The characteristic quasi-polynomials for exceptional well-generated complex reflection groups

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Joint work with
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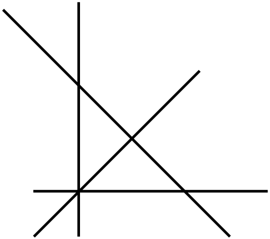
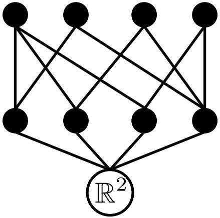
Plan:

1. Hyperplane arrangements and characteristic polynomials
2. Characteristic quasi-polynomials
3. Generalization of characteristic quasi-polynomials
4. Result for complex reflection groups

$\mathcal{A}$: Hyperplane arrangement (finite collection of hyperplanes)

$$L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{B}} H \neq \varnothing \;\middle|\; \mathcal{B} \subseteq \mathcal{A} \right\}$$: The intersection poset





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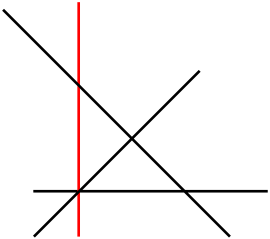

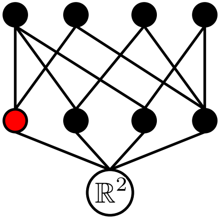

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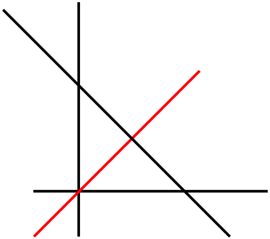

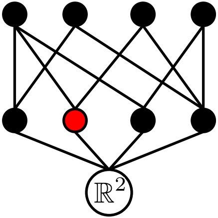

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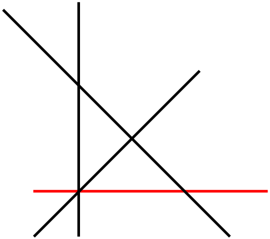

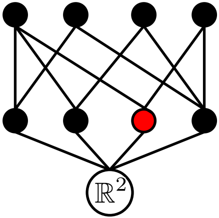

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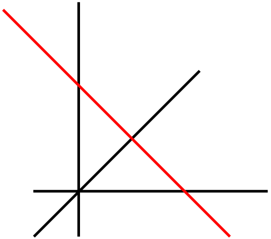

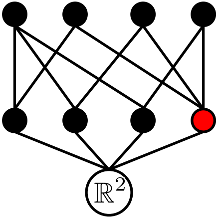

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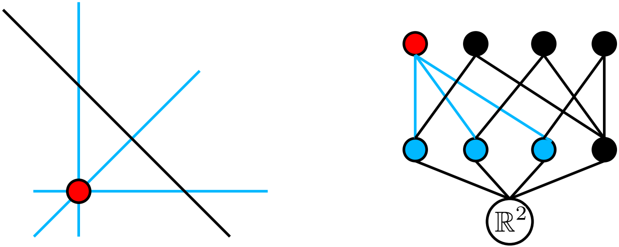

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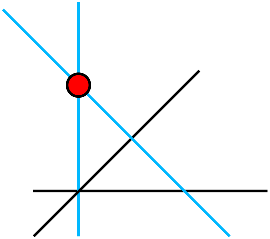

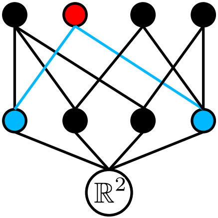

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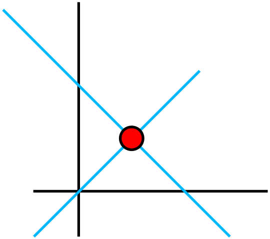

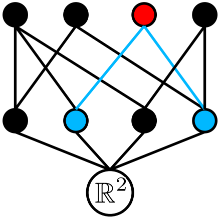

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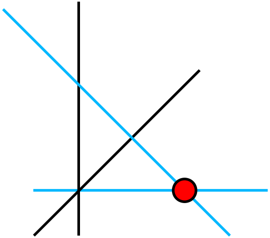

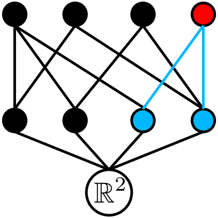

The **characteristic polynomial** is defined by

$$\chi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X},$$

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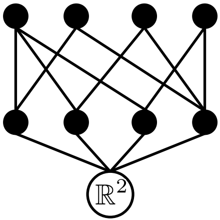

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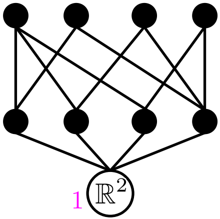

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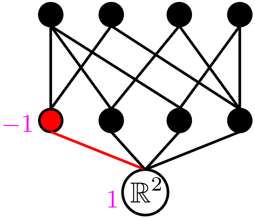

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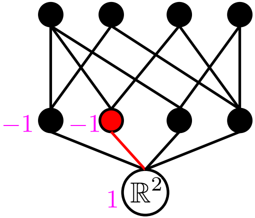

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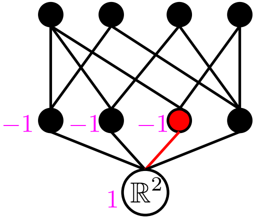

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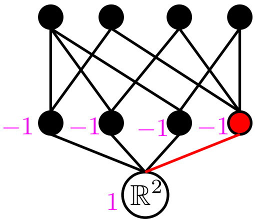

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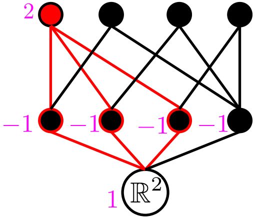

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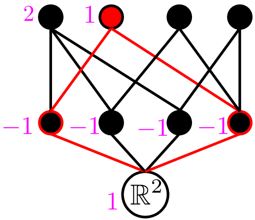

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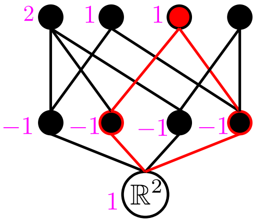

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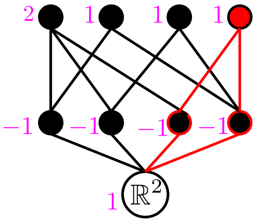

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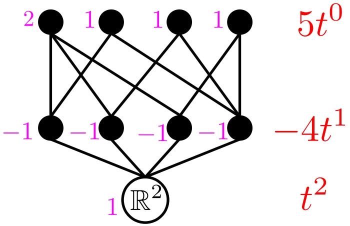

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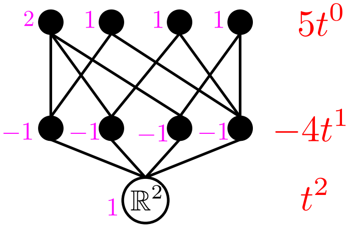
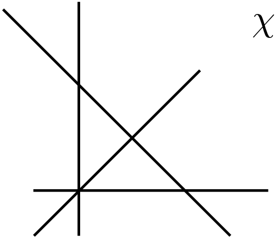


$$\boxed{\chi(\mathcal{A}, t) = t^2 - 4t + 5}$$

Theorem (Zaslavsky)

Let $\mathcal{A}$ be an arrangement over $\mathbb{R}$.

$$|\chi(\mathcal{A},-1)| = \#\textit{chambers}$$
$$|\chi(\mathcal{A},1)| = \#\textit{bounded chambers}$$



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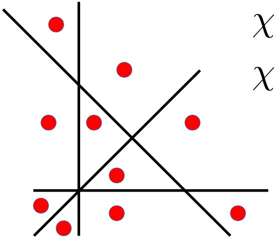

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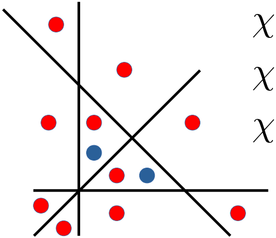


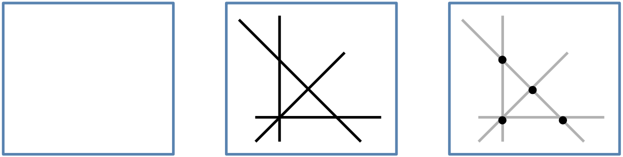
$$\chi(\mathcal{A}, t) = t^2 - 4t + 5$$
$$\chi(\mathcal{A}, -1) = 1 + 4 + 5 = 10$$
$$\chi(\mathcal{A}, 1) = 1 - 4 + 5 = 2$$

Proposition

Let $\mathcal{A}$ be an arrangement in $\mathbb{F}_p^\ell$. Then

$$\chi(\mathcal{A}, p) = \# \left(\mathbb{F}_p^\ell \setminus \bigcup_{H \in \mathcal{A}} H \right)$$

$$\chi(\mathcal{A}, p) = p^2 - 4p + 5$$



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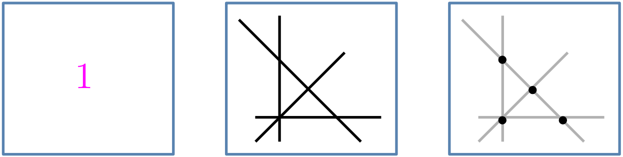

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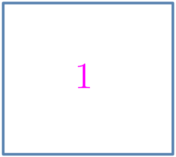

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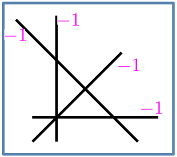
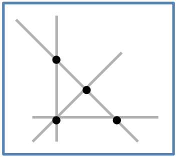


$-4p$



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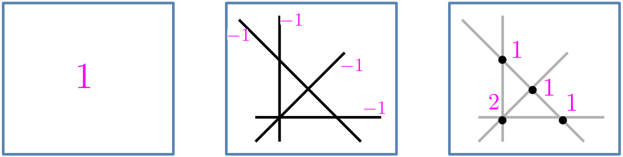

p^2 $\quad -4p \quad$ 5

Let $\mathcal{A}$ be an arrangement over $\mathbb{Q}$.
We may suppose that every hyperplane is defined by a linear equation with integer coefficients.
Let p be a prime large enough. Taking modulo p of the coefficients yields the arrangement $\mathcal{A}_p$ over $\mathbb{F}_p$ such that

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Proposition (Finite Field Method)

Let $\mathcal{A}$ be an arrangement in $\mathbb{Q}^\ell$. Then there are infinitely many primes p such that

$$\chi(\mathcal{A}, p) = \# \left(\mathbb{F}_p^\ell \setminus \bigcup_{H \in \mathcal{A}_p} H \right)$$

Let q be a positive integer.
We can consider "hyperplane" arrangement $\mathcal{A}_q$ in $(\mathbb{Z}/q\mathbb{Z})^\ell$ by taking modulo q.

Question

Does the counting function in q

$$\#\left((\mathbb{Z}/q\mathbb{Z})^\ell \setminus \bigcup_{H \in \mathcal{A}_q} H\right)$$

have interesting property?

Remark

The equations $x = 0$ and $2x = 0$ define the same hyperplane but if we take modulo 2, then they become different equations $x = 0$ and $0 = 0$. Therefore we must fix the coefficients to consider the question above.

From now on, let $\mathcal{A} = \{c_1, \ldots, c_n\}$ be a finite subset in $\mathbb{Z}^\ell$ (coefficient column vectors).
For any $\mathbb{Z}$-module M, we define "hyperplane arrangement" $\mathcal{A}(M)$ by

$$H_i(M) \coloneqq \left\{ x \in M^\ell \mid xc_i = 0 \right\},$$
$$\mathcal{A}(M) \coloneqq \{ H_1(M), \ldots, H_n(M) \}.$$

Define the **characteristic quasi-polynomial** $\chi_{\mathcal{A}}^{\mathrm{quasi}}$ by

$$\chi_{\mathcal{A}}^{\mathrm{quasi}}(q) \coloneqq \# \left((\mathbb{Z}/q\mathbb{Z})^\ell \setminus \bigcup_{H \in \mathcal{A}(\mathbb{Z}/q\mathbb{Z})} H \right)$$

Theorem (Kamiya-Takemura-Terao (2008))

*$\chi_{\mathcal{A}}^{\text{quasi}}$ is a quasi-polynomial in q. Namely, there exists a positive integer ρ (**period**) and polynomials $f_{\mathcal{A}}^1(t), \ldots, f_{\mathcal{A}}^{\rho}(t) \in \mathbb{Z}[t]$ (**constituents**) such that*

$$q \equiv k \pmod{\rho} \Longrightarrow \chi_{\mathcal{A}}^{\text{quasi}}(q) = f_{\mathcal{A}}^k(q).$$

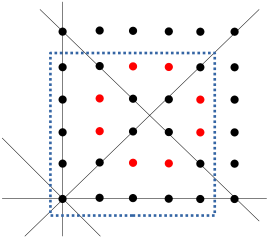
Moreover,

- The first constituent $f_{\mathcal{A}}^1(t)$ coincides with the characteristic polynomial $\chi(\mathcal{A}(\mathbb{Q}), t)$.
- We can compute a period **by calculating elementary divisors** of some submatrices of $(c_1 \cdots c_n)$.
- **GCD-property:**
  $\gcd(k_1, \rho) = \gcd(k_2, \rho) \Longrightarrow f_{\mathcal{A}}^{k_1}(t) = f_{\mathcal{A}}^{k_2}(t)$
  Hence $\chi_{\mathcal{A}}^{\text{quasi}}$ is determined by constituents $f^k(t)$ such that k is a divisor of ρ.

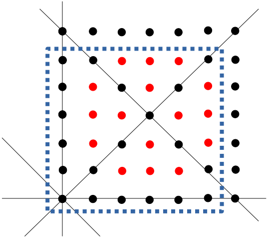
Example

$$\mathcal{A} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}, \qquad \rho = 2$$

$$f^1_{\mathcal{A}}(t) = (t-1)(t-3) = \chi(\mathcal{A}(\mathbb{Q}), t),$$
$$f^2_{\mathcal{A}}(t) = (t-2)^2$$



$q = 5$



$q = 6$

Question

Does every constituent $f^k_{\mathcal{A}}(t)$ have combinatorial meaning?

Theorem (Liu-Tran-Yoshinaga, Tran-Yoshinaga)

Every constituent comes from the **poset of layers** *of the corresponding* **toric arrangement**.

$$\mathcal{A}(\mathbb{C}^\times) = \{H_1(\mathbb{C}^\times), \dots, H_n(\mathbb{C}^\times)\}\text{: } \textbf{toric arrangement}$$
$$H_i(\mathbb{C}^\times) = \left\{ x \in (\mathbb{C}^\times)^\ell \;\middle|\; x_1^{c_{1i}} \cdots x_n^{c_{ni}} = 1 \right\}$$

We call a connected component of the intersection of some $H_i(\mathbb{C}^\times)$'s a **layer**.
Let $L(\mathcal{A}(\mathbb{C}^\times))$ denote the **poset of layers**. (The order is the reverse inclusion.)

Theorem (Liu-Tran-Yoshinaga, Tran-Yoshinaga)

The k-th constituent $f_{\mathcal{A}}^k(t)$ is the characteristic polynomial of the poset $L(\mathcal{A}(\mathbb{C}^\times))[k]$ defined by

$$L(\mathcal{A}(\mathbb{C}^\times))[k] := \left\{ Z \in L(\mathcal{A}(\mathbb{C}^\times)) \;\middle|\; Z \text{ has a } k\text{-torsion} \right\}.$$

More precisely,

$$f_{\mathcal{A}}^k = \sum_{Z \in L(\mathcal{A}(\mathbb{C}^\times))[k]} \mu(Z) t^{\dim Z}.$$

Let Φ be a crystallographic root system and $\{\alpha_1, \dots, \alpha_\ell\}$ its simple system.
Every root is an integral linear combination of simple roots.
Let $\mathcal{A}_\Phi \subseteq \mathbb{Z}^\ell$ be the collection of the coefficient vectors.

Example

$\Phi = \Phi_{\mathrm{E}_6}$. The period is $\rho = 6$.

$$
\begin{aligned}
f^1(t) &= (t-1)(t-4)(t-5)(t-7)(t-8)(t-11)\\
f^2(t) &= (t-2)(t-4)(t-8)(t-10)(t^2-12t+26)\\
f^3(t) &= (t-3)(t-9)(t^4-24t^3+195t^2-612t+480)\\
f^6(t) &= (t-6)^2(t^4-24t^3+186t^2-504t+480)
\end{aligned}
$$

$$t \longleftrightarrow 12 - t \qquad \chi^{\mathrm{quasi}}_{\mathrm{E}_6}(q) > 0 \iff q \geq 12$$

$$h_{\mathrm{E}_6} = 12$$ **Coxeter number**

Theorem (Kamiya–Takemura–Terao)

$$\chi_\Phi^{\text{quasi}}(q) > 0 \iff q \geq h$$

Theorem (Kamiya–Takemura–Terao, Suter, Yoshinaga)

$$\chi_\Phi^{\text{quasi}}(q) = (-1)^\ell \chi_\Phi^{\text{quasi}}(h - q)$$

type	period	Coxeter number
A_ℓ	1	$\ell + 1$
B_ℓ	2	2ℓ
C_ℓ	2	2ℓ
D_ℓ	2	$2\ell - 2$
E_6	6	12
E_7	12	18
E_8	60	30
F_4	12	12
G_2	6	6

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Theorem (Kamiya–Takemura–Terao, Suter)
The radical of the period ρ divides the Coxeter number h.

From this property we have

$$\gcd(q,\rho)=1 \iff \gcd(h-q,\rho)=1$$

$$\chi(\mathcal{A}_\Phi(\mathbb{Q}),t)=(-1)^\ell\chi(\mathcal{A}_\Phi(\mathbb{Q}),h-t)$$

Thus, the characteristic quasi-polynomial is **very interesting**.
However, it can be defined only for arrangements over $\mathbb{Z}$.

Question

How can we generalize the characteristic quasi-polynomial?

A generalization must be related with

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$\rightsquigarrow$ We consider **Dedekind domains**!

Let $\mathcal{O}$ be a Dedekind domain such that the residue ring $\mathcal{O}/\mathfrak{a}$ is finite for any nonzero ideal $\mathfrak{a}$ of $\mathcal{O}$.

Example

$\mathbb{Z}, \mathbb{Z}[\sqrt{-1}], \mathbb{Z}[\sqrt{2}], \mathbb{Z}[\frac{-1+\sqrt{-3}}{2}], \mathbb{F}_q[t]$

Let $\mathcal{A} = \{c_1, \dots, c_n\} \subseteq \mathcal{O}^\ell$ be a finite subset (coefficient column vectors).
Given an $\mathcal{O}$-module M,

$$H_i(M) := \left\{ x \in M^\ell \mid xc_i = 0 \right\},$$
$$\mathcal{A}(M) := \left\{ H_1(M), \dots, H_n(M) \right\}.$$

Let $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}$. Define $\chi_{\mathcal{A}}^{\text{quasi}}(\mathfrak{a})$ by

$$\chi_{\mathcal{A}}^{\text{quasi}}(\mathfrak{a}) = \#\left((\mathcal{O}/\mathfrak{a})^\ell \setminus \bigcup_{H \in \mathcal{A}(\mathcal{O}/\mathfrak{a})} H \right)$$

Theorem (Kuroda-T (2024))

$\chi_{\mathcal{A}}^{\text{quasi}}(\mathfrak{a})$ *behaves like a quasi-polynomial with GCD-property. Namely, there exist a nonzero ideal* ρ *(**period**) and polynomials* $f_{\mathcal{A}}^{\kappa}(t) \in \mathbb{Z}[t]$ *for each divisor* κ *of* ρ *(**constituent**) such that*

$$\mathfrak{a} + \rho = \kappa \Longrightarrow \chi_{\mathcal{A}}^{\text{quasi}}(\mathfrak{a}) = f_{\mathcal{A}}^{\kappa}(N(\mathfrak{a})),$$

where $N(\mathfrak{a})$ *denotes the absolute norm of* $\mathfrak{a}$ *defined by*

$$N(\mathfrak{a}) = \#\left(\mathcal{O}/\mathfrak{a}\right)$$

Moreover,

- $f_{\mathcal{A}}^{\langle 1 \rangle}(t) = \chi(\mathcal{A}(K), t)$, where K is the field of fractions $\mathcal{O}$.
- Every constituent comes from the poset $L(\mathcal{A}(K/\mathcal{O}))$.

Non-crystallographic root systems H_3 and H_4 are defined over the Dedekind domain $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$.
Moreover, every complex reflection group admit a "root system" over a Dedekind domain defined by Lehrer and Taylor.
Now, we can consider the characteristic quasi-polynomials for these root systems.
We are **lucky** if they have interesting properties.

Example

Consider G_{33}. The ring of definition is $\mathcal{O} = \mathbb{Z}[\frac{-1+\sqrt{-3}}{2}]$ and the period is $\rho = \langle 2\sqrt{-3}\rangle$. The Coxeter number is $h = 18$.

$$\begin{aligned}
f^{\langle 1\rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 17169t - 12285. \\
&= (t-1)(t-7)(t-9)(t-13)(t-15). \\
f^{\langle 2\rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 17574t - 18360. \\
&= (t-4)(t-15)(t^3 - 26t^2 + 196t - 306). \\
f^{\langle \sqrt{-3}\rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 18129t - 20925. \\
&= (t-3)(t-9)(t^3 - 33t^2 + 327t - 775). \\
f^{\langle 2\sqrt{-3}\rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 18534t - 27000.
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Unlucky?

We computed the period for the case G is exceptional, well-generated, and irreducible.

G	$\mathcal{O}$	ρ	h
G_4	$\mathbb{Z}[\omega]$	$\langle 1 \rangle$	6
G_5	$\mathbb{Z}[\omega]$	$\langle 2\sqrt{-3} \rangle$	12
G_6	$\mathbb{Z}[i,\omega]$	$\langle 1+i \rangle$	12
G_8	$\mathbb{Z}[i]$	$\langle 1+i \rangle$	12
G_9	$\mathbb{Z}[\zeta_8]$	$\langle 6 \rangle$	24
G_{10}	$\mathbb{Z}[i,\omega]$	$\langle (1+i)\sqrt{-3} \rangle$	24
G_{14}	$\mathbb{Z}[\omega,\sqrt{-2}]$	$\langle 6 \rangle$	24
G_{16}	$\mathbb{Z}[\zeta_5]$	$\langle 1-\zeta_5 \rangle$	30
G_{17}	$\mathbb{Z}[i,\zeta_5]$	$\langle 6\sqrt{5} \rangle$	60
G_{18}	$\mathbb{Z}[\omega,\zeta_5]$	$\langle 2\sqrt{-3}(1-\zeta_{15}^3) \rangle$	60
G_{20}	$\mathbb{Z}[\omega,\tau]$	$\langle 2\sqrt{-3} \rangle$	30
G_{21}	$\mathbb{Z}[i,\omega,\tau]$	$\langle 6\sqrt{5} \rangle$	60

$G_{23} = H_3$	$\mathbb{Z}[\tau]$	$\langle 2 \rangle$	10
G_{24}	$\mathbb{Z}[\lambda]$	$\langle 4 \rangle$	14
G_{25}	$\mathbb{Z}[\omega]$	$\langle \sqrt{-3} \rangle$	12
G_{26}	$\mathbb{Z}[\omega]$	$\langle 6 \rangle$	18
G_{27}	$\mathbb{Z}[\omega,\tau]$	$\langle 4\sqrt{-3} \rangle$	30
$G_{28} = F_4$	$\mathbb{Z}$	$\langle 12 \rangle$	12
G_{29}	$\mathbb{Z}[i]$	$\langle 10(1+i) \rangle$	20
$G_{30} = H_4$	$\mathbb{Z}[\tau]$	$\langle 6\sqrt{5} \rangle$	30
G_{32}	$\mathbb{Z}[\omega]$	$\langle 2\sqrt{-3} \rangle$	30
G_{33}	$\mathbb{Z}[\omega]$	$\langle 2\sqrt{-3} \rangle$	18
G_{34}	$\mathbb{Z}[\omega]$	$\langle 84 \rangle$	42
$G_{35} = E_6$	$\mathbb{Z}$	$\langle 6 \rangle$	12
$G_{36} = E_7$	$\mathbb{Z}$	$\langle 12 \rangle$	18
$G_{37} = E_8$	$\mathbb{Z}$	$\langle 60 \rangle$	30

$$i = \sqrt{-1},\ \omega = \frac{-1+\sqrt{-3}}{2},\ \tau = \frac{1+\sqrt{5}}{2},\ \lambda = \frac{-1+\sqrt{-7}}{2},\ \zeta_k = e^{2\pi i/k}.$$

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Theorem (Kuroda-T)

Every exceptional well-generated irreducible complex reflection group G admits "root system" such that the radical of the period divides the Coxeter number.