The Erdős–Ginzburg–Ziv Problem in Discrete Geometry

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Introduction

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We are looking for m points (among the given s points) such that $(p_1 + \dots + p_m)/m \in \mathbb{Z}^n$, i.e. such that all coordinates of the sum $p_1 + \dots + p_m$ are divisible by m.

Let n and m be given positive integers.

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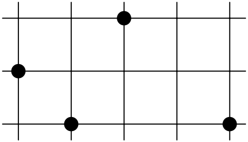

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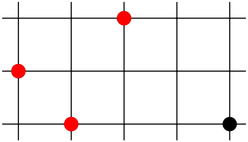

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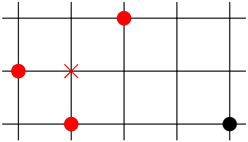

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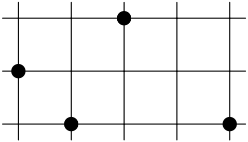

Among these four points in $\mathbb{Z}^2$ it is possible to find $m = 3$ points whose centroid is also a lattice point in $\mathbb{Z}^2$.

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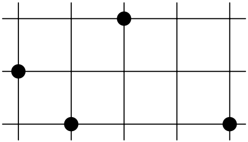

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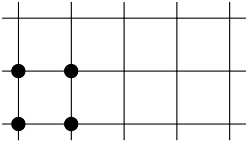

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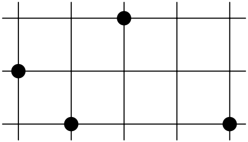

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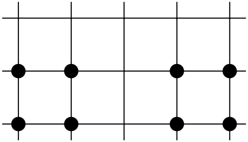

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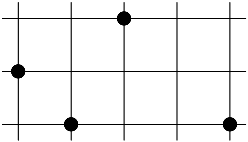

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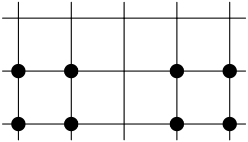

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For $n = 2$ and $m = 3$, one needs $s = 9$ points.

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- $n = 1$: $\mathfrak{s}(\mathbb{Z}_m^1) = 2m - 1$ (Erdős–Ginzburg–Ziv, 1961).
- $n = 2$: $\mathfrak{s}(\mathbb{Z}_m^2) = 4m - 3$ (Reiher, 2007).
- $n = 3$ and m has only certain prime factors : $\mathfrak{s}(\mathbb{Z}_m^3) = 9m - 8$.
- $n = 4$ and m is a power of 3: $\mathfrak{s}(\mathbb{Z}_m^4) = 20m - 19$ (Edel et al., 2007).
- m is a power of 2: $\mathfrak{s}(\mathbb{Z}_m^n) = (m - 1)2^n + 1$ (Harborth, 1973).

Definition

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Theorem (Alon, Dubiner, 1995)

For fixed dimension n, the function $\mathfrak{s}(\mathbb{Z}_m^n)$ grows linearly with m.

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Open Problem

What happens in the opposite regime, when m is fixed and the dimension n is large?

Reducing to $m = p$ prime

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If $m = k\ell$, the following lemma gives an upper bound for $\mathfrak{s}(\mathbb{Z}_m^n)$ in terms of $\mathfrak{s}(\mathbb{Z}_k^n)$ and $\mathfrak{s}(\mathbb{Z}_\ell^n)$.

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$\mathfrak{s}(\mathbb{Z}_{k\ell}^n) \leq \ell \cdot (\mathfrak{s}(\mathbb{Z}_k^n) - 1) + \mathfrak{s}(\mathbb{Z}_\ell^n)$.

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This lemma can be used in all of the previously mentioned upper bounds for $\mathfrak{s}(\mathbb{Z}_m^n)$ in order to reduce to the case where m is a prime.

In other words, for these upper bounds it suffices to study $\mathfrak{s}(\mathbb{Z}_p^n) = \mathfrak{s}(\mathbb{F}_p^n)$ for a prime p.

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In total, we have found $\mathfrak{s}(\mathbb{Z}_k^n)$ groups of ℓ points, so that each group has a centroid in $\mathbb{Z}^n$. Among these $\mathfrak{s}(\mathbb{Z}_k^n)$ centroid points, we can find k centroid points whose centroid is a lattice point in $\mathbb{Z}^n$.

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Considering the corresponding k groups of size ℓ, gives $k\ell$ points in S whose centroid is a lattice point in $\mathbb{Z}^n$.

Proof techniques for small dimension n

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For dimension $n = 1$, we have $\mathfrak{s}(\mathbb{F}_p^1) = 2p - 1$.

This can be proved as an easy application of the Combinatorial Nullstellensatz (due to Alon, 1999), although the original proof of Erdős, Ginzburg, and Ziv was different.

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Theorem (Reiher, 2007)

For dimension $n = 2$, we have $\mathfrak{s}(\mathbb{F}_p^2) = 4p - 3$.

This also be proved with the Combinatorial Nullstellensatz (or the Chevalley–Warning Theorem), but the proof is much more involved.

Theorem (Alon, Dubiner, 1995)

For fixed dimension n, the function $\mathfrak{s}(\mathbb{F}_p^n)$ grows linearly in p.

The proof is by induction on the dimension n. The induction step relies on results from additive combinatorics, as well as arguments using spectral graph theory (i.e. studying eigenvalues of certain matrices).

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None of these approaches seems to work for large dimension n.

Open Problem

How large is $\mathfrak{s}(\mathbb{F}_p^n)$ for a fixed prime $p \geq 3$ and large dimension n?

$\mathfrak{s}(\mathbb{F}_p^n)$ for a fixed prime p

Assume from now on that $p \geq 3$ is a fixed prime and n is large.

Definition

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We can consider the points as (possibly repeated) elements of $\mathbb{Z}_p^n = \mathbb{F}_p^n$. We are then trying to find p elements of $\mathbb{F}_p^n$ whose sum is the zero vector in $\mathbb{F}_p^n$.

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So, up to a factor of at most $p-1$, this problem is equivalent to:

Problem

What is the maximum size of a subset of $\mathbb{F}_p^n$ without p distinct elements summing to zero?

Let $p \geq 3$ be a fixed prime and n large.

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We have seen that $\mathfrak{s}(\mathbb{F}_p^n)$ agrees with the answer to this problem up to a constant factor (if p is fixed).

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An easy lower bound for the problem above is 2^n (consider the subset $\{0,1\}^n \subseteq \mathbb{F}_p^n$).

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The best known lower bound is roughly 2.1398^n for $p \geq 5$ (Edel, 2008) and roughly 2.2202^n for $p = 3$ (Romera-Paredes et al., 2024).

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For $p = 3$, we are asking about the maximum size of a subset $A \subseteq \mathbb{F}_3^n$ not containing distinct vectors $x, y, z \in A$ with $x + y + z = 0$.

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In characteristic 3, having $x + y + z = 0$ is equivalent to $x - 2y + z = 0$, i.e. to x, y, z forming an arithmetic progression.

For $p = 3$, determining $\mathfrak{s}(\mathbb{F}_3^n)$ is equivalent to the following problem.

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What is the maximum size of a subset of $\mathbb{F}_3^n$ without a three-term arithmetic progression
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This is a famous problem, called the *Cap-Set Problem*.

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More generally, studying the maximum size of progression-free subsets of $\mathbb{F}_p^n$ or $\{1, \ldots, N\}$ is a fundamental problem in additive combinatorics.

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Theorem (Szemerédi, 1975)

For any fixed $k \geq 3$, the maximum size of a subset of $\{1, \ldots, N\}$ without a k-term arithmetic progression is of the form $o(N)$.

This was the main result described in Szemerédi's 2012 Abel Prize citation.

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The behavior of the $o(N)$-term is still not understood, despite a lot of attention. For $k = 3$, a revolutionary new upper bound was shown by Kelley and Meka (2023+).

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In 2017, Ellenberg and Gijswijt achieved a breakthrough on this problem, improving *exponentially* upon the trivial upper bound 3^n.

Theorem (Ellenberg, Gijswijt, 2017)

If $A \subseteq \mathbb{F}_3^n$ does not contain a three-term arithmetic progression, then $|A| \leq 2.756^n$.

Ellenberg and Gijswijt actually proved a more general result for $\mathbb{F}_p^n$ for any fixed prime $p \geq 3$:

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The proof gives the the following value for Γ_p:

$$\Gamma_p = \min_{0<t<1} \frac{1 + t + \cdots + t^{p-1}}{t^{(p-1)/3}},$$

This Γ_p satisfies $0.8414p \leq \Gamma_p \leq 0.9184p$.

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However, Γ_p is tight for a certain "multi-colored" generalization of this result (Kleinberg–Sawin–Speyer, Norin, Pebody).

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The result above gives the following corollary concerning $\mathfrak{s}(\mathbb{F}_3^n)$ for $p = 3$.

Corollary

$\mathfrak{s}(\mathbb{F}_3^n) \leq 1 + 2 \cdot 2.756^n$.

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Again, the proof relies on an algebraic technique (using polynomials), but in a very different way than the results on $\mathfrak{s}(\mathbb{F}_p^n)$ for small dimension n.

For fixed $p \geq 3$, determining $\mathfrak{s}(\mathbb{F}_p^n)$ is equivalent to the following problem (up to constant factors depending on p).

Problem

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For $p = 3$, this problem is the Cap-Set Problem, and by the result of Ellenberg–Gijswijt we have an upper bound of 2.756^n.

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It is natural to also try to apply the slice rank polynomial method to this problem for $p \geq 5$.

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It is natural to also try to apply the slice rank polynomial method to this problem for $p \geq 5$.

However, this does not work. The problem is that the natural tensor associated with this problem is not a diagonal tensor, and so one does not have a good lower bound for its slice rank.

The fact that the tensor is not necessarily diagonal is due to the distinctness condition in the problem above.

Let $p \geq 5$ be a fixed prime, and let n be large.

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In other words, we are asking for the maximum size of a subset $A \subseteq \mathbb{F}_p^n$ with no solution for $x_1 + \cdots + x_p = 0$ with $x_1, \ldots, x_p \in A$ being distinct.

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Similar-looking problem

What is the maximum size of a subset of $A \subseteq \mathbb{F}_p^n$ with no solution for $x_1 + \cdots + x_p = 0$ with $x_1, \ldots, x_p \in A$ being not all equal.

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Here, we have $|A| < 4^n$. This is a straightforward application of the slice rank polynomial method.

However, this argument fails for the top problem because we do not have a diagonal tensor anymore.

Results for fixed $p \geq 5$ and large n

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Clearly, $|A| \leq p^n$.

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Clearly, $|A| \leq p^n$. Naslund proved that $|A|$ must be *exponentially smaller* than p^n.

Theorem (Naslund, 2020)

If $A \subseteq \mathbb{F}_p^n$ does not contain p distinct elements summing to zero, then $|A| \leq (2^p - p - 2) \cdot \Gamma_p^n$.

Here, $\Gamma_p < p$ is the constant in the Ellenberg–Gijswijt bound for progression-free subsets of $\mathbb{F}_p^n$. It satisfies $0.8414p \leq \Gamma_p \leq 0.9184p$.

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So this bound is exponentially better than the trivial bound $|A| \leq p^n$, but the base Γ_p is still is linear in p.

Let $p \geq 5$ be a fixed prime and n large.

Theorem (S., 2021)

If $A \subseteq \mathbb{F}_p^n$ does not contain p distinct elements summing to zero, then $|A| \leq C_p \cdot \left(2\sqrt{p}\right)^n$ for some constant C_p only depending on p.

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The same proof gives a "multi-colored" generalization (which is somewhat technical to state).

Interestingly, in this multi-colored generalization the bound is almost tight (there is a lower bound $\sqrt{p}^n$ for even n).

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This is similar to the situation for the Cap-Set Problem, where the bound is known to be tight for the "multi-colored" generalization.

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One barrier to improving the result above or the bound for the Cap-Set Problem is that one needs an approach which does not generalize to the "multi-colored" setting.

Overcoming this barrier, in joint work with Zakharov, improved the bound as follows.

Theorem (S., Zakharov, 2023+)

For every fixed $\varepsilon > 0$, for all primes p and all n, one has a bound $|A| \leq D_{\varepsilon,p} \cdot (C_\varepsilon p^\varepsilon)^n$ for any subset $A \subseteq \mathbb{F}_p^n$ without p distinct elements summing to zero.

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The proof combines the slice rank polynomial method with combinatorial and probabilistic arguments, as well as a higher uniformity version of the Balog–Szemerédi–Gowers Theorem due to Borenstein–Croot (2011).

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Corollary (S., Zakharov, 2023+)

For every fixed $\varepsilon > 0$, for all primes p and all n, one has $\mathfrak{s}(\mathbb{F}_p^n) \leq D_{\varepsilon,p} \cdot (C_\varepsilon p^\varepsilon)^n$.

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In particular, $\mathfrak{s}(\mathbb{F}_p^n) \leq D_p \cdot (C \cdot p^{0.01})^n$ for some absolute constant C.

A similar bound holds when replacing 0.01 by any fixed $\varepsilon > 0$.

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For every fixed $\varepsilon > 0$, for all primes p and all n, one has $\mathfrak{s}(\mathbb{F}_p^n) \leq D_{\varepsilon,p} \cdot (C_\varepsilon p^\varepsilon)^n$.

The best lower bound for $\mathfrak{s}(\mathbb{F}_p^n)$ for fixed p and large n is roughly 2.1398^n (Edel, 2008).

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The best lower bound for $\mathfrak{s}(\mathbb{F}_p^n)$ for fixed p and large n is roughly 2.1398^n (Edel, 2008).

Thus, there is still a significant gap between the upper and lower bounds. In particular, the following question is open.

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Is there a bound of the form $\mathfrak{s}(\mathbb{F}_p^n) \leq C_p \cdot c^n$ for some absolute constant c?

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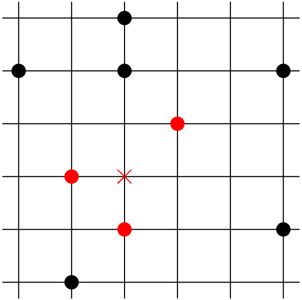
In the opposite parameter regime, where n is fixed and p is large with respect to n, Zakharov (2020+) proved $\mathfrak{s}(\mathbb{F}_p^n) \leq p \cdot 4^n$.

However, his methods do not apply for fixed p and large n.

Thank you very much for your attention!



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