A Diagram Model for the Okada Algebra and Monoid

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Introduction: Stanley's **Young-Fibonacci lattice** $\mathbb{YF}$

The $\mathbb{YF}$-lattice is close analogue of **Young's lattice** $\mathbb{Y}$:

- $\mathbb{YF}$ governs the representation theory of a tower of semi-simple algebras (counterpart to the tower of **symmetric groups**)
- $\mathbb{YF}$ supports a version of the **RS-correspondence**
- $\mathbb{YF}$ version of the RS-correspondence reflected in the **structure theory** of algebras

The $\mathbb{YF}$-lattice vis-à-vis **Fibonacci Sets**

Definition

A Fibonacci set of rank N is as subset $S = \{s_1 < \cdots < s_k\}$ of $\{1, \ldots, N\}$ whose size $|S| = k$ has the same parity as N and such that s_ℓ has the same parity as ℓ for each $1 \leq \ell \leq k$.

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Remark

We distinguish Fibonacci sets by a subscript indicating their rank. For example $\{1, 2, 5\}_{\mathbf{5}}$ *and* $\{1, 2, 5\}_{\mathbf{7}}$ *are distinct Fibonacci sets of ranks* 5 *and* 7 *respectively.*

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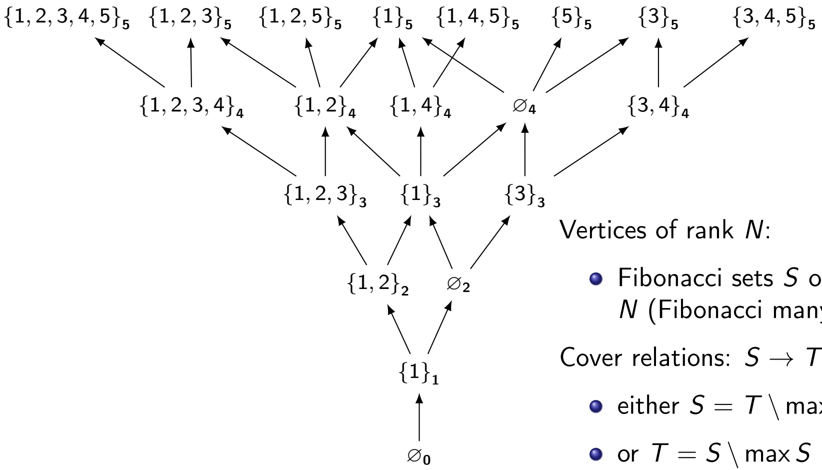
Lemma

The number Fibonacci set S of rank N is exactly the Fibonacci number F_N*, with the convention that* $F_0 = F_1 = 1$*.*

Young-Fibonacci lattice $\mathbb{YF}$ (Stanley 1988)



Vertices of rank N:

- Fibonacci sets S of rank N (Fibonacci many)

Cover relations: $S \to T$

- either $S = T \setminus \max T$
- or $T = S \setminus \max S$

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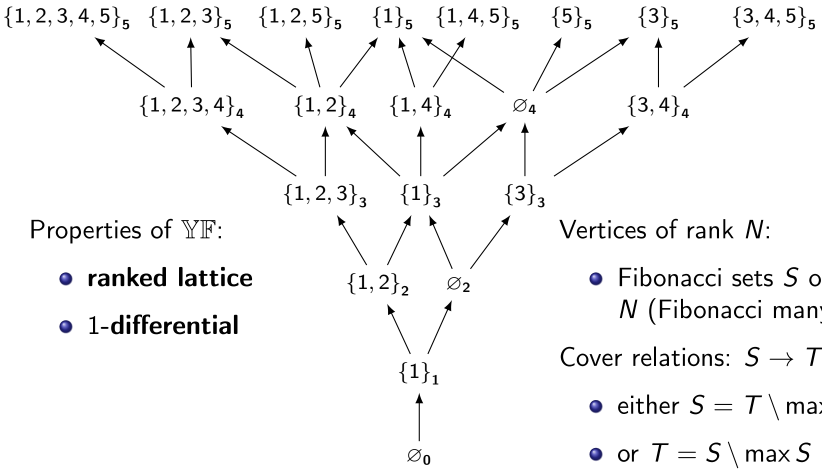

Properties of $\mathbb{YF}$:

- **ranked lattice**
- 1-**differential**

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Differential posets

Definition

The **Up** and **Down** operators for a locally finite poset $\mathbb{P}$ are

$$U(v) := \sum_{w \text{ covers } v} w \qquad D(v) := \sum_{v \text{ covers } w} w$$

for any $v \in \mathbb{P}$.

Definition (Stanley 1988)

A **differential poset** is a locally finite, ranked poset with a unique minimum element $\varnothing$ whose **Up** and **Down** operators satisfy

$$DU - UD = \mathrm{Id}\,.$$

Enumerative properties of differential posets

In analogy to $Uf = x\,f$ and $Df = \partial_x f$ and the identity:

$$D^N U^N(1) = \partial_x^N x^N = N!$$

Theorem (Stanley 1988)

If $\mathbb{P}$ is differential with minimum element $\varnothing$ then

$$D^N U^N(\varnothing) = N!\,\varnothing$$

Otherwise said,

$$\sum_{\mathrm{rank}(w) = N} \dim^2(w) = N!$$

where $\dim(w)$ *is the number of* **saturated chains** *from $\varnothing$ to $w \in \mathbb{P}$.*

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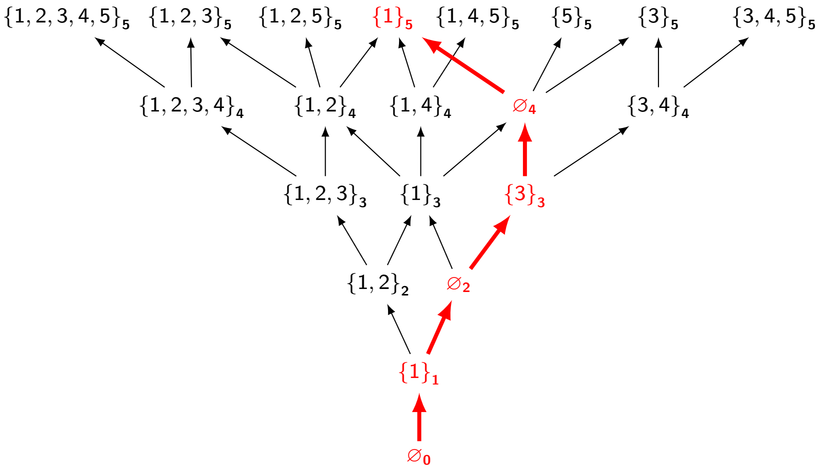
where $\dim(w)$ *is the number of* saturated chains *from $\varnothing$ to $w \in \mathbb{P}$.* **Indicates possibility of an RS-correspondence.**

Saturated chain from $\varnothing_0$ to $S = \{1\}_5$ in $\mathbb{YF}$-lattice



From Young-Fibonacci to Okada algebra

- Stanley 1975-1977-1988: Notion of differential poset
  Example: **Young, Young-Fibonacci,...**
- Fomin 1994,1995: Any differential poset supports an analogue of the **Robinson-Schensted correspondance** that is a bijection between permutations and pairs of saturated chains
- Roby 1991, Killpatrick 2005, Nzeutchap 2009: Various notions of Young-Fibonacci **tableaux**
- Okada 1994: Bratelli diagram of the tower of Okada algebras (analogues of the symmetric groups $\mathfrak{S}_n$)

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Today: Diagrammatic understanding of Okada algebras and $\mathbb{YF}$ RS-correspondence

Okada algebras

Definition (Okada-1994)

Let $N \in \mathbb{N}$ and let $X = (x_1, \ldots, x_{N-1})$ and $Y = (y_1, \ldots, y_{N-2})$ be parameters in a field $\mathbb{K}$.

The **Okada algebra** $\mathbf{O}_N(X, Y)$ is the algebra generated by $\{\mathrm{E}_i \mid i = 1, \ldots, N-1\}$ subject to the relations

$$\begin{aligned} \mathrm{E}_i^2 &= x_i \mathrm{E}_i && 1 \leq i \leq N-1, \\ \mathrm{E}_i \mathrm{E}_j &= \mathrm{E}_j \mathrm{E}_i && |i-j| \geq 2, \\ \mathrm{E}_{i+1} \mathrm{E}_i \mathrm{E}_{i+1} &= y_i \mathrm{E}_{i+1} && 1 \leq i \leq N-2. \end{aligned}$$

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Remark

After specializing the X's and Y's we get the **Temperely-Lieb algebra** $\mathbf{TL_N(\delta)}$ *and the* **Blob algebra** $\mathbf{Blob_N(\gamma, \delta)}$ *as quotients of* $\mathrm{O}_N(X, Y)$

Okada's theorems

Theorem (Okada 1994)

If X and Y are **generic** *then*

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$$\left\{ \begin{array}{c} \textit{Fibonacci sets} \\ S \textit{ of rank } N \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{iso-classes of simple} \\ \mathrm{O}_N(X, Y) \textit{ -modules} \end{array} \right\}$$

and $\textbf{dim}\, \mathbf{V}_{\boldsymbol{S}} = \#$ *saturated chains from* $\varnothing_0$ *to* S *in* $\mathbb{YF}$

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- *The* $\mathbb{YF}$*-lattice governs how simple* $\mathrm{O}_N(X, Y)$*-modules* *restrict and decompose* *as* $\mathrm{O}_{N-1}(X, Y)$*-modules, i.e.*

$$\mathrm{Res}^N_{N-1}\, \mathrm{V}_S \cong \bigoplus_{S \text{ covers } T} \mathrm{V}_T$$

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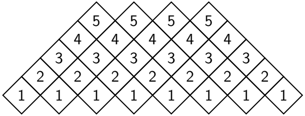
- *Analogues of symmetric functions, Kostka numbers, induction product and Littlewood-Richardson rule, …*

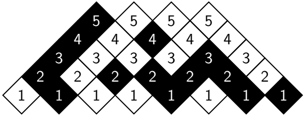
Vizualizing monomials in $O_N(X, Y)$

Diamond grid of height $N - 1$



Vizualizing monomials in $O_N(X, Y)$

Filling of diamond grid of height $N - 1$



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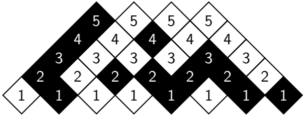

$$\underbrace{\mathbf{E_2E_1 \cdot E_3 \cdot E_4E_2 \cdot E_5E_2E_1 \cdot E_4E_2 \cdot E_3E_2E_1 \cdot E_1}}_{\text{monomial in } O_N(X, Y)}$$

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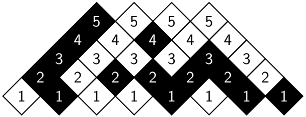

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Replace:  $\to$  and  $\to$ 

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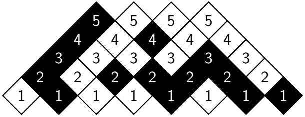

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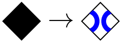
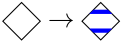
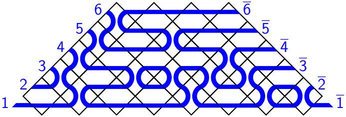
Replace:  and 



Fully packed loop configuration (FPLC)

Vizualizing monomials in $O_N(X, Y)$

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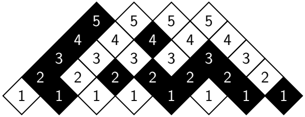

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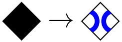
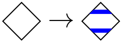
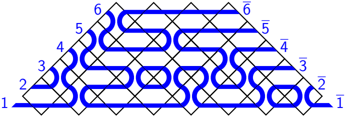
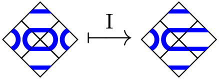
Replace:  and 

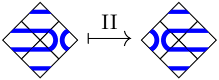


Fully packed loop configuration (FPLC)

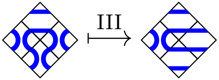
Okada relations correspond to local moves: (Setting X's, Y's $= 1$)



$E_i^2 = x_i E_i$



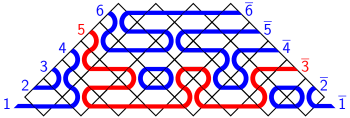
$E_i E_j = E_j E_i$



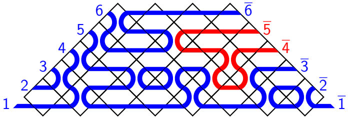
$E_i E_{i-1} E_i = y_{i-1} E_i$

Height of arcs

The lowest level to which an arc/loop descends is called its **height**:



height $= 1$



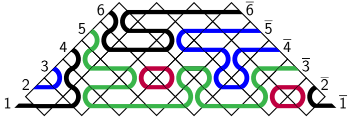
height $= 2$

Observation

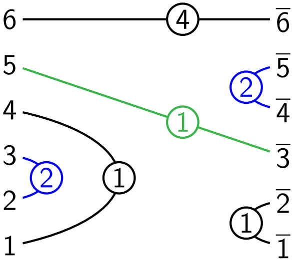
The Height of an arc/loop is invariant under local transformation (types I, II, III) of the fully packed loop configuration (FPLC).

Okada arc-diagrams from FPLCs

Given FPLC



Okada arc-diagram



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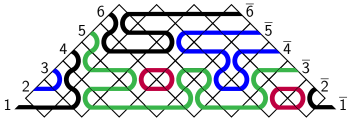

- label each arc by its Height

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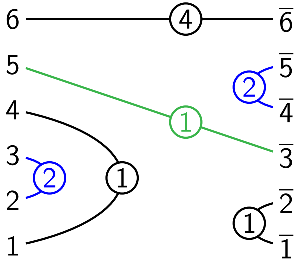

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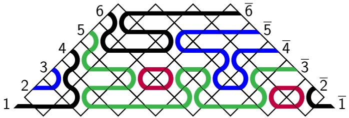
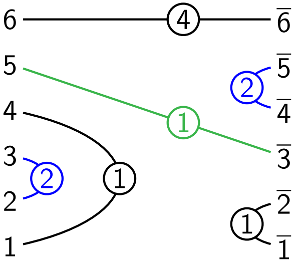


- label each arc by its Height
- remove all loops

Okada arc-diagram



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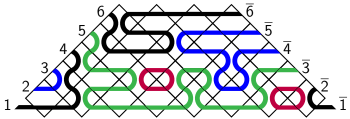
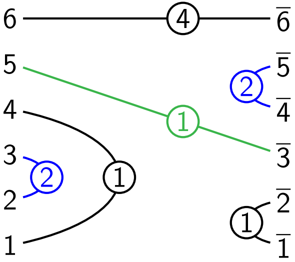


- label each arc by its Height
- remove all loops
- take isotopy class of arcs

Okada arc-diagram



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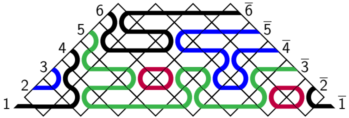
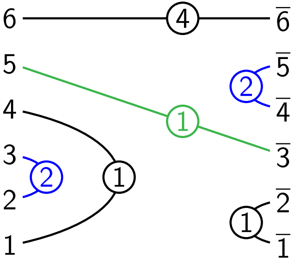


- $1 \leq \text{height} \leq \min(a, b)$

Okada arc-diagram



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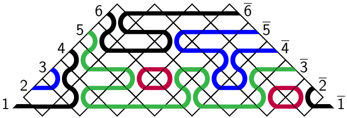

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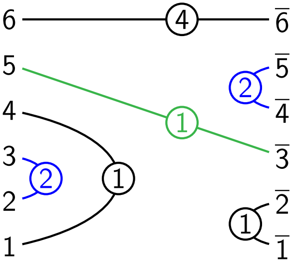
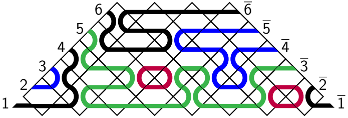
- $\text{height} = \min(a, b) \bmod 2$

Okada arc-diagram



Okada arc-diagrams from FPLCs

Given FPLC



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- heights increase with nesting order

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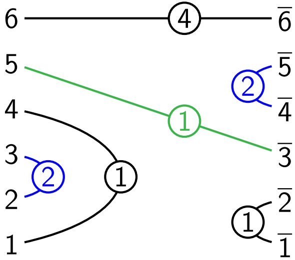

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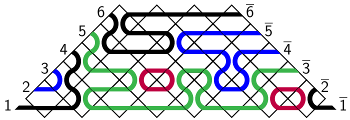

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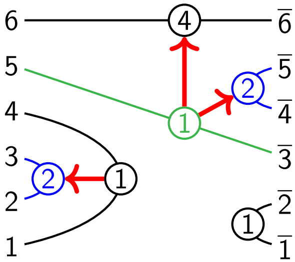


Okada arc-diagrams: Formal definition

Definition

A rank N **Okada arc-diagram** is an isotopy class of a perfect matching joining vertices $\{1, \ldots, N\}$ and $\{\overline{1}, \ldots, \overline{N}\}$ on the left and right boundaries of a rectangle by **non-crossing arcs** $\boldsymbol{a \longmapsto b}$ with $a, b \in \{1, \ldots, N\} \cup \{\overline{1}, \ldots, \overline{N}\}$, each having an **$h$-label** satisfying:

- $1 \leq \mathrm{h}(a \longmapsto b) \leq \min(a, b)$ for each arc $a \longmapsto b$
- $\mathrm{h}(a \longmapsto b) = \min(a, b) \bmod 2$ for each arc $a \longmapsto b$
- $\mathrm{h}(a \longmapsto b) < \mathrm{h}(c \longmapsto d)$ whenever $c \longmapsto d$ is **nested in** $a \longmapsto b$

Definition

An arc $c \longmapsto d$ is **nested in** an arc $a \longmapsto b$ if $a < c < d < b$ where we employ the total order $\{1 < 2 < \cdots < N < \overline{N} < \cdots \overline{2} < \overline{1}\}$.

Examples of Okada arc-diagrams ($N = 8$)

$\mathbb{I}_N :=$

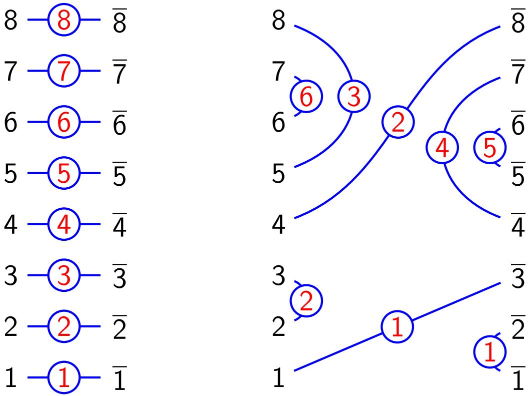


Figure: Two Okada arc-diagrams

Examples of Okada arc-diagrams ($N = 8$)

$\mathbb{I}_N :=$

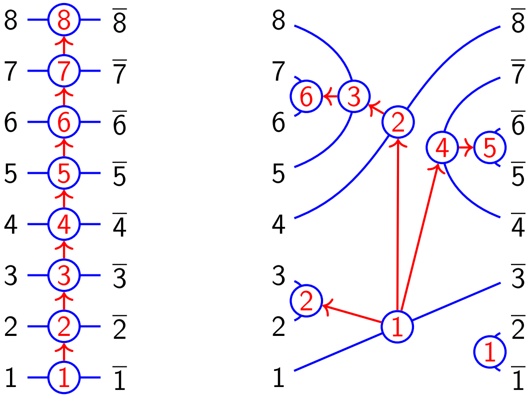
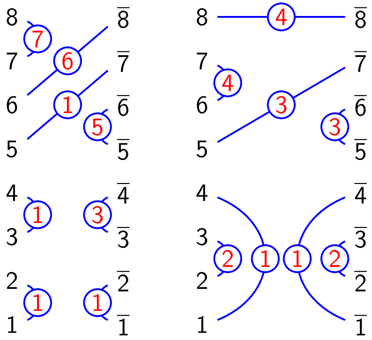
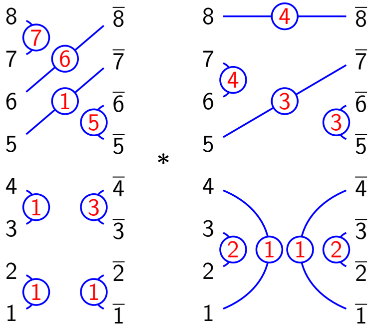


Figure: Okada arc-diagrams with nesting relations

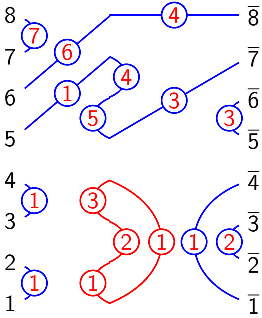
Composing Okada arc-diagrams



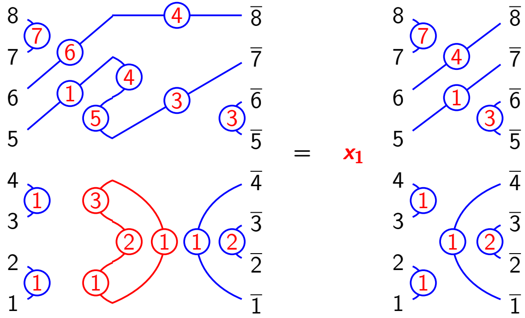
Composing Okada arc-diagrams



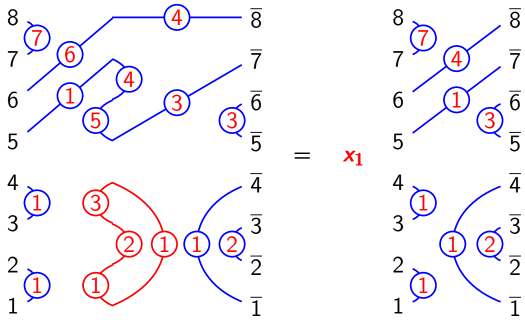
Composition of Okada arc-diagrams: Example



Composition of Okada arc-diagrams: Example



Composition of Okada arc-diagrams: Example



Lemma

Up to the creation of a scalar, the composition $\mathcal{D} * \mathcal{D}'$ of two rank N Okada arc-diagrams is a rank N Okada arc-diagram.

Diagrammatic model for the Okada algebras

Proposition (Hivert-S.)

The $\mathbb{K}$-vector space $\mathcal{A}_N(X, Y)$ spanned by all rank N Okada arc-diagrams becomes an associative algebra, with unit $\mathbb{I}_N$, under the composition product $*$.

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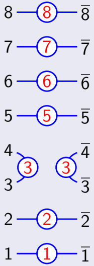
The $\mathbb{K}$-vector space $\mathcal{A}_N(X, Y)$ spanned by all rank N Okada arc-diagrams becomes an associative algebra, with unit $\mathbb{I}_N$, under the composition product $*$.

Theorem (Hivert-S.)

The map $\mathrm{E}_k \mapsto \mathcal{E}_k$ for $1 \leq k \leq N-1$ extends to an algebra isomorphism φ from $\mathrm{O}_N(X, Y)$ to $\mathcal{A}_N(X, Y)$ where $\mathcal{E}_k$ is the Okada arc-diagram containing the labeled arcs

- $\mathrm{h}(\ell \longmapsto \overline{\ell}) = \ell$ for $\ell \neq k, k+1$
- $\mathrm{h}(k \longmapsto k+1) = k$
- $\mathrm{h}(\overline{k} \longmapsto \overline{k+1}) = k$

$\mathcal{E}_3 =$



Comments of proof

Remark

East to check that $\mathcal{E}_1, \ldots, \mathcal{E}_{N-1}$ satisfy the defining relations of the Okada algebra $\mathrm{O}_N(X, Y)$ (i.e. φ is well-defined).

Comments of proof

Remark

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Remark

With a bit of work one shows that $\mathcal{E}_1, \ldots, \mathcal{E}_{N-1}$ generate the algebra $\mathcal{A}_N(X, Y)$ of rank N Okada arc-diagrams (i.e. φ is onto).

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$\dim \mathcal{A}_N(X, Y) = N!$ *(so φ is an isomorphism).*

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Proposition

$\dim \mathcal{A}_N(X, Y) = N!$ *(so φ is an isomorphism).* **Proof uses a diagram version of $\mathbb{YF}$ RS-correspondence.**

Diagram version of the $\mathbb{YF}$ RS-correspondence

Definition

An arc $a \longmapsto b$ of a rank N Okada arc-diagram $\mathcal{D}$ is **propagating** if $a \in \{1, \ldots, N\}$ and $b \in \{\overline{1}, \ldots, \overline{N}\}$ or the reverse. Let $\mathbf{PLab}(\mathcal{D})$ denote the set of **h-labels** of all propagating arcs of $\mathcal{D}$.

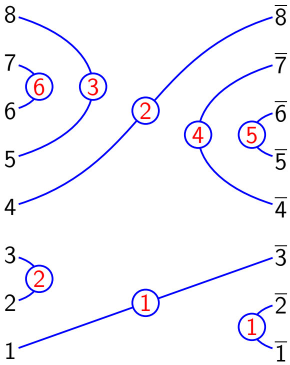
Observation

$\mathbf{PLab}(\mathcal{D})$ is a **rank N fibonacci set** *for any Okada arc-diagram $\mathcal{D}$ of rank N.*

Idea: Given a rank N fibonacci set $\boldsymbol{S}$ we construct a **bijection**

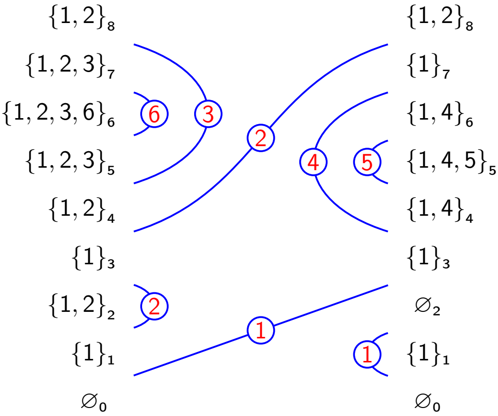
$$\left\{ \begin{array}{c} \text{rank } N \text{ Okada} \\ \text{arc-diagrams } \mathcal{D}, \\ \mathrm{PLab}(\mathcal{D}) = S \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} \text{pairs of saturated chains in } \mathbb{YF} \\ \text{sharing } S \text{ as common endpoint} \end{array} \right\}$$

From Okada arc-diagrams to pairs of saturated chains: 1st



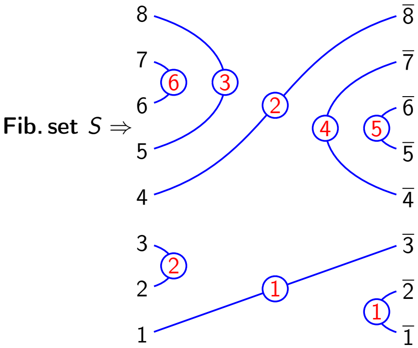
Okada arc-diagram

From Okada arc-diagrams to pairs of saturated chains: 2nd



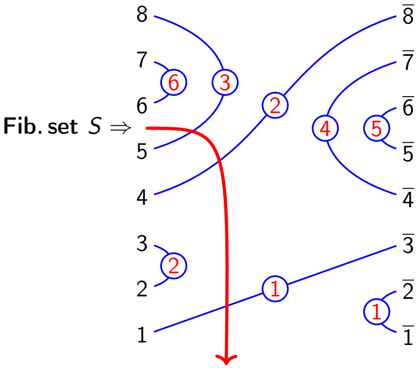
Saturated chains

From Okada arc-diagrams to pairs of saturated chains: 3rd



Draw curve downward

From Okada arc-diagrams to pairs of saturated chains: 3rd



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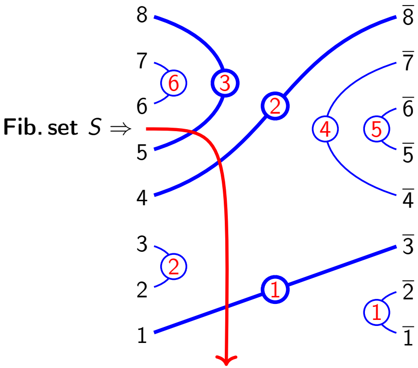
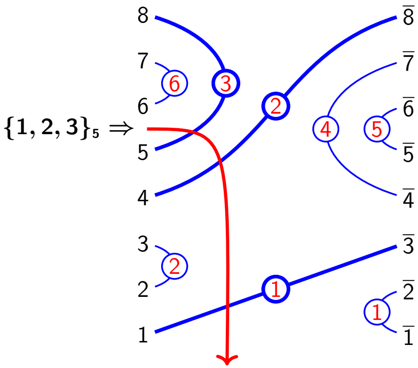


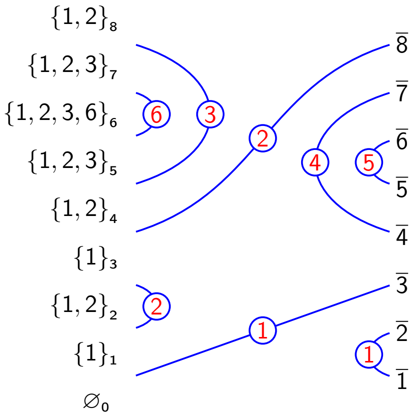
h-labels $= \{1, 2, 3\}$

From Okada arc-diagrams to pairs of saturated chains: 3rd



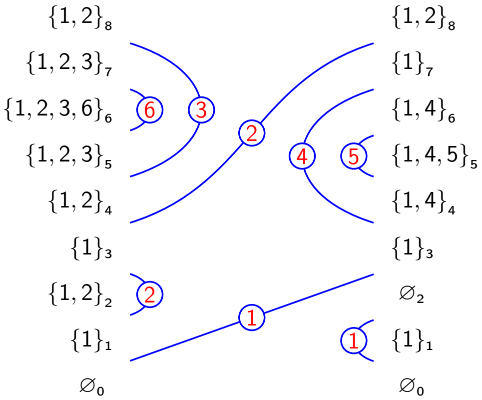
$$\text{h-labels} = \{1,2,3\}$$

From Okada arc-diagrams to pairs of saturated chains: 4th



$\Uparrow$ **1st saturated chain**

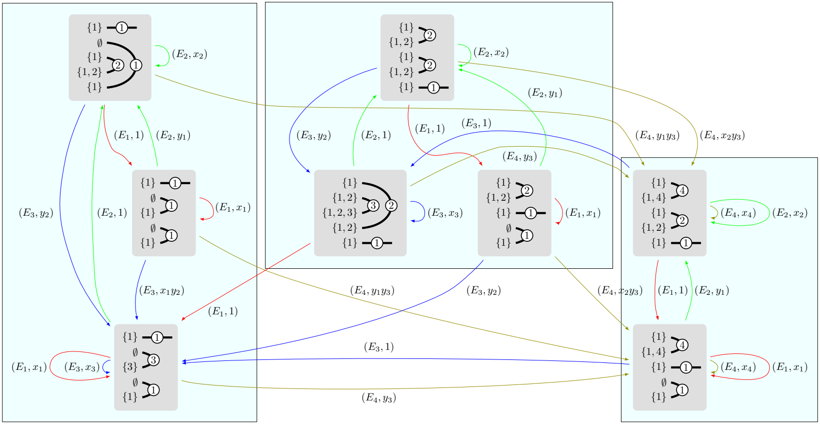
From Okada arc-diagrams to pairs of saturated chains: 5th



Saturated chains ending at $\{1,2\}_8$

Cell modules V_S: Example $N = 5$ and $S = \{1\}_5$

Basis of V_S: rank N Okada **half** arc-diagrams $\mathcal{H}$ with $\mathrm{PLab}(\mathcal{H}) = S$



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Riesen Dank und Glück auf!