Framing lattices and flow polytopes

Matias von Bell and Cesar Ceballos

07.23.2024

FPSAC 2024, Bochum, Germany

Why care about framing lattices?

Tamari lattices
(Tamari, 1962)

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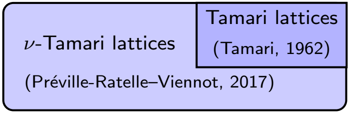

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ν-Tamari lattices (Préville-Ratelle–Viennot, 2017)

Tamari lattices (Tamari, 1962)

ν-Dyck lattices

Dyck lattices

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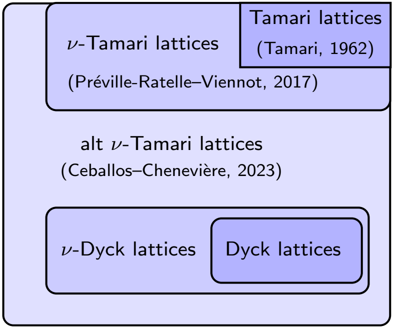

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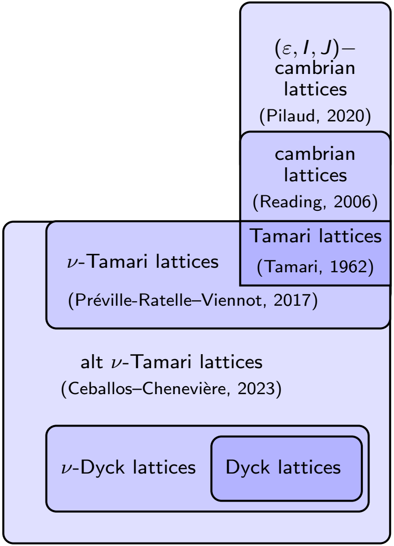

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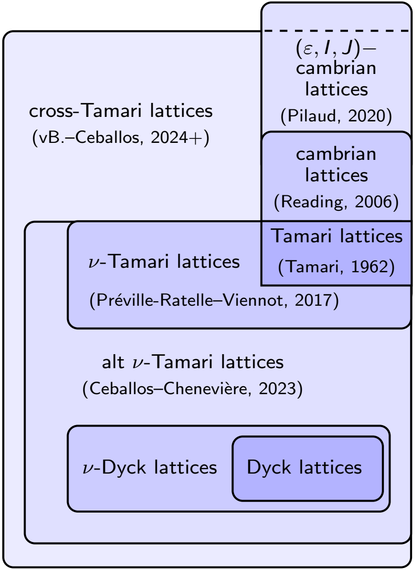

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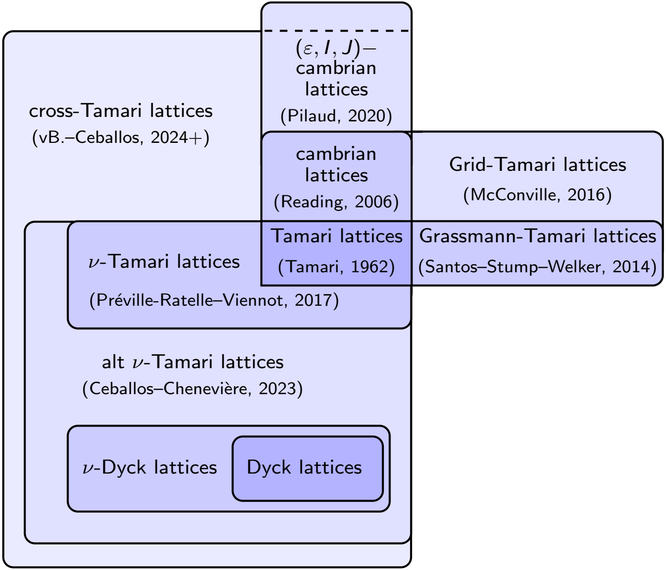

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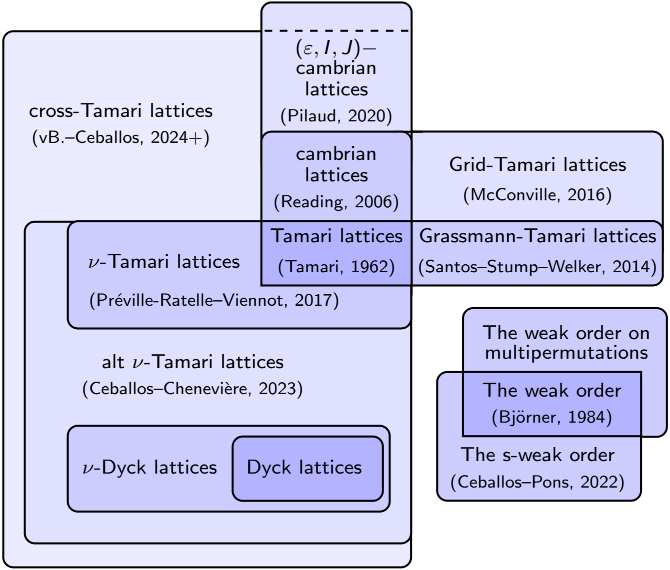

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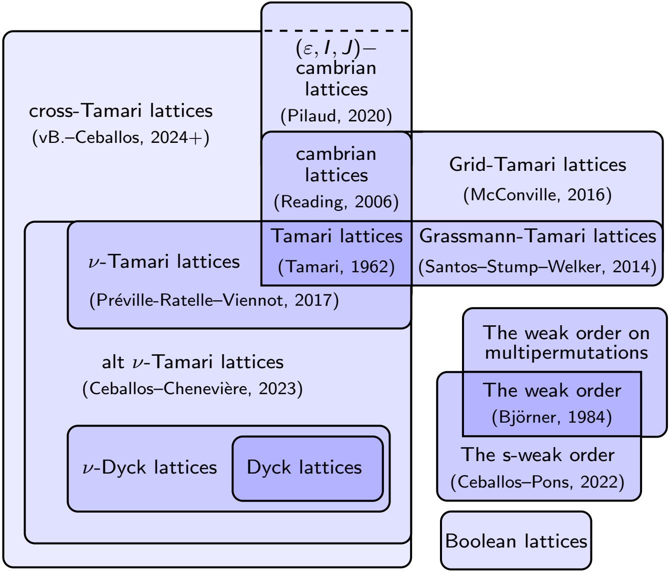

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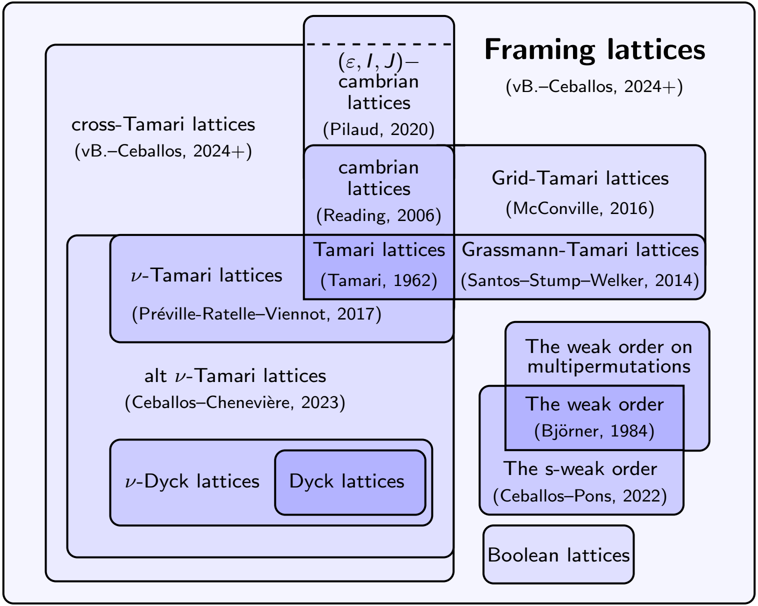

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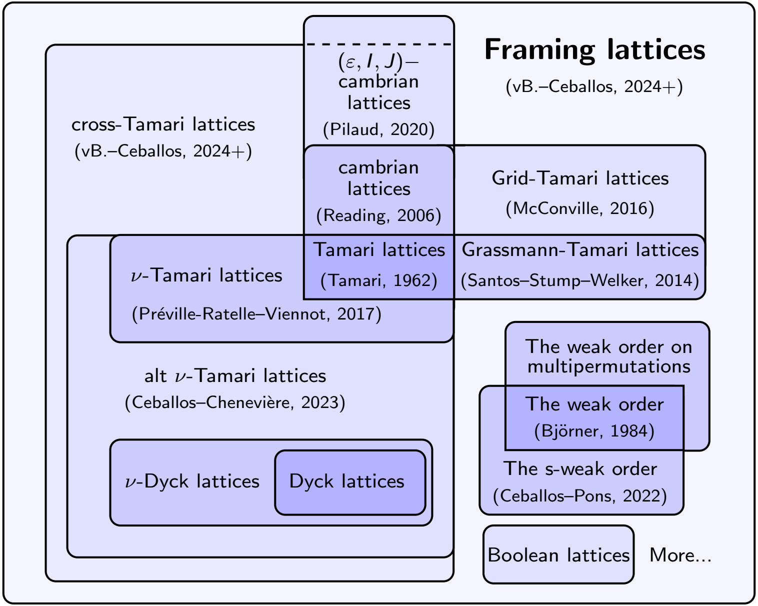

Flow graphs and flow polytopes

Definition

Let G be a directed graph on vertex set $V = \{1, \ldots, n\}$ and edge multiset E with edges directed from smaller to larger vertices.

G_1

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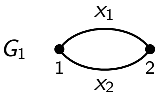
- A **route** in G is a path from a source to a sink.



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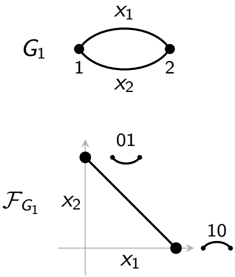
- The **(unit) flow polytope** of G is $\mathcal{F}_G := \mathrm{conv}\{\mathbf{x}_R \mid R \text{ is a route in } G\}$, where $\mathbf{x}_R$ denotes the indicator vector of R.



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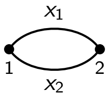
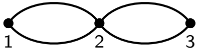
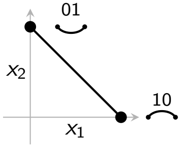


G_2



$\mathcal{F}_{G_1}$



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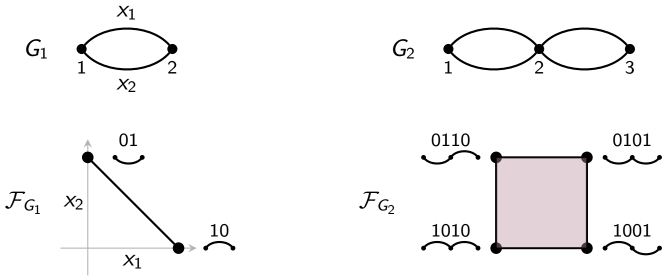

Framed graphs

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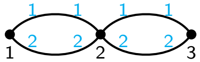
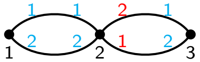
- For a vertex v in a directed graph G, a **framing** of G is a collection F of linear orders $\leq_{\mathrm{In}(v)}, \leq_{\mathrm{Out(v)}}$ on the incoming and outgoing edges at each v.
- A **framed graph** (G, F) is a graph with a framing F.





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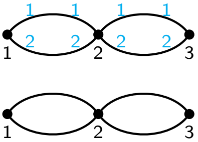

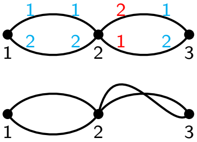

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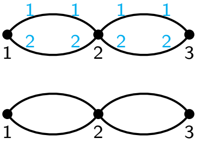
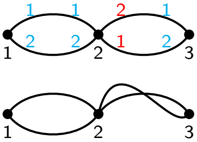
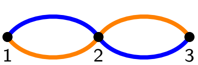
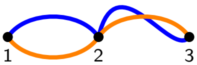




Routes P and Q **conflict** (in a framed graph) if they enter and exit a vertex in different orders. They are **coherent** otherwise.



P and Q conflict



P and Q are coherent

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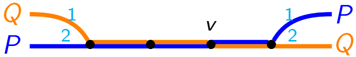
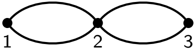


P and Q conflict

Framing lattices

Definition

A **maximal clique** is a maximal collection of pairwise coherent routes.

$(G, F) =$ 

Δ_1 Δ_2

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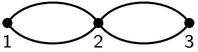
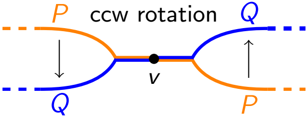
$(G, F) =$ 

Define a cover relation:
$\Delta_1 \prec \Delta_2 \iff \Delta_2$ can be obtained from Δ_1 by a ccw rotation of **a single route**.



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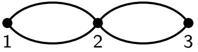
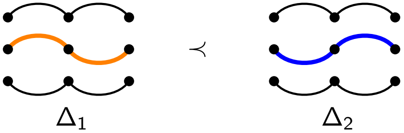
$(G, F) =$ 



$\mathscr{L}_{G,F} =$ 

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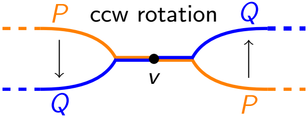

The transitive closure of $\prec$ is a poset $\mathscr{L}_{G,F}$ on maximal cliques!

Framing lattices

Theorem (vB.–Ceballos, 2024+)

The poset $\mathscr{L}_{G,F}$ is a semidistributive, polygonal, and congruence uniform lattice. Moreover, the polygons appearing in $\mathscr{L}_{G,F}$ are squares, pentagons, or hexagons.

- Semidistributive: $x \vee (y \wedge z) = x \vee y$ whenever $x \vee y = x \vee z$; and
  $x \wedge (y \vee z) = x \wedge y$ whenever $x \wedge y = x \wedge z$.
- Polygonal: The interval $[x \wedge y, x \vee y]$ is a polygon when x and y cover $x \wedge y$.
- Congruence uniform: Obtainable from the singleton lattice via Day doublings.

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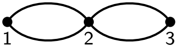
Proof outline:

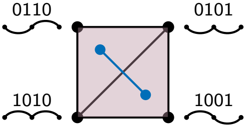
1. Utilize the BEZ lemma for the lattice property and check cases.
2. Only squares, pentagons, and hexagons arise from these cases.
3. Use another BEZ lemma to prove semidistributivity.
4. We show the lattice is an $\mathcal{HH}$-lattice [Caspard–de Poly-Barbut–Morvan, '04], which implies congruence uniformity.

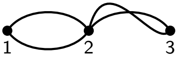
The flow polytope connection

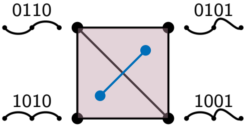
Framing lattices live in flow polytopes!

- Maximal cliques of routes in (G, F) are top-dimensional simplices in a regular unimodular triangulation of $\mathcal{F}_G$. [Danilov–Karzanov–Koshevoy,'12]

$(G, F) =$ 

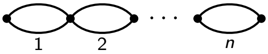
$\mathcal{F}_G$ 

$(G, F) =$ 

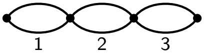
$\mathcal{F}_G$ 

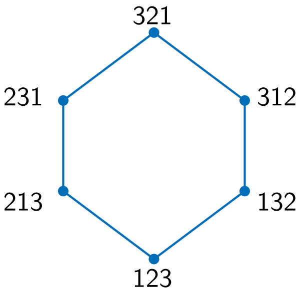
The weak order on $\mathfrak{S}_n$

The graph:

$(\mathrm{oru}(n), F)$ 

Example: $n = 3$

$(\mathrm{oru}(3), F)$ 



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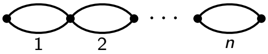
$(\mathrm{oru}(n), F)$ 

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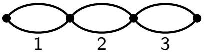
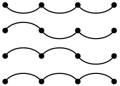
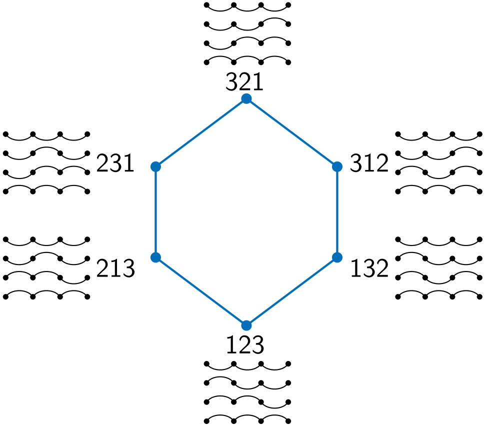
$(\mathrm{oru}(3), F)$ 

Bijection:

$123 \longleftrightarrow$ 



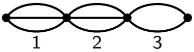
A weak order on multipermutations

Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ be a composition of some integer m.
The weak order can be extended to multipermutations of $1^{s_1}2^{s_2}\cdots n^{s_n}$.
[Bennett–Birkhoff,'94]

The graph:

$\mathrm{multi\text{-}oru}(\mathbf{s}) := \mathrm{oru(n)}$ with $s_i + 1$ edges in segment i.

Example: $\mathbf{s} = (2, 2, 1)$

$(\mathrm{multi\text{-}oru}(\mathbf{s}), F)$ 

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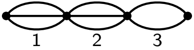

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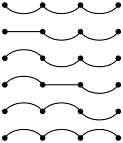
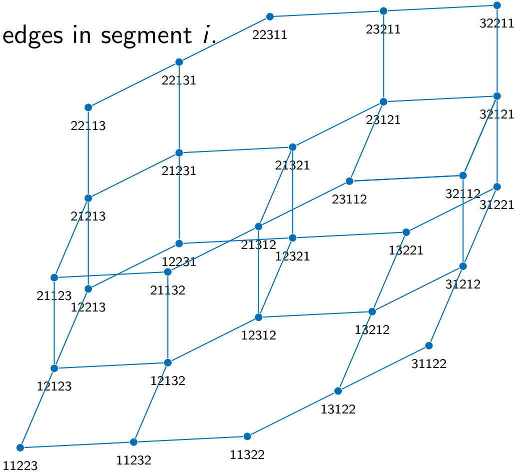
$11223 \longleftrightarrow$





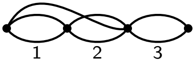
The s-weak order

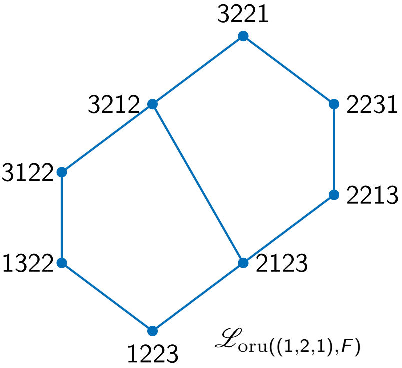
The **s**-weak order is a lattice of 121-avoiding multipermutations.
[Ceballos–Pons,'22]

The graph:

$\mathrm{oru}(\mathbf{s}) := \mathrm{oru}(\mathrm{n})$ with "whiskers".

Example: $\mathbf{s} = (1, 2, 1)$

$(\mathrm{oru}(\mathbf{s}), F)$ 



$\mathscr{L}_{\mathrm{oru}((1,2,1),F)}$

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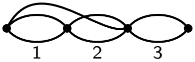
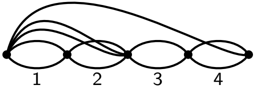
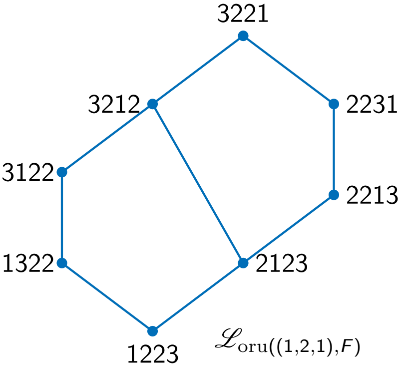
$(\mathrm{oru}(\mathbf{s}), F)$ 

Bigger example: $\mathbf{s} = (1, 3, 1, 2)$

$(\mathrm{oru}(\mathbf{s}), F)$ 



$\mathscr{L}_{\mathrm{oru}((1,2,1),F)}$

[González D'León, Morales, Philippe, Tamayo Jiménez, Yip, 2023]

The Tamari Family

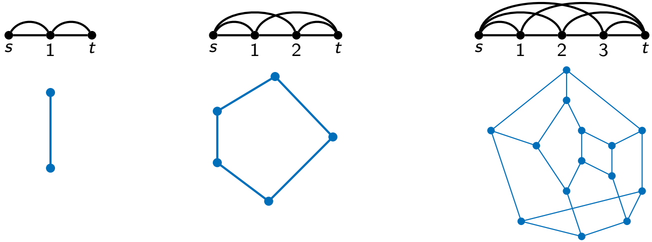
Tamari lattices:

The graph:

$\mathrm{car}(n) :=$ the path $s, 1, \ldots, n, t$ with added edges (s, i) and (i, t) for $i \in [n]$.

$(\mathrm{car}(n), F)$

$\mathscr{L}_{\mathrm{car}(n),F}$



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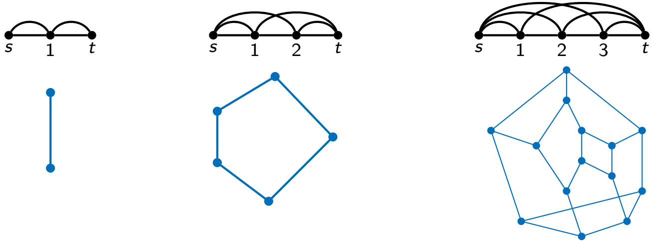
Tamari lattices:

The graph:

$\mathrm{car}(n) :=$ the path $s, 1, \ldots, n, t$ with added edges (s, i) and (i, t) for $i \in [n]$.

$(\mathrm{car}(n), F)$

$\mathscr{L}_{\mathrm{car}(n),F}$

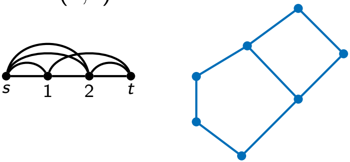


s-Tamari lattices:

The graph:

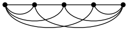
$\mathrm{car}(\mathbf{s}) := \mathrm{car}(n)$, but with s_i copies of (s, i).
$\mathbf{s} = (s_1, s_2, \ldots, s_n)$

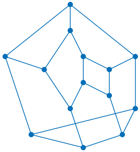
Example: $\mathbf{s} = (1, 2)$



A gallery of examples

Various framing lattices for $\mathrm{car}(3)$:





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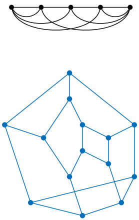

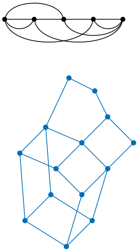

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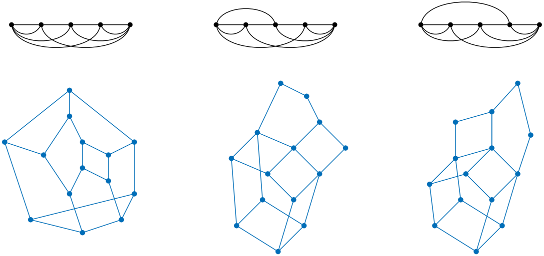

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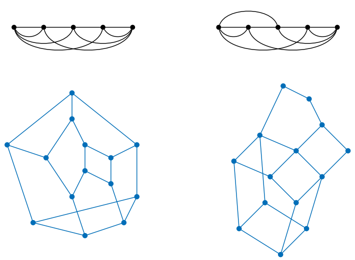

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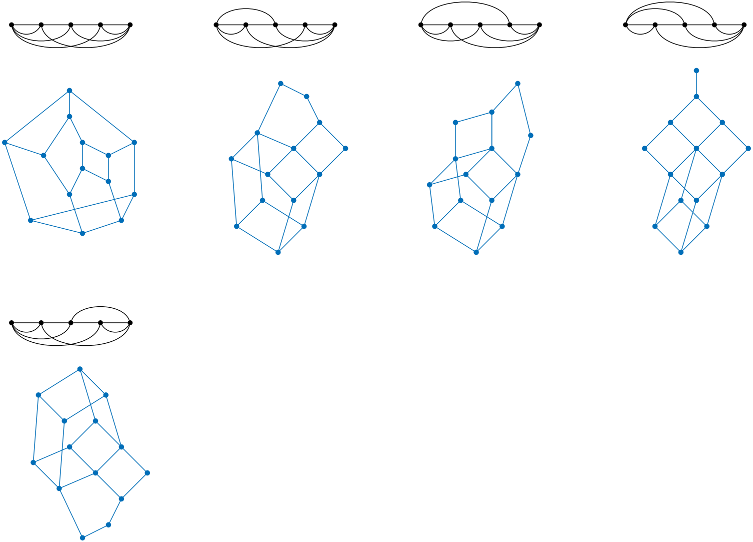

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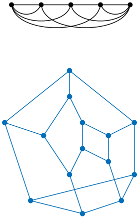

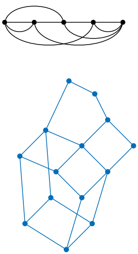

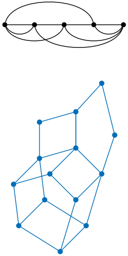

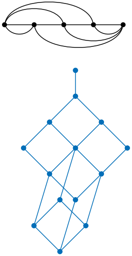

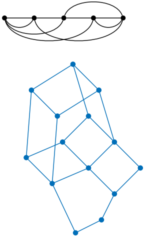

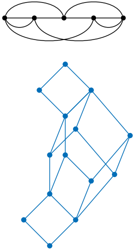

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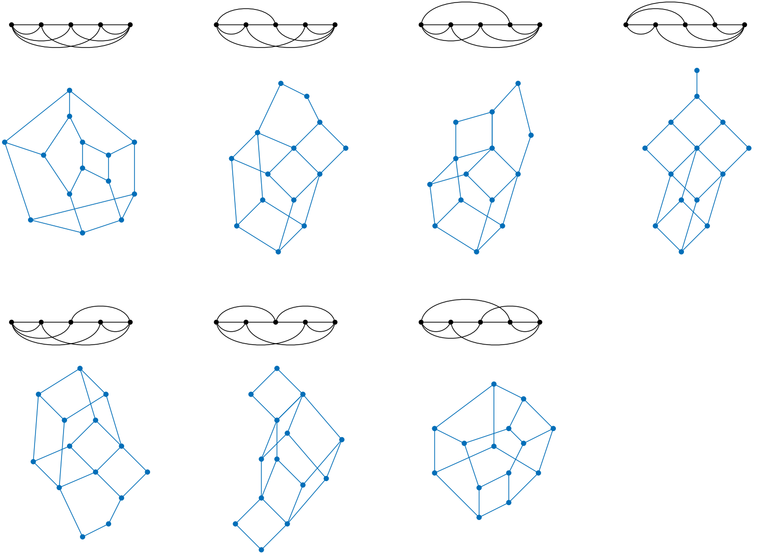

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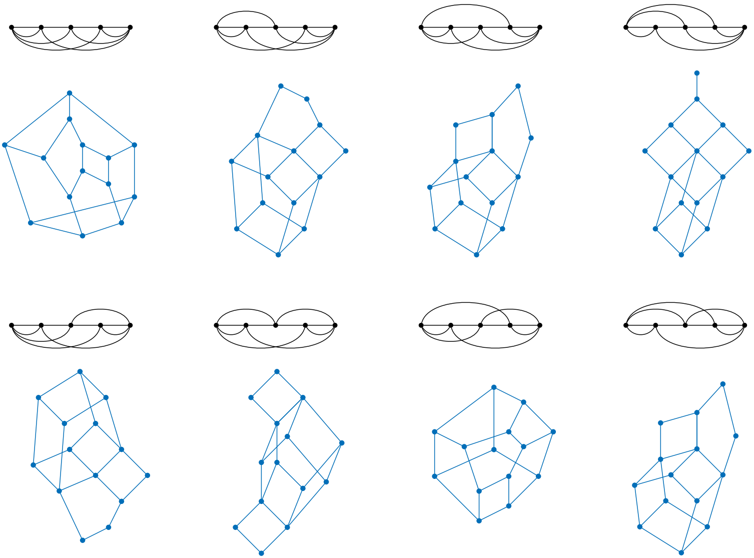

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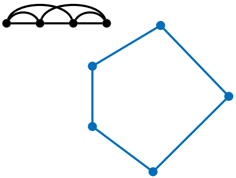
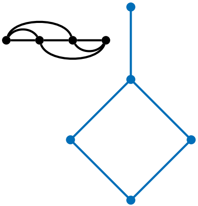
3. What conditions on G and F make the Hasse-diagram n-regular?

Conjecture

For a fixed graph, all framing lattices have the same number of linear intervals of length k for every $k \geq 0$.

Example: The linear interval counts of the following are $(5, 5, 2, 0, \ldots)$.





It holds for alt ν-Tamari lattices. [Ceballos–Chenevière,'23]

Thank you!

References:

- Bennett, Mary K., and Garrett Birkhoff. "Two families of Newman lattices." Algebra Universalis 32 (1994): 115-144.
- Caspard, Nathalie, de Poly-Barbut, Claude LC, and Michel Morvan. "Cayley lattices of finite Coxeter groups are bounded." Adv Appl Math. 33.1 (2004): 71-94.
- Ceballos, Cesar and Chenevière, Clément. "On linear intervals in the alt ν-Tamari lattices." arXiv preprint arXiv:2305.02250 (2023).
- Danilov, Vladimir I., Alexander V. Karzanov, and Gleb A. Koshevoy. "Coherent fans in the space of flows in framed graphs." Discrete Mathematics & Theoretical Computer Science Proceedings (2012).
- D'León, Rafael S. González, et al. "Realizing the **s**-permutahedron via flow polytopes." arXiv preprint arXiv:2307.03474 (2023).
- Gratzer, George A., and Friedrich Wehrung. Lattice theory: special topics and applications. Birkhäuser/Springer International Publishing Switzerland, 2016.