$qt\mathrm{RSK}^*$: A probabilistic dual RSK correspondence for Macdonald polynomials

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Our aim is a tableaux theoretic proof of

$$\sum_{\lambda} P_{\lambda}(\mathbf{x}; q, t) P_{\lambda'}(\mathbf{y}; t, q) = \prod_{i,j \geq 1} (1 + x_i y_j).$$

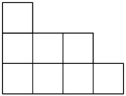
→ The Schur case – classical RSK*

→ Macdonald polynomials

→ A probabilistic dual RSK correspondence: $qt\mathrm{RSK}^*$

→ A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a weakly decreasing sequence of positive integers. We identify partitions with their Young diagrams.

$$\lambda = (4, 3, 1) \qquad \leftrightarrow$$



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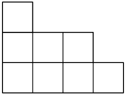
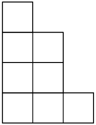


→ The conjugate $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_l)$ of a partition is obtained by reflecting the Young diagram of λ along the line $x = y$.

$$\lambda' = (3, 2, 2, 1) \qquad \leftrightarrow$$



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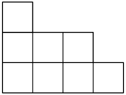

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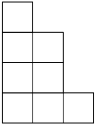

→ For $\mu \subseteq \lambda$ the skew diagram λ/μ is obtained by deleting all boxes of μ from the Young diagram of λ.

A semistandard Young tableau (SSYT) is a filling of the cells of λ with positive integers such that

- → rows are weakly increasing,
- → and columns are strictly increasing.

4			
2	3	4	
1	2	2	3

The weight of an SSYT T is $\mathbf{x}^T = \prod_i x_i^{\#i\text{'s in } T}$. In the above examples, we have $\mathbf{x}^T = x_1 x_2^3 x_3^2 x_4^2$.

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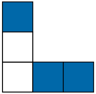
Let λ be a partition and $\mathbf{x} = (x_1, \ldots, x_n)$. The Schur polynomial $s_\lambda(\mathbf{x})$ is the multivariate generating function of all SSYT of shape λ and with entries at most n

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}_\lambda(n)} \mathbf{x}^T.$$

We say that the diagram λ/μ is a

- horizontal strip ($\mu \prec \lambda$) if it contains at most one cell in each column,
- vertical strip ($\mu \prec' \lambda$) if it contains at most one cell in each row.



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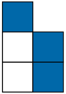

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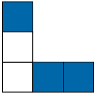

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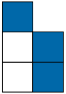

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We define the up operator U_x and the dual down operator D^*_y as

$$U_x\lambda = \sum_{\nu \succ \lambda} x^{|\nu/\lambda|}\nu, \qquad D^*_y\lambda = \sum_{\mu \prec' \lambda} y^{|\lambda/\mu|}\mu.$$

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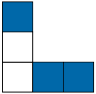

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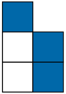

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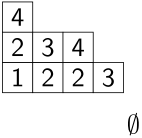
The Schur polynomials can be rewritten as

$$s_\lambda(x_1, \ldots, x_n) = \langle U_{x_n} \cdots U_{x_1} \emptyset, \lambda \rangle ,$$



where $\langle \lambda, \rho \rangle := \delta_{\lambda,\rho}$ for all partitions λ, ρ.

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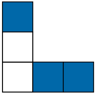
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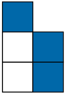
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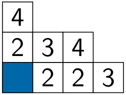

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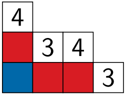

$U_{x_2} U_{x_1} \emptyset$

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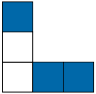

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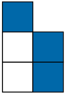

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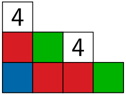

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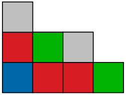

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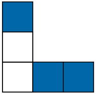

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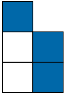

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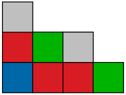

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2. *The commutation relation is equivalent to the skew version of the dual Cauchy identity*

$$\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (1 + x_i y_j) \sum_{\mu} s_{\lambda/\mu}(\mathbf{x}) s_{\rho'/\mu'}(\mathbf{y}) = \sum_{\nu} s_{\nu/\rho}(\mathbf{x}) s_{\nu'/\lambda'}(\mathbf{y}).$$

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→ Ad 2) This follows from a simple linear algebra argument.

→ Ad 1) We give a combinatorial proof via dual local growth rules. Using dual growth diagrams, we obtain a combinatorial proof of the dual Cauchy identity.

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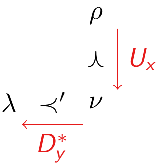

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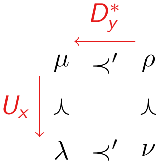

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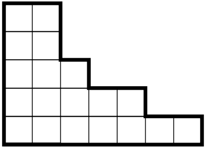
$$\begin{array}{ccc} \mu & \prec' & \rho \\ \curlywedge & \searrow & \curlywedge \\ \lambda & \prec' & \nu \end{array}$$

It suffices to find a bijection

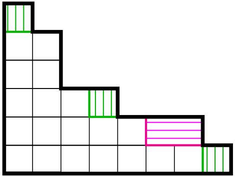
$$F_{\lambda,\rho,k} : \mathcal{D}^{k-1}(\lambda, \rho) \cup \mathcal{D}^k(\lambda, \rho) \to \mathcal{U}^k(\lambda, \rho),$$

for all λ, ρ and non-negative k. We call a family of such bijections a set of dual local growth rules.

An inner corner (resp., outer corner) of λ is a cell c which we can remove (resp., add) and still obtain a partition.



$\lambda \cap \rho$



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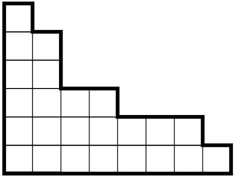

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$\lambda = (7, 7, 3, 2, 2)$ and $\rho = (8, 5, 4, 2, 2, 1)$.

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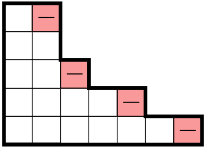

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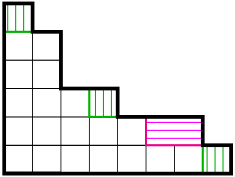

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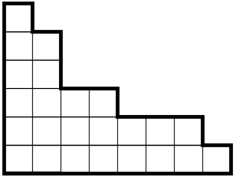

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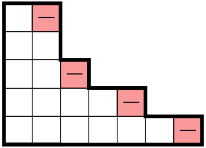

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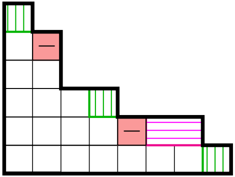

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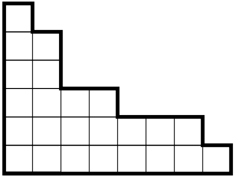

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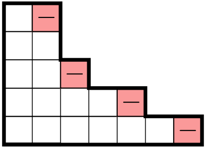

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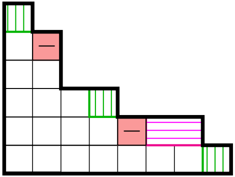

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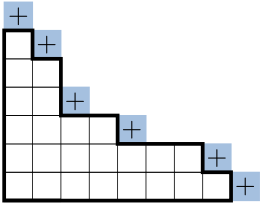

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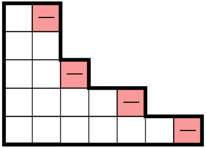
- → An outer corner of $\lambda \cup \rho$ is called addable if the obtained partition ν satisfies $\lambda \prec' \nu \succ \rho$.



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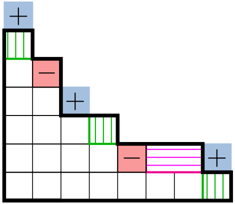

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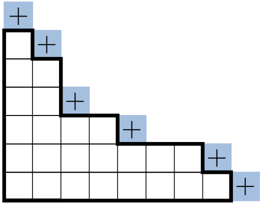
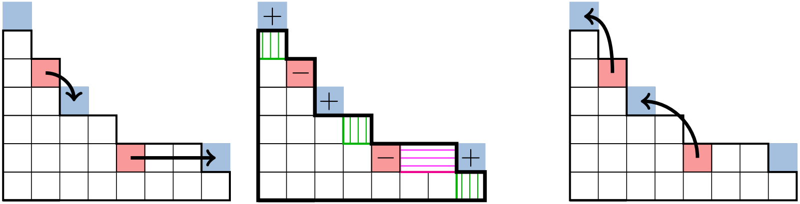


$\lambda \cup \rho$

$\lambda = (7, 7, 3, 2, 2)$ and $\rho = (8, 5, 4, 2, 2, 1)$.

An inner corner (resp., outer corner) of λ is a cell c which we can remove (resp., add) and still obtain a partition.

- → An inner corner of $\lambda \cap \rho$ is called removeable if the obtained partition μ satisfies $\lambda \succ \mu \prec' \rho$.
- → An outer corner of $\lambda \cup \rho$ is called addable if the obtained partition ν satisfies $\lambda \prec' \nu \succ \rho$.



$F^{*\mathrm{col}}_{\lambda,\rho,1}$ $\qquad$ $\lambda \cup \rho$ $\qquad$ $F^{*\mathrm{row}}_{\lambda,\rho,1}$

$\lambda = (7,7,3,2,2)$ and $\rho = (8,5,4,2,2,1)$.

The (q,t)-up and dual down operator are defined as

$$U_x(q,t)\lambda = \sum_{\nu \succ \lambda} x^{|\nu/\lambda|}\psi_{\nu/\lambda}(q,t)\nu, \qquad D_y^*(q,t)\lambda = \sum_{\mu \prec' \lambda} y^{|\lambda/\mu|}\varphi^*_{\lambda/\mu}(q,t)\mu,$$

where $\psi_{\nu/\lambda}, \varphi^*_{\lambda/\mu}$ are products over terms of the form $\dfrac{1-q^a t^b}{1-q^c t^d}$.

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Theorem (Macdonald)

The Macdonald polynomials P_λ can be expressed via

$$P_\lambda(x_1,\dots,x_n;q,t) = \left\langle U_{x_n}(q,t)\cdots U_{x_1}(q,t)\emptyset, \lambda\right\rangle,$$
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The Macdonald polynomials $P_\lambda(\mathbf{x};q,t)$ specialise to

- → Schur polynomials for $q = t$,
- → Hall–Littlewood polynomials for $q = 0$,
- → q-Whittaker polynomials for $t = 0$,
- → Jack polynomials for $q = t^\alpha$ and taking the limit $t \to 1$.

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1. *For two sets of variables* $\mathbf{x} = (x_1, \dots, x_n)$ *and* $\mathbf{y} = (y_1, \dots, y_m)$ *holds*

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$$\sum_{\mu \in \mathcal{D}^k(\lambda,\rho) \cup \mathcal{D}^{k-1}(\lambda,\rho)} \psi_{\lambda/\mu}(q,t) \varphi^*_{\rho/\mu}(q,t) = \sum_{\nu \in \mathcal{U}^k(\lambda,\rho)} \psi_{\nu/\rho}(q,t) \varphi^*_{\nu/\lambda}(q,t),$$

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for all partitions λ,ρ and non-negative integers k where $\binom{\mathbf{X}}{k}$ the set of k-subsets of $\mathbf{X}$ and

$$\omega_{\lambda,\rho}(\mu)=\psi_{\lambda/\mu}(q,t)\varphi^*_{\rho/\mu}(q,t),$$
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Let A be an algebra and X, Y be two sets equipped with weight functions

$$\omega : X \to A, \qquad \overline{\omega} : Y \to A.$$

A probabilistic bijection from (X, ω) to $(Y, \overline{\omega})$ is a pair of A-valued "probability distributions" $\mathcal{P}(x \to y), \overline{\mathcal{P}}(x \leftarrow y)$ such that

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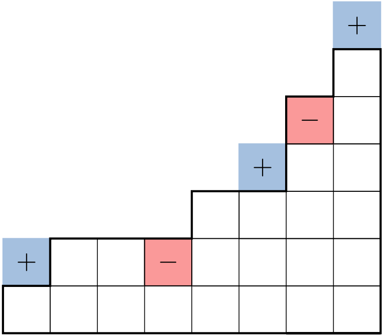
Let λ, ρ be partitions and regard the removable and addable corners of $\lambda \cap \rho$ and $\lambda \cup \rho$ respectively. Denote by d the number of removable corners.

→ We regard the Young diagram right-justified (Quebecois notation).



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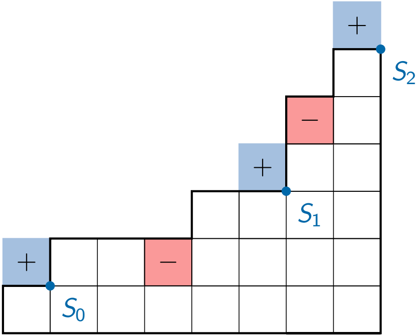

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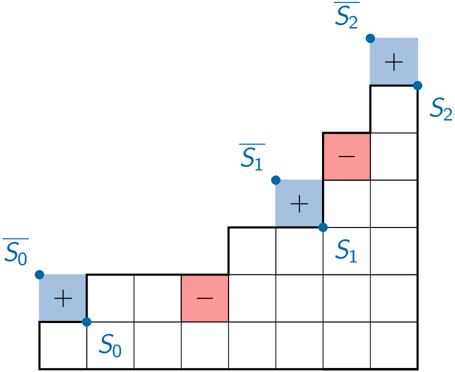

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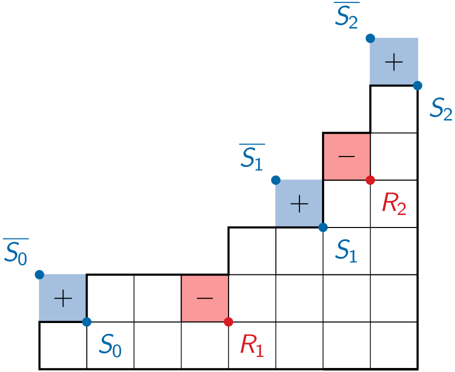

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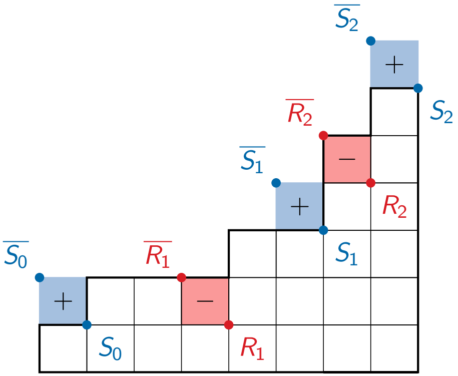

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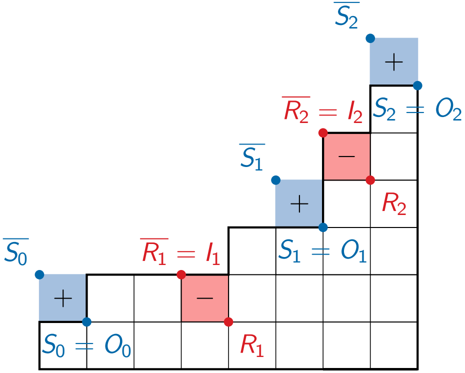

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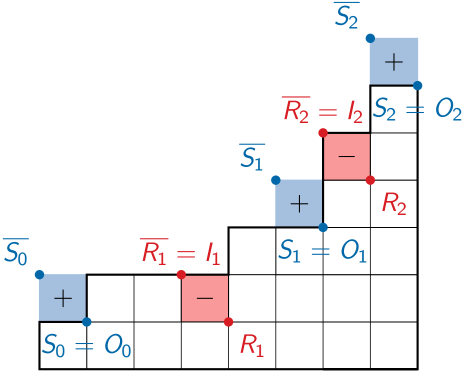

We identify a point (x, y) with the monomial $q^x t^y$.

Definition (Frieden-SA)

Let λ, ρ be partitions, d the number of removable corners of $\lambda \cap \rho$. For $\mathbf{R} \subseteq [d]$ and $\mathbf{S} \subseteq [0, d]$, we define the forward probability

$$\mathcal{P}_{\lambda,\rho}(\mathbf{R} \to \mathbf{S}) = \prod_{s \in \mathbf{S}} \frac{\prod_{i \in [d] \setminus \mathbf{R}} (S_s - I_i)}{\prod_{j \in [0,d] \setminus \mathbf{S}} (S_s - O_j)} \prod_{r \in \mathbf{R}} \frac{\prod_{j \in [0,d] \setminus \mathbf{S}} (R_r - O_j)}{\prod_{i \in [d] \setminus \mathbf{R}} (R_r - I_i)},$$

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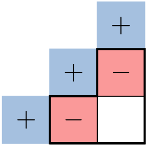

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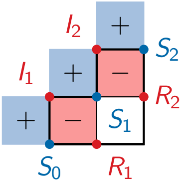

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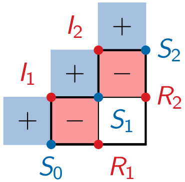

$\mathcal{P}_{\lambda,\rho}(\{2\} \to \{0,1\})$

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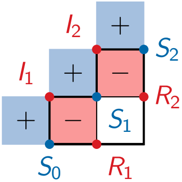

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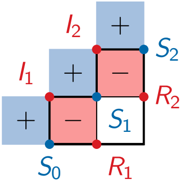

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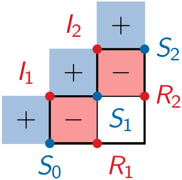
$$= \frac{(1-t)(qt-t)(q^2t - q^2t^2)}{(1-q^2t^2)(qt-q^2t^2)(q^2t-t)}$$

$$\mathcal{P}_{\lambda,\rho}(\mathbf{R} \to \mathbf{S}) = \prod_{s\in\mathbf{S}} \frac{\prod_{i\in[d]\setminus\mathbf{R}} (S_s - I_i)}{\prod_{j\in[0,d]\setminus\mathbf{S}} (S_s - O_j)} \prod_{r\in\mathbf{R}} \frac{\prod_{j\in[0,d]\setminus\mathbf{S}} (R_r - O_j)}{\prod_{i\in[d]\setminus\mathbf{R}} (R_r - I_i)}.$$

→ Let $\lambda = \rho = (2,1)$. We have $d = 2$.

→ Remember $O_i = S_i$.



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$$= \frac{(1-t)(qt-t)(q^2t - q^2t^2)}{(1-q^2t^2)(qt-q^2t^2)(q^2t-t)} = q\frac{(1-q)(1-t)^2}{(1-q^2)(1-qt)(1-q^2t^2)}.$$

Theorem (Frieden–SA)

1. *The probabilities $\mathcal{P}_{\lambda,\rho}$ and $\overline{\mathcal{P}}_{\lambda,\rho}$ form a probabilistic bijection.*
2. *The probabilistic insertion algorithm, $qt\mathrm{RSK}^*$, building on these growth rules is a probabilistic bijection which allows to proof the dual Cauchy identity for Macdonald polynomials.*

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Lemma (Frieden–SA)

The probabilities satisfy

$$\mathcal{P}_{\lambda,\rho}(\mu \to \nu)[q^{-1},t^{-1}] = \mathcal{P}_{\rho',\lambda'}(\mu' \to \nu')[t,q],$$
$$\overline{\mathcal{P}}_{\lambda,\rho}(\mu \leftarrow \nu)[q^{-1},t^{-1}] = \overline{\mathcal{P}}_{\rho',\lambda'}(\mu' \leftarrow \nu')[t,q].$$

Macdonald polynomials

$qt\mathrm{RSK}^*$ (Frieden–SA)

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$t = 0$ / $t \to \infty$

$q = 0$ / $q \to \infty$

q-Whittaker polynomials

row / col q-RSK* (Matveev–Petrov)

Hall–Littlewood polynomials

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Macdonald polynomials
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Schur polynomials
row / col RSK*

Macdonald polynomials
$qt\mathrm{RSK}^*$ (Frieden–SA)

$t = 0 \ / \ t \to \infty$ $q = t^\alpha, t \to 1$ $q = 0 \ / \ q \to \infty$

q-Whittaker polynomials
row / col q-RSK* (Matveev–Petrov)

Jack polynomials

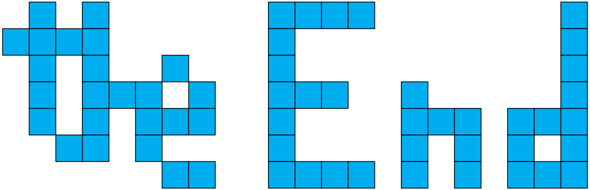
Hall–Littlewood polynomials
row / col t-RSK*

$q = 0 \ / \ q \to \infty$ $t = 0 \ / \ t \to \infty$

Schur polynomials
row / col RSK*

Theorem (Frieden–SA)

Restricting to matrices with at most one nonzero entry per column, the P-tableaux distribution in the Jack limit ($q = t^\alpha$ and $t \to 1$) is invariant under interchanging two columns.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