Patchworking in higher codimension and oriented matroids

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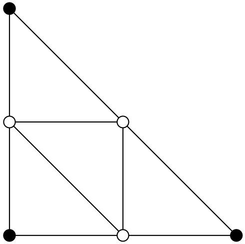
Viro's unimodular combinatorial patchworking

Input:

1) a **regular** unimodular subdivision of $d\Delta_{n+1}$ (the $n+1$ dim simplex of size d).
2) a choice of signs $\epsilon(a) \in \{+, -\}$ for each $a \in \mathbb{Z}^{n+1} \cap d\Delta_{n+1}$.

Output:

an n-dimensional polyhedral complex $\mathcal{P} \subset \mathbb{R}P^{n+1}$.



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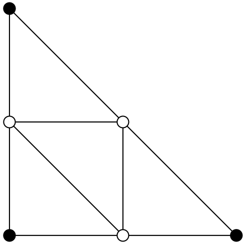
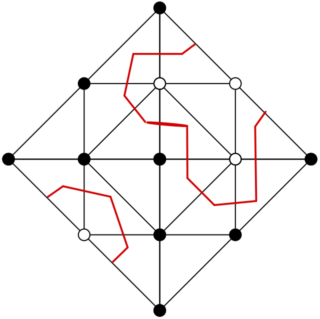


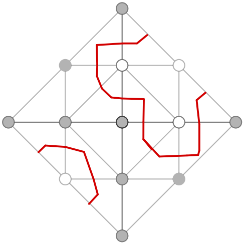


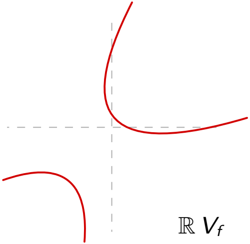
The patchworking theorem

Theorem (Viro 1984)

There exists a homogeneous polynomial $f(x_0, \ldots, x_{n+1})$ of degree d and a homeomorphism of pairs

$$(\mathbb{R}P^{n+1}, \mathbb{R}V_f) \cong (\mathbb{R}P^{n+1}, \mathcal{P}).$$





Take t sufficiently large and f to be the homogenization of

$$F_t(x_0, \ldots, x_{n+1}) = \sum_{a \in \mathbb{Z}^{n+1} \cap d\Delta_{n+1}} \epsilon(a) t^{\alpha(a)} \underline{x}^a.$$

Topology of patchworked hypersurfaces

Theorem (Bertrand 2006)

If $\mathbb{R} V_f \subset \mathbb{R} P^{n+1}$ is obtained by unimodular patchworking then

$$\chi(\mathbb{R} V_f) = \sigma(\mathbb{C} V_f) := \sum_{n=p+q} (-1)^q \dim \mathrm{H}^{p,q}(\mathbb{C} V_f).$$

Theorem (Renaudineau-S. 2023)

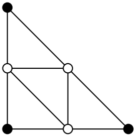
If $\mathbb{R} V_f \subset \mathbb{R} P^{n+1}$ is obtained by unimodular patchworking then

$$b_q(\mathbb{R} V_f) \leq \sum_p \dim \mathrm{H}^{p,q}(\mathbb{C} V_f)$$

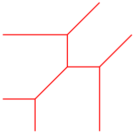
where $\mathrm{H}^n(\mathbb{C} V_f; \mathbb{C}) = \bigoplus_{q=0}^n \mathrm{H}^{n-q,q}(\mathbb{C} V_f)$ *is the* **Hodge decomposition**.

A tropical approach to patchworking

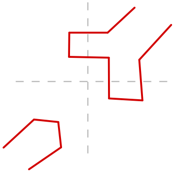
The family of hypersurfaces V_{F_t} has tropicalisation dual to the subdivision of $d\Delta_{n+1}$ and **real tropicalisation** (Jell-Scheiderer-Yu) homeomorphic to $\mathcal{P}$.



Regular subdivision



$\text{Trop}(V_{F_t})$



$\text{Trop}_{\mathbb{R}}(V_{F_t})$

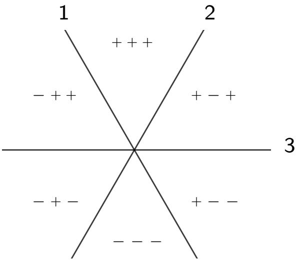
Motivational question: What encodes a real structure on a tropical variety?

For a tropical hypersurface: the sign choices $\epsilon : d\Delta_{n+1} \cap \mathbb{Z}^{n+1} \to \{+, -\}$.
Similarly for transversal intersections of tropical hypersurfaces (Sturmfels 1994).

Oriented matroids in tropical geometry

Oriented matroids

An oriented matroid $\mathcal{M}$ on E is a collection $\mathcal{C} \subseteq \{0,+,-\}^E$ "covectors" satisfying **some axioms**.



$$\mathcal{C}_{\mathcal{A}} = \{(\mathrm{sgn}(f_1(\underline{x})), \ldots, \mathrm{sgn}(f_n(\underline{x})) \mid x \in \mathbb{R}^d\}$$

$$\{000, 0++, +0+, +-0, 0--, -0-, -+0, \\ +++, +-+, +--, ---, -+-, -++\}$$

The **topes** of $\mathcal{M}$ are the covectors in $\{+,-\}^E$.

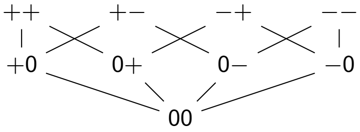
Theorem (Folkman-Lawrence 1978)

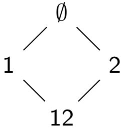
Every oriented matroid can be represented by an arrangement of pseudohyperplanes.

From oriented matroids to matroids

Every oriented matroid $\mathcal{M}$ has an underlying matroid $M = \underline{\mathcal{M}}$ with lattice of flats

$$\mathcal{L}_M = \{\mathrm{Supp}(C)^c \mid C \in \mathcal{C}\}.$$





- Graphs and vector configurations/hyperplane arrangements $\rightsquigarrow$ matroids.
- Oriented graphs and vector configurations/hyperplane arrangements over $\mathbb{R}$ $\rightsquigarrow$ oriented matroids.

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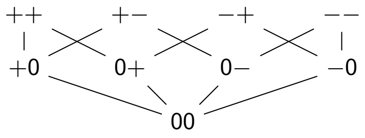

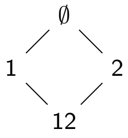

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Not every matroid is orientable!

Theorem (Richter-Gebert 1999)

Determining orientability of a matroid is an NP-complete problem.

Tropicalisation of linear spaces and matroids

Theorem (Sturmfels 2002)

For every matroid M, there is a tropical variety $Trop(M)$.
The tropicalisation of a linear space $L \subset \mathbb{K}^N$ depends only on its matroid:

$$Trop(L) = Trop(M_L).$$

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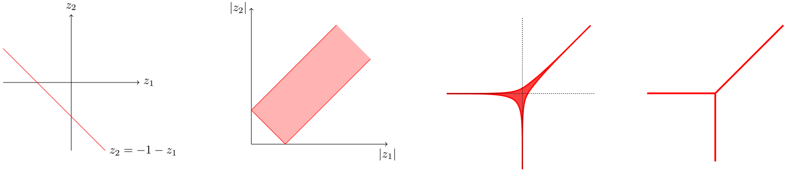


When $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, the tropicalisation is

$$\text{Trop}(L) := \lim_{t \to \infty} \text{Log}_t(L),$$

where $\text{Log}_t(z_1, \ldots, z_N) = (log_t|z_1|, \ldots, log_t|z_N|)$

Matroid fans

The set Trop(M) is the support of the Ardila-Klivans fan.

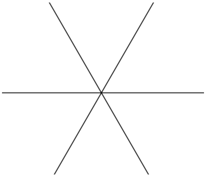
Definition (Ardila-Klivans 2006)

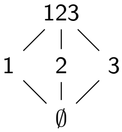
The Ardila-Klivans fan of M is the unimodular fan

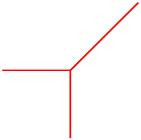
$$\tilde{\Sigma}_M = \{\sigma_{\mathcal{F}} = \langle v_{F_1}, \ldots, v_{F_k}, \pm v_E \rangle_{\geq 0}\} \text{ in } \mathbb{R}^E$$

where $\mathcal{F} = \emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E$ is a chain of flats and $v_I = -\sum_{i \in I} e_i$.

The projective fan Σ_M is the image of $\tilde{\Sigma}_M$ in $\mathbb{R}^E / \langle v_E \rangle$.







The projective Ardila-Klivans fan Σ_M is a Minkowski weight (i.e. cohomology class) of the permutahedral toric variety Y_{Π_E}.

Encoding a matroid orientation

- If $\mathcal{M}$ is an orientation of M, define for $\sigma_{\mathcal{F}} \in \tilde{\Sigma}_M$
$$\mathcal{E}_{\mathcal{M}}(\sigma_{\mathcal{F}}) := \left\{\text{topes of } \mathcal{M} \text{ adjacent to } \mathcal{F}\right\} \subset \{+,-\}^E.$$

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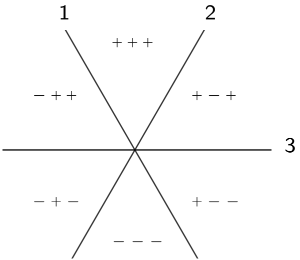
- For a linear space $L \subset \mathbb{R}^E$, define for $\sigma_{\mathcal{F}} \in \tilde{\Sigma}_{M_L}$

$$\mathcal{E}_L(\sigma_{\mathcal{F}}) = \{\varepsilon \in \{+,-\}^E \mid \sigma_{\mathcal{F}} \subset \mathrm{Trop}(L \cap \mathbb{R}^E_\varepsilon)\},$$

where $\mathbb{R}^E_\varepsilon = \{(\varepsilon_1 x_1, \dots, \varepsilon_{|E|} x_{|E|}) \mid x_i > 0\} \subset \mathbb{R}^E$. See **real tropicalization** of Jell-Scheiderer-Yu.



$$\mathcal{E}(\sigma_{123}) = \{\text{all topes}\}$$
$$\mathcal{E}(\sigma_{1,123}) = \{+++, -++, ---, +--\}$$
$$\mathcal{E}(\sigma_{3,123}) = \{-+-, -++, -+-, +--\}$$

Properties of $\mathcal{E}$

Identify $\{+,-\}$ with the field $\mathbb{Z}_2 = \{0,1\}$ via:

$$+ \mapsto 0 \qquad \text{and} \qquad - \mapsto 1.$$

Then $\{+,-\}^E$ inherits the structure of a vector space.

Notice:
$\mathcal{E}(\sigma_{1,123}) = \{+++, -++, ---, +--\} \cong \langle v_1, v_E \rangle_{\mathbb{Z}_2}$
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Lemma

If $\sigma_{\mathcal{F}}$ is a facet of $\tilde{\Sigma}_M$, then $\mathcal{E}_\bullet(\sigma_{\mathcal{F}})$ is an affine subspace of $\{+,-\}^E \cong \mathbb{Z}_2^E$ and

$$\mathcal{E}_\bullet(\tau_{\mathcal{F}}) = \bigcup_{\substack{\tau_{\mathcal{F}} \subset \sigma_{\mathcal{F}'} \\ \sigma_{\mathcal{F}'} \in Facets(\tilde{\Sigma}_M)}} \mathcal{E}_\bullet(\sigma_{\mathcal{F}'}).$$

Real phase structures on fans

Definition (Rau-Renaudineau-S. 2022)

A **real phase structure** on a d-dimensional rational polyhedral fan $\Sigma \subset \mathbb{R}^N$ is a map:

$$\mathcal{E} : \text{Facets}(\Sigma) \to \text{Aff}_d((\mathbb{Z}_2)^N)$$

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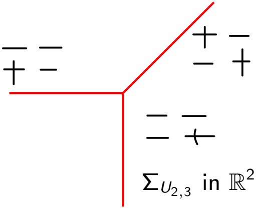
satisfying

R1 $\forall \sigma \in \mathrm{Facets}(\Sigma)$, $\mathcal{E}(\sigma)$ is parallel to $\langle \sigma \rangle_{\mathbb{Z}_2}$.

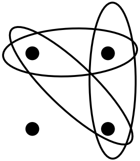
R2 $\forall \tau$ of codimension 1 in Σ with facets $\sigma_1, \ldots, \sigma_k \supset \tau$, the multiset

$$\mathcal{E}(\sigma_1) \dot{\cup} \ldots \dot{\cup} \mathcal{E}(\sigma_k)$$

is an even covering. **"Real balancing condition"**.



$\Sigma_{U_{2,3}}$ in $\mathbb{R}^2$



$(\mathbb{Z}_2)^2$

Real phase structures and matroid orientations

Theorem (RRS 2022)

A real phase structure on $\tilde{\Sigma}_M$ (or Σ_M) is cryptomorphic to an orientation of M.

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A real phase structure on $\tilde{\Sigma}_M$ (or Σ_M) is cryptomorphic to an orientation of M.

Proof.

The map $\mathcal{E}_{\mathcal{M}}$ satisfies R1 and also R2 by the diamond property of $\mathcal{C}$.

To recover $\mathcal{M}$ from a real phase structure $\mathcal{E}$ on Σ_M, use the oriented matroid extension property and a deletion-contraction argument:

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- $\mathcal{E}$ induces real phase structures on $\Sigma_{M/e}$ and $\Sigma_{M\setminus e}$,

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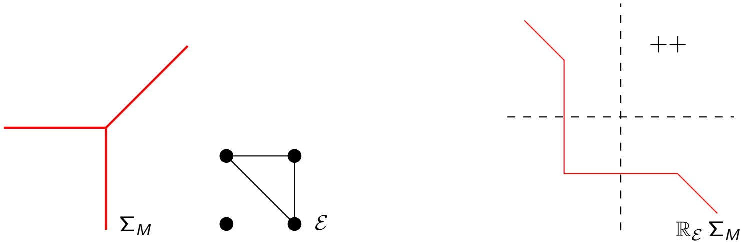
- verify that the orientation of M/e is a quotient of the orientation of $M\backslash e$,
- use the corank one oriented matroid extension property to cook up an orientation $\mathcal{M}$ of M,
- show $\mathcal{E}_{\mathcal{M}} = \mathcal{E}$ (up to reorientation).

□

Patchworking the real part

For Σ equipped with a real phase structure $\mathcal{E}$, the **real part** is

$$\mathbb{R}_{\mathcal{E}}\,\Sigma = \bigsqcup_{\substack{\sigma\in\mathrm{Facets}(\Sigma)\\ \varepsilon\in\mathcal{E}(\sigma)}} \sigma(\varepsilon) \subseteq \bigsqcup_{\varepsilon\in\mathbb{Z}_2^N} \mathbb{R}^N(\varepsilon) \cong (\mathbb{R}^*)^N.$$



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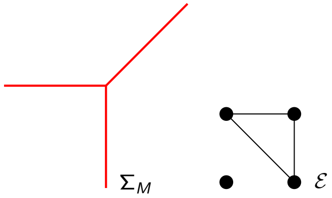

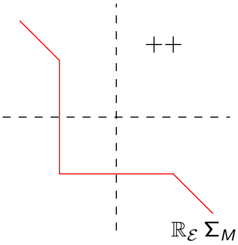

Proposition

If $M = M_L$ for $L \subset \mathbb{R}^N$, then $\mathbb{R}_{\mathcal{E}_L}\Sigma_{M_L}$ is the real tropicalisation of L

$$Trop_{\mathbb{R}}(L) := \bigsqcup_{\varepsilon\in\mathbb{Z}_2^N} Trop(L\cap\mathbb{R}^N_{\varepsilon}) \subset \bigsqcup_{\varepsilon\in\mathbb{Z}_2^N} \mathbb{R}^N(\varepsilon).$$

The real part of a matroid fan

Theorem (Las Vergnas, Ardila-Klivans-Williams, Celaya, RRS)

The positive part $\mathbb{R}_{\mathcal{E}}\,\tilde{\Sigma}_M \cap \mathbb{R}^E(+\cdots+)$ *is homeomorphic to* $\mathbb{R}^{d+1}$.
The compactification $\overline{\mathbb{R}_{\mathcal{E}}\,\tilde{\Sigma}_M} \subset \mathbb{R}^E$ *is homeomorphic to* $\mathbb{R}^{d+1}$.
The subcomplexes $\overline{\mathbb{R}_{\mathcal{E}}\,\tilde{\Sigma}}_M \cap \{x_i = 0\}$ *form a pseudohyperplane arrangement with associated oriented matroid* $\mathcal{M}_{\mathcal{E}}$.

There is also a **valuated matroid** version (Celaya-Loho-Yuen 2022 and Olarte-Rincón-Smith 2024+).

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Theorem (RRS 2023)

Let $M = M_L$ *for a* $\mathbb{R}$*-linear space* L*, then*

$$\overline{\mathbb{R}_{\mathcal{E}}\,\Sigma_M}_{\geq 0} \subset \mathbb{R}\,P^N_{\geq 0}$$

is a standard pair of discs.
(Similarly for $\overline{\mathbb{R}_{\mathcal{E}}\,\Sigma_M}_{\geq 0} \subset \mathbb{R}\,Y_{\Sigma \geq 0}$ *when* Σ_M *is a union of cones of* Σ*.)*

Question: Do there exist wild embeddings $\overline{\mathbb{R}_{\mathcal{E}}\,\Sigma}_M \subseteq \overline{\mathbb{R}_{\mathcal{E}'}\,\Sigma}_{M'}$?

The real part and real toric varieties

The space $\bigsqcup_{\varepsilon \in \mathbb{Z}_2^N} \mathbb{R}^N(\varepsilon)$ is homeomorphic to the real torus $(\mathbb{R}^*)^N$ which is a subset of a real toric variety $\mathbb{R}\, Y_{\Sigma'}$ from a fan Σ' in $\mathbb{R}^N$.

Consider $\bigsqcup_{\varepsilon \in \mathbb{Z}_2^N} \mathbb{R}^N(\varepsilon) \cong (\mathbb{R}^*)^N \subset \mathbb{R}\, Y_{\Sigma'}$ and the closure $\overline{\mathbb{R}_{\mathcal{E}}\, \Sigma} \subset \mathbb{R}\, Y_{\Sigma'}$.

E.g. Obtain closures of $\mathbb{R}_{\mathcal{E}}\, \Sigma$ in $\mathbb{R}^N, \mathbb{R}\, P^N$, $(\mathbb{R}\, P^1)^N$, $\mathbb{R}\, Y_{\Pi_E} \dots$

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E.g. Obtain closures of $\mathbb{R}_{\mathcal{E}}\, \Sigma$ in $\mathbb{R}^N, \mathbb{R}\, P^N$, $(\mathbb{R}\, P^1)^N$, $\mathbb{R}\, Y_{\Pi_E} \dots$

Proposition (RRS 2022)

Suppose Σ is a d-dimensional subfan of Σ' and let $\mathcal{E}$ be a real phase structure on Σ. Then $\mathbb{R}_{\mathcal{E}}\, \Sigma$ is a closed cellular chain in $C_d(\mathbb{R}\, Y_{\Sigma'}; \mathbb{Z}_2)$ and we have

$$[\mathbb{R}_{\mathcal{E}}\, \Sigma] \in H_d(\mathbb{R}\, Y_{\Sigma'}; \mathbb{Z}_2).$$

Obstruction to matroid orientability from toric geometry

If M is a matroid on E, then Σ_M is a subfan of the permutahedral fan Π_E. There is an inclusion of toric varieties:

$$i : \mathbb{R}\, Y_{\Sigma_M} \to \mathbb{R}\, Y_{\Pi_E}.$$

Theorem (S.)

Let $d = \dim \Sigma_M$. *If the map*

$$i_* : H_d(\mathbb{R}\, Y_{\Sigma_M}; \mathbb{Z}_2) \to H_d(\mathbb{R}\, Y_{\Pi_E}; \mathbb{Z}_2)$$

is 0 *then* M *is not orientable.*

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- The image of i_* is either 0 or 1 dimensional.
- The map is 0 for the Fano matroid (first known non-orientable matroid).
- Unknown if the converse holds!

Patchworking beyond codimension one

Patchworking for locally matroidal spaces

A real phase structure on a d-dimensional polyhedral complex X in $\mathbb{R}^N$ is a map

$$\mathcal{E} : \mathrm{Facets}(X) \to \mathrm{Aff}_d(\mathbb{Z}_2^N)$$

satisfying R1 and R2.

Theorem (RRS 2023)

Let X be a locally matroidal tropical variety equipped with a real phase structure $\mathcal{E}$. Then $\mathbb{R}_{\mathcal{E}} X$ is a PL-manifold.

Moreover,

$$b_q(\mathbb{R}_{\mathcal{E}} X) \leq \sum \dim H_{p,q}^{trop}(X; \mathbb{Z}_2),$$

where $H_{p,q}^{trop}(X; \mathbb{Z}_2)$ are **tropical homology groups***.*

Proof summary

1) Rewrite the cellular chain groups

$$C_\bullet(\mathbb{R}_\mathcal{E} X; \mathbb{Z}_2) = \bigoplus_{\sigma \in X} \Big(\bigoplus_{\varepsilon \in \mathcal{E}(\sigma)} \mathbb{Z}_2 \Big),$$

and identify $(\bigoplus_{\varepsilon \in \mathcal{E}(\sigma)} \mathbb{Z}_2)$ as the "tope space" of an oriented matroid $\mathcal{M}_\sigma$.

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2) Adapt Quillen's filtration to the tope space of a matroid, or equivalently Kalinin's or Gelfand-Varchenko's filtration (Yuen-S. 2024+).

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2) Adapt Quillen's filtration to the tope space of a matroid, or equivalently Kalinin's or Gelfand-Varchenko's filtration (Yuen-S. 2024+).

3) Identity the first page of this spectral sequence as the tropical homology groups of X:

$$E^1_{p,q} = H^{trop}_{p,q}(X; \mathbb{Z}_2).$$

[In the case of **hypersurfaces** we can prove that

$$\dim H^{trop}_{p,q}(X; \mathbb{Z}_2) = \dim H^{trop}_{p,q}(X; \mathbb{Q}) = \dim H^{p,q}(\mathbb{C}\, V_f)$$

using (Arnal-Renaudineau-S. 2021) and (Itenberg-Katzarkov-Mikhalkin-Zharkov 2019). See also (Brugallé-López de Medrano-Rau 2022) for generalisations.]

A generalised patchworking homeomorphism

Let $\mathcal{X} \to \mathcal{D}^*$ be a real family of subvarieties of a toric variety Y_Σ over the punctured disc $\mathcal{D}^*$.
If the tropicalisation $X = \text{Trop}(\mathcal{X})$ is locally matroidal (or more generally weight 1), then it comes with a real phase structure $\mathcal{E}$.

Theorem (RRS 2023)

If $Trop(\mathcal{X}) = X$ is locally matroidal, then for t sufficiently large there is a homeomorphism of pairs

$$(\mathbb{R}\, Y_\Sigma, \mathbb{R}\, \mathcal{X}_t) \cong (\mathbb{R}\, Y_\Sigma, \mathbb{R}_\mathcal{E}\, X).$$

Moreover,

$$b_q(\mathbb{R}\, \mathcal{X}_t) \leq \sum \dim H_{p,q}^{trop}(X; \mathbb{Z}_2) \qquad \textit{and} \qquad \chi(\mathbb{R}\, \mathcal{X}_t) = \sigma(\mathbb{C}\, \mathcal{X}_t).$$

Thank you!