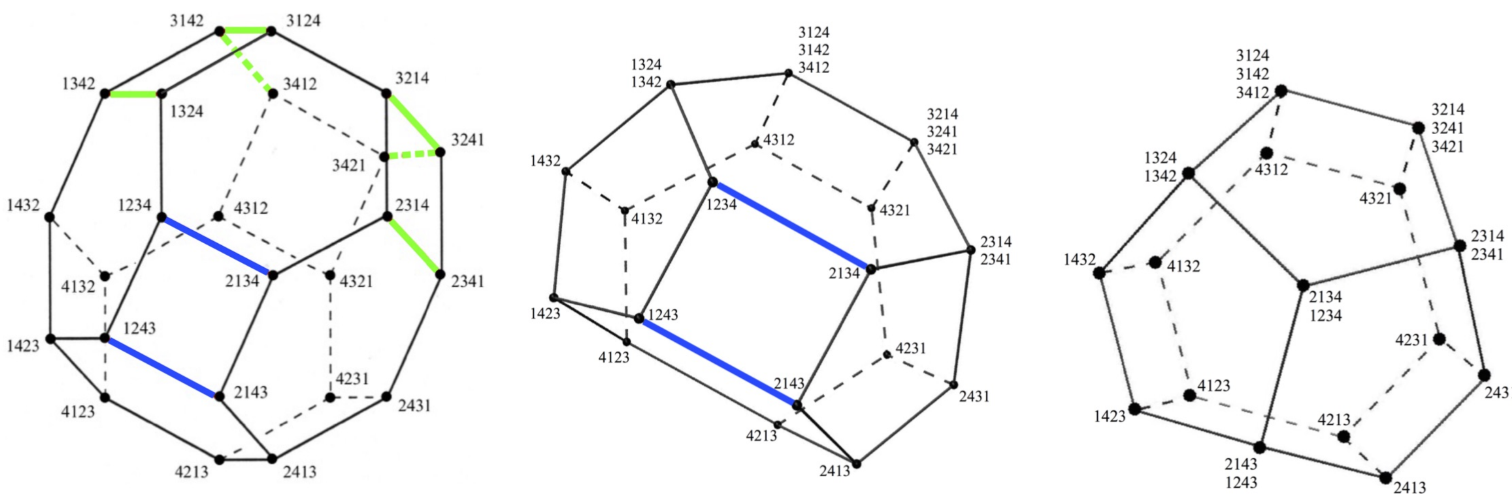


The is no stable polynomial of the form

$$a\,x_2x_4 + b\,x_1x_4 + c\,x_2x_3 + d\,x_3x_4, \quad a,b,c,d>0$$

Theorem (Brändén, 2010). ~~multivariate~~

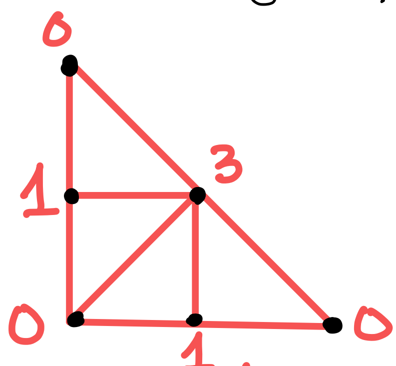
The Newton polytope of a homogeneous stable polynomial is a generalized permutohedron.



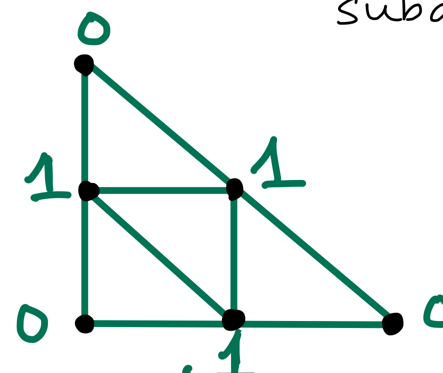
Brändener: More generally:

for a stable polynomial over $\mathbb{R}\{\{t\}\}_{\geq 0}$ the valuation of coefficients form an $M^\natural$-concave function on its support.

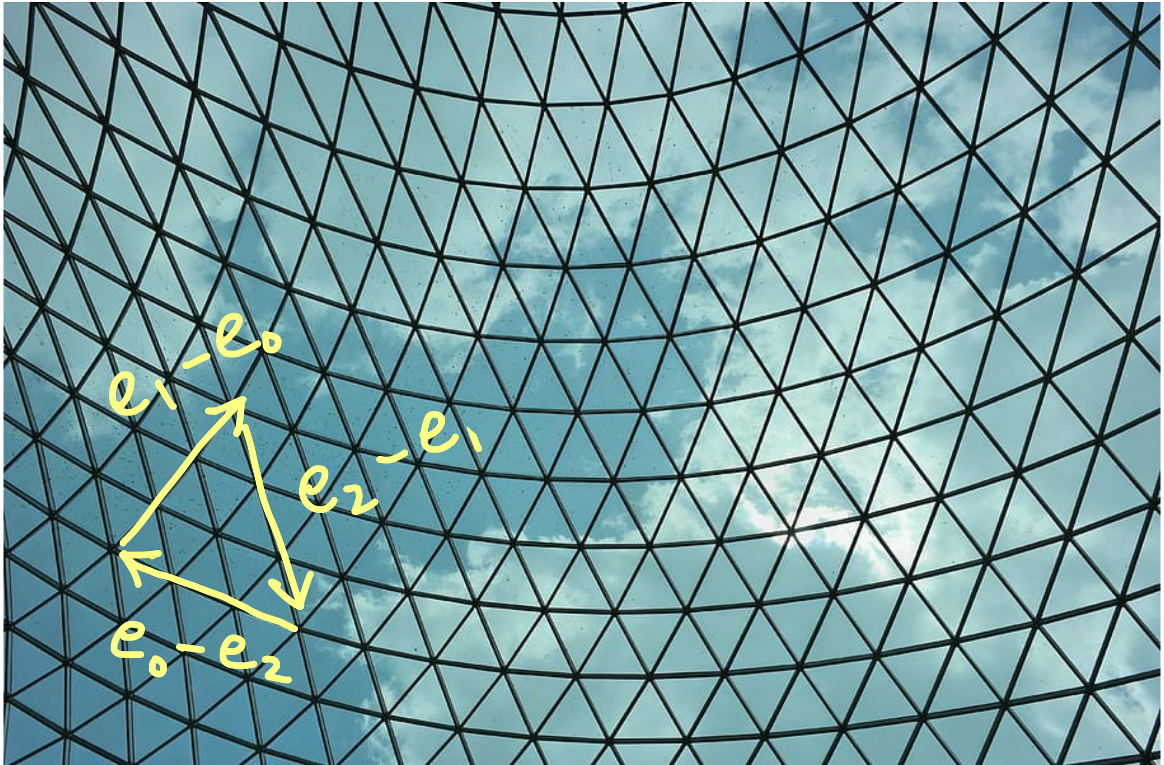
↖ edges // $e_i - e_j$ or e_i in the subdivision.



NOT $M^\natural$-concave



$M^\natural$-concave



M-concave

$M^\natural$-concave

Back to <u>principal minors</u>.

For any square matrix A, the coefficients of $\det\left(A + \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}\right)$ are principal minors of A.

<u>Recall:</u> If A_0 is Hermitian and $A_1, \dots, A_n$ are PSD, then $\det(A_0 + x_1 A_1 + \cdots + x_n A_n)$ is stable.

If A is PSD, then

$\det\left(A + \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix}\right)$ is stable <u>and</u> has nonnegative coefficients.

Brändën $\Longrightarrow$ $\boxed{\text{The tropicalization (valuation) of principal minors of a PD matrix forms an } M^\natural\text{-concave function on } \{0,1\}^n.}$

Q: Do we get all M-concave functions this way?

A: No for large n, by dimension count.

The principal minors have dim $\leq \binom{n+1}{2}$.

The $M^\natural$-concave functions have full dimension 2^n.

$$\boxed{\mathrm{trop}(PM(PD)) \subset \underset{\text{on } \{0,1\}^n}{M^\natural\text{concave}} \subset \text{submodular}}$$

Problem: (Tropical PM assignment)

Characterize $M^\natural$-concave functions arising as tropical principal minors of PD matrices.

(Analogous to valuated matroid representability problem.)

Example: $B = \begin{bmatrix} 1 & t & t^3 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$, $A = B^T B = \begin{bmatrix} 1 & t & t^2 \\ t & t^2+1 & t^3+t \\ t^2 & t^3+t & t^4+t^2+1 \end{bmatrix}$

$A = B^T B$ is positive definite

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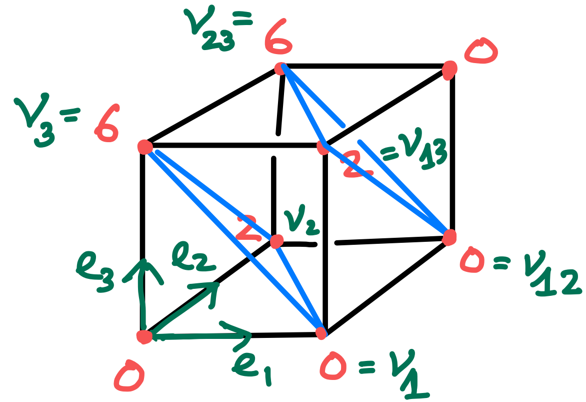
3	t^6+t^2+1	6
12	1	0
13	t^2+1	2
23	$t^6-2t^5+t^4+t^2+1$	6
123	1	0



$$\max\left(\underset{0+6}{\nu_1+\nu_{23}},\ \underset{2+2}{\nu_2+\nu_{13}},\ \underset{6+0}{\nu_3+\nu_{12}}\right)$$

is attained twice.

The hidden tropical Grassmannian

$$B = \begin{bmatrix} 1 & t & t^3 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{generic (random) } U = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 5 & 7 \\ -4 & 2 & 1 \end{bmatrix}$$

Then the valuations of Plücker coords of $[UB \mid I]$ are:

$123 \mapsto 0$
$124 \mapsto 0$
$125 \mapsto 0$
$126 \mapsto 0$
$134 \mapsto 1$
$135 \mapsto 1$
$136 \mapsto 1$

$145 \mapsto 0$
$146 \mapsto 0$
$156 \mapsto 0$
$234 \mapsto 3$
$235 \mapsto 3$
$236 \mapsto 3$

$245 \mapsto 1$
$246 \mapsto 1$
$345 \mapsto 3$
$346 \mapsto 3$
$356 \mapsto 3$
$456 \mapsto 0$

What do you notice?

valuations of Plücker coords of 3×6 matrix $[UB \mid I]$

$123 \mapsto 0$	$145 \mapsto 0$	$245 \mapsto 1$
$124 \mapsto 0$	$146 \mapsto 0$	$246 \mapsto 1$
$125 \mapsto 0$	$156 \mapsto 0$	$345 \mapsto 3$
$126 \mapsto 0$	$234 \mapsto 3$	$346 \mapsto 3$
$134 \mapsto 1$	$235 \mapsto 3$	$356 \mapsto 3$
$135 \mapsto 1$	$236 \mapsto 3$	$456 \mapsto 0$
$136 \mapsto 1$		

valuations of PM of $A = B^T B$

S	A_S	$\nu(A_S)$
ϕ	1	0
1	1	0
2	t^2+1	2
3	t^6+t^2+1	6
12	1	0
13	t^2+1	2
23	$t^6-2t^5+t^4+t^2+1$	6
123	1	0

What do you notice?

Main Theorem (ARVY '2024$^+$)

trop(PM(PD $n \times n$ matrices))

$= $ a linear slice of $\frac{1}{2}$ trop $Gr_{\mathbb{R}}(n, 2n)$ *

$= $ tropical complete flag variety$_{\mathbb{R}}$ $\cap$ submodular cone

* Slice with linear subspace $L \subseteq \mathbb{R}^{\binom{2n}{n}}$

$p(C) = p(D)$ if $C \cap [n] = D \cap [n]$, for $C, D \in \binom{[2n]}{n}$

An analogous statement holds for PD Hermitian matrices.

Representability of M-concave functions

For M-concave functions on $\{0,1\}^n$

three notions of representability coincide:

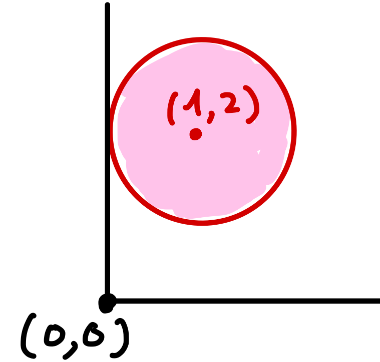
1. principal minors of PD
2. slice of trop $Gr(n, 2n)$
3. submodular part of trop $Fl(n)$

may be of independent interest

Remark

The tropical principal minors of PD matrices satisfy tropical Grassman-Plücker equations, although the principal minors do not satisfy equations of this form.

Lifting Inequalities.



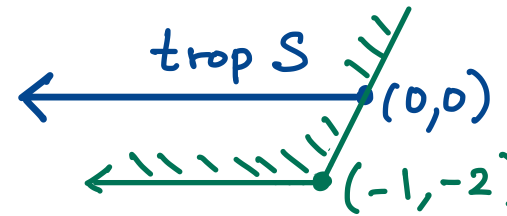
Ex. $S =$ unit circle centered at $(1,2) \subseteq \mathbb{R}^2_+$

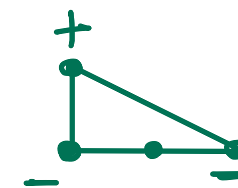
$$(x-1)^2 + (y-2)^2 \leq 1$$

$$x^2 - 2x + y^2 - 4y + 4 \leq 0$$

$$\Downarrow$$

$$\max(\underbrace{2v(x), 2v(y), 0}_{\text{val of positive terms}}) \geq \max(\underbrace{v(x), v(y)}_{\text{val of negative terms.}})$$





trop S satisfies: $\quad y \geq \max(2x, -2)$

Does this come from a usual inequality of the form $A\, y \geq B\, x^2 + C\, t^{-2}$?

$A, B, C > 0$

val $= 0$

Yes!

Let $B = C = 1$.

Map $S \longrightarrow \mathbb{R}\{\{t\}\}$, $(x,y) \mapsto \dfrac{x^2 + t^{-1}}{y}$

The image is a semialgebraic subset of $\mathbb{R}\{\{t\}\}_{>0}$ with valuation ≥ 0, so it satisfies $X \geq A$ for some $A > 0$, val 0.

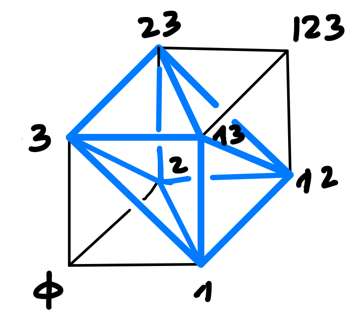
<u>Lifting Lemma</u> (Jell-Scheiderer-Y., 2019)

Every tropical inequality valid on $\operatorname{trop}(S)$ can be lifted to a usual inequality valid on S.

i.e. $\operatorname{trop}(S)$ sees all monomial supports of ineqs on S.

Lifting $M^\natural$-concavity inequalities

$a : \{0,1\}^n \to \mathbb{R}$ is $M^\natural$-concave $\Longrightarrow$



$\max(a_1 + a_{23},\ a_2 + a_{13},\ a_3 + a_{12})$ is attained twice. $\Longrightarrow$

$\max(a_1 + a_{23},\ a_2 + a_{13}) \geq a_3 + a_{12}$, and

$\max(a_1 + a_{23},\ a_1 + a_{23}) \geq a_2 + a_{13}$, and

$\max(a_2 + a_{13},\ a_3 + a_{12}) \geq a_1 + a_{23}$.

Lifting Lemma $\Longrightarrow$ there are inequalities

$? \, A_1 A_{23} + \, ? \, A_2 A_{13} \geq \, ? \, A_3 A_{12}$

positive real numbers

Proposition (ARVY)

The principal minors of a 3x3 PD matrix satisfy:

$$(c+1)\,A_1 A_{23} + c(c+1)\,A_2 A_{13} \geq c\,A_3 A_{12}$$

for any real number c.

They generalize to larger matrices & minors, and to coefficients of stable and Lorentzian polynomials.

Summary

Tropicalization of varieties and semialgebraic sets reveal

- combinatorial structures, matroidal structures
- monomial supports of inequalities

* Principal minors of PD matrices have a matroidal & a hidden trop. Grassmannian structure.
* 3 types of representability agree for $M^\natural$-concave functions on $\{0,1\}^n$.