Ian G. Macdonald

Works of Art

Arun Ram
University of Melbourne

SFB

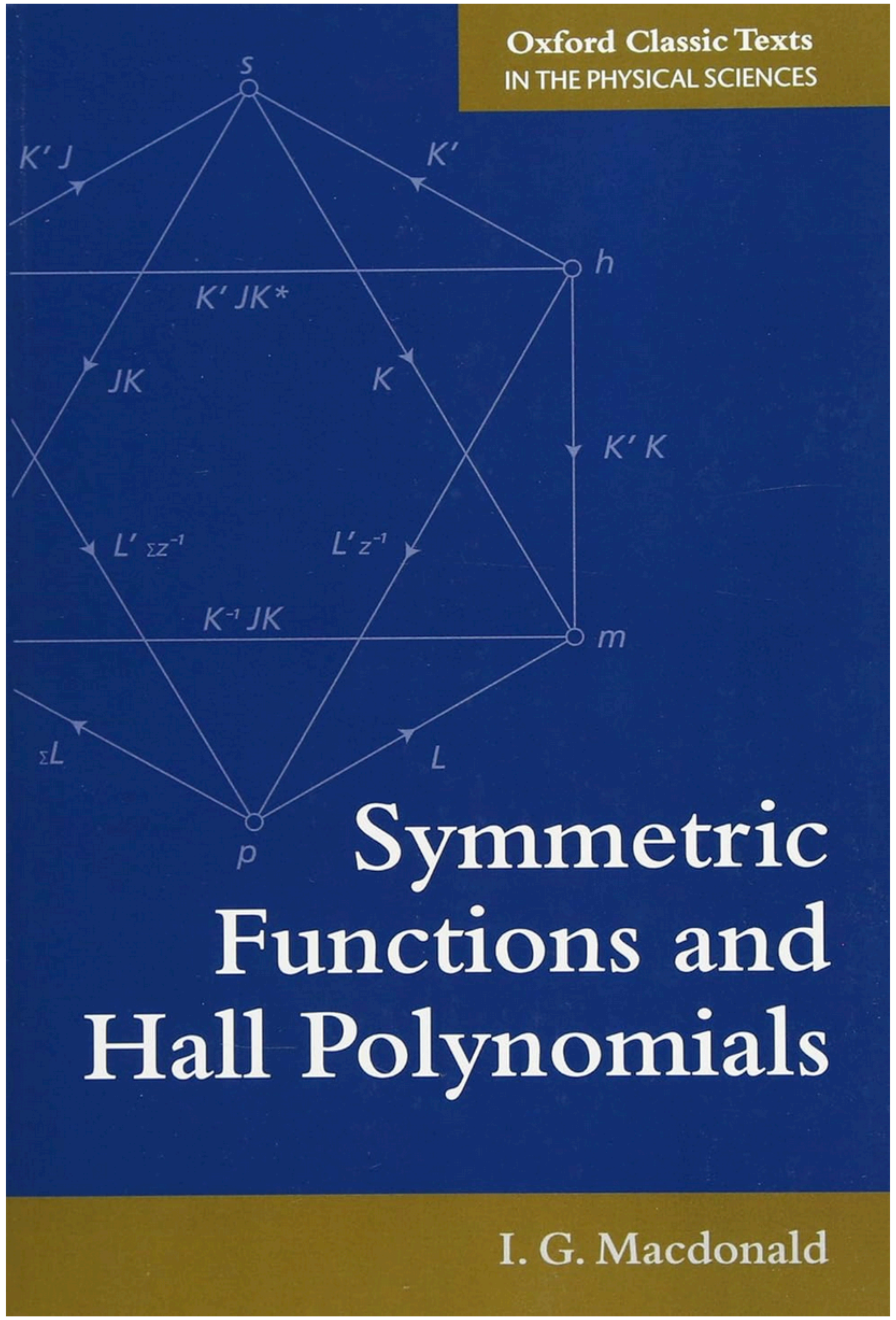
SFB

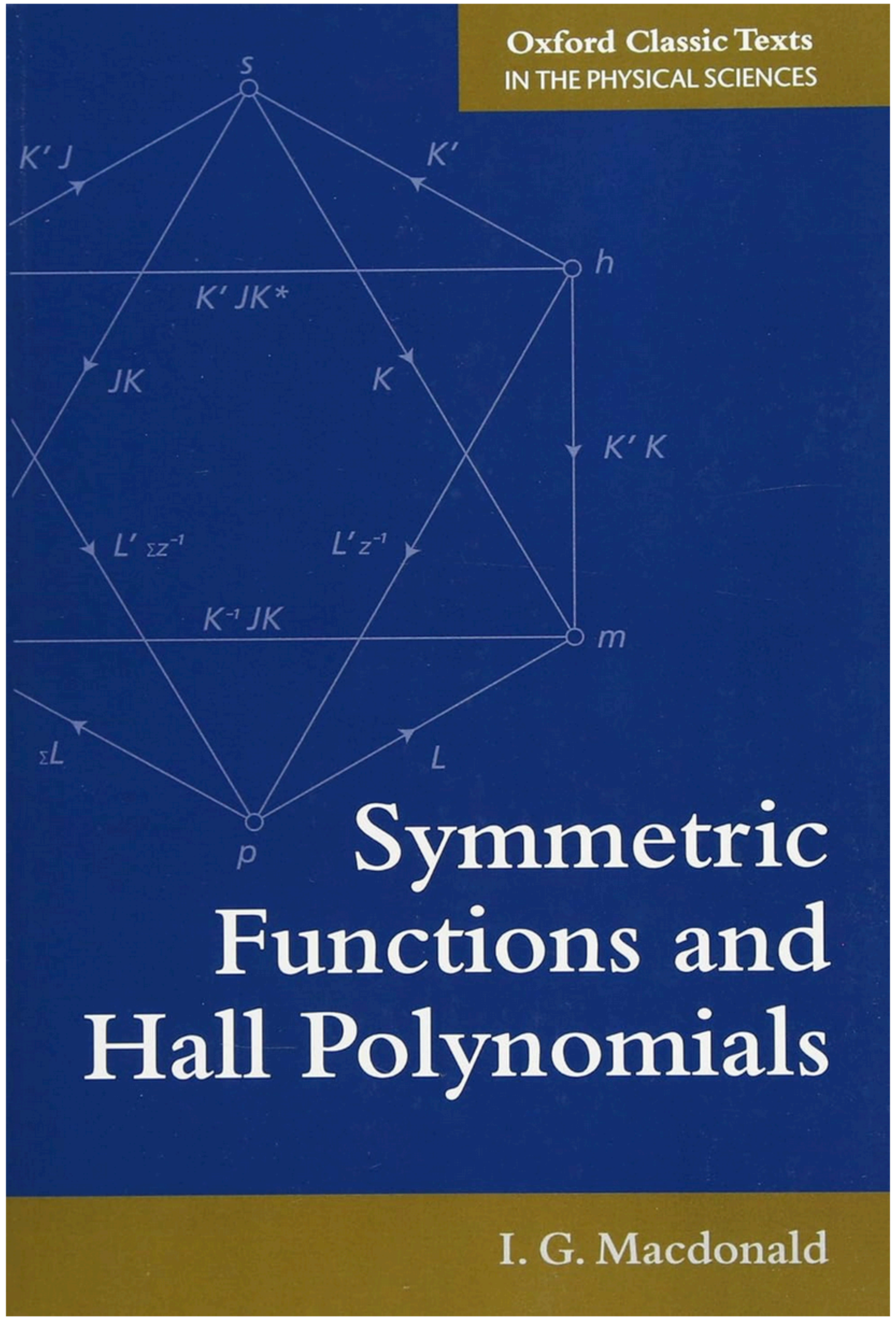
Sonderforschungsbereiche

(Collaborative Research Centres
funded by the
Deutsche Forschungsgemeinschaft)

SFB

Symmetric Functions Bible




Oxford Classic Texts
IN THE PHYSICAL SCIENCES
Symmetric Functions and Hall Polynomials
I. G. Macdonald

Macdonald polynomials

Favourite formulas for the Schur function s_λ

$$s_\lambda = \sum_{T \in B(\lambda)} x^T$$

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

Favourite formulas for the Schur function s_λ

$$s_\lambda = \sum_{T \in B(\lambda)} x^T \qquad\qquad s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

$T =$

1	1	1	2	2	2
2	2	3	3	5	8
3	4	6	6	7	
5	7	7	9	10	
6	9	9	10	11	
9					

is a semistandard Young tableau (SSYT)
of shape $\lambda = (6,6,5,5,5,1)$

$$x^T = x_1^{\#\text{ 1s in }T} x_2^{\#\text{ 2s in }T} \cdots x_n^{\#\ n\text{s in }T}$$

Favourite formulas for the Schur function s_λ

$$s_\lambda = \sum_{T \in B(\lambda)} x^T$$

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

$$a_\mu = \sum_{w \in S_n} (-1)^{\ell(w)} w x^\mu$$

$$x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$$

$$\delta = (n-1, \ldots, 2, 1, 0)$$

Favourite formulas for the Macdonald polynomial $P_\lambda(q,t)$

$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t)$$

$$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}$$

which, when $q = t = 0$, are

$$s_\lambda = \sum_{T\in B(\lambda)} x^T$$

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

$$a_\mu = \sum_{w\in S_n} (-1)^{\ell(w)} w x^\mu$$

$$x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$$

Favourite formulas for the Macdonald polynomial $P_\lambda(q,t)$

$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t)$$

$$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}$$

$$A_\mu(q,t) = \sum_{w\in S_n} (-t^{-\frac{1}{2}})^{\ell(w)} T_w E_\mu(q,t)$$

$E_\mu(q,t)$ are the electronic Macdonald polynomials

which, when $q = t = 0$, are

$$s_\lambda = \sum_{T\in B(\lambda)} x^T$$

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

$$a_\mu = \sum_{w\in S_n} (-1)^{\ell(w)} w x^\mu$$

$$x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$$

Favourite formulas for the Macdonald polynomial $P_\lambda(q,t)$

$$P_\lambda(q,t) = \sum_{T \in B(\lambda)} x^T \psi_T(q,t)$$

$$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}$$

$T =$

1	1	1	2	2	2
2	2	3	3	5	8
3	4	6	6	7	
5	7	7	9	10	
6	9	9	10	11	
9					

is a semistandard Young tableau (SSYT) of shape $\lambda = (6,6,5,5,5,1)$

$$x^T = x_1^{\#\ 1\text{s in } T} x_2^{\#\ 2\text{s in } T} \cdots x_n^{\#\ n\text{s in } T}$$

Favourite formulas for the Macdonald polynomial $P_\lambda(q,t)$

$$P_\lambda(q,t) = \sum_{T \in B(\lambda)} x^T \psi_T(q,t)$$

$$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}$$

$T =$

1	1	1	2	2	2
2	2	3	3	5	8
3	4	6	6	7	
5	7	7	9	10	
6	9	9	10	11	
9					

$$\psi_T(q,t) = \prod_{b \in \lambda} h_T(b)$$

For $i > T(b)$,

$$h_T(b) = \prod_{\substack{i \in T(\mathrm{arm}_\lambda(b)) \\ i \notin T(\mathrm{leg}_\lambda(b))}} \frac{h_T(b, < i)}{h_T(b, \le i)}$$

Favourite formulas for the Macdonald polynomial $P_\lambda(q,t)$

$$P_\lambda(q,t) = \sum_{T \in B(\lambda)} x^T \psi_T(q,t)$$

$T =$

1	1	1	2	2	2
2	2	3	3	5	8
3	4	6	6	7	
5	7	7	9	10	
6	9	9	10	11	
9					

$$\psi_T(q,t) = \prod_{b \in \lambda} h_T(b)$$

For $i > T(b)$,

$$h_T(b) = \prod_{\substack{i \in T(\mathrm{arm}_\lambda(b)) \\ i \notin T(\mathrm{leg}_\lambda(b))}} \frac{h_T(b, < i)}{h_T(b, \leq i)}$$

$$h_T(b, < i) = \frac{1 - t \cdot q^{a(b,<i)} t^{l(b,<i)}}{1 - q \cdot q^{a(b,<i)} t^{l(b,<i)}}$$

Favourite formulas for the Macdonald polynomial $P_\lambda(q,t)$

$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t)$$

$$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}$$

$$A_\mu(q,t) = \sum_{w\in S_n} (-t^{-\frac{1}{2}})^{\ell(w)} T_w E_\mu(q,t)$$

$E_\mu(q,t)$ are the electronic Macdonald polynomials

which, when $q = t = 0$, are

$$s_\lambda = \sum_{T\in B(\lambda)} x^T$$

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

$$a_\mu = \sum_{w\in S_n} (-1)^{\ell(w)} w x^\mu$$

$$x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$$

Favourite formulas for the Macdonald polynomial $P_\lambda(q,t)$

$$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}$$

$$A_\mu(q,t) = \sum_{w\in S_n} (-t^{-\frac{1}{2}})^{\ell(w)} T_w E_\mu(q,t)$$

For $f \in \mathbb{C}[x_1,\ldots,x_n]$, $(s_i f)(x_1,\ldots,x_n) = f(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_n)$,

$$T_i = -t^{-\frac{1}{2}} + (1+s_i)\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}x_i^{-1}x_{i+1}}{1-x_i^{-1}x_{i+1}},$$

If $w \in S_n$ and $w = s_{i_1}\cdots s_{i_\ell}$ is a reduced word then

$$T_w = T_{i_1}\cdots T_{i_\ell}$$

Favourite formulas for the Macdonald polynomial $P_\lambda(q,t)$

$$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}$$

$$A_\mu(q,t) = \sum_{w\in S_n} (-t^{-\frac{1}{2}})^{\ell(w)} T_w E_\mu(q,t)$$

$$\partial_i = (1+s_i)\frac{1}{x_i - x_{i+1}}$$

The *electronic Macdonald polynomial* $E_\mu(q,t)$ is recursively determined by

$$E_{(0,0,\ldots,0)} = 1, \qquad E_{(\mu_n+1,\mu_1,\ldots,\mu_{n-1})} = q^{\mu_n} x_1 E_\mu(x_2,\ldots,x_n,q^{-1}x_1),$$

$$E_{s_i\mu} = \left(\partial_i x_i - t x_i \partial_i + \frac{(1-t)q^{\mu_i-\mu_{i+1}}t^{\nu_\mu(i)-\nu_\mu(i+1)}}{1-q^{\mu_i-\mu_{i+1}}t^{\nu_\mu(i)-\nu_\mu(i+1)}}\right)E_\mu, \qquad \text{if } \mu_i > \mu_{i+1}.$$

Favourite formulas for the Macdonald polynomial $P_\lambda(q,t)$

$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t)$$

$$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}$$

$$A_\mu(q,t) = \sum_{w\in S_n} (-t^{-\frac{1}{2}})^{\ell(w)} T_w E_\mu(q,t)$$

$E_\mu(q,t)$ are the electronic Macdonald polynomials

which, when $q = t = 0$, are

$$s_\lambda = \sum_{T\in B(\lambda)} x^T$$

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

$$a_\mu = \sum_{w\in S_n} (-1)^{\ell(w)} w x^\mu$$

$$x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$$

Favourite formulas for the Macdonald polynomial $P_\lambda(q,t)$

$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t) \qquad\qquad P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}$$

which, when $q = t = 0$, are

$$s_\lambda = \sum_{T\in B(\lambda)} x^T \qquad\qquad s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

Can you do type B?

Can you do type B?

Here Macdonald has something interesting to say.

Which type B?

Can you do type B?

Here Macdonald has something interesting to say.

Which type B?

Because, as Macdonald worked out in his 1972 paper on affine root systems,

there are 9 different type Bs.

Can you do type B?

Here Macdonald has something interesting to say.

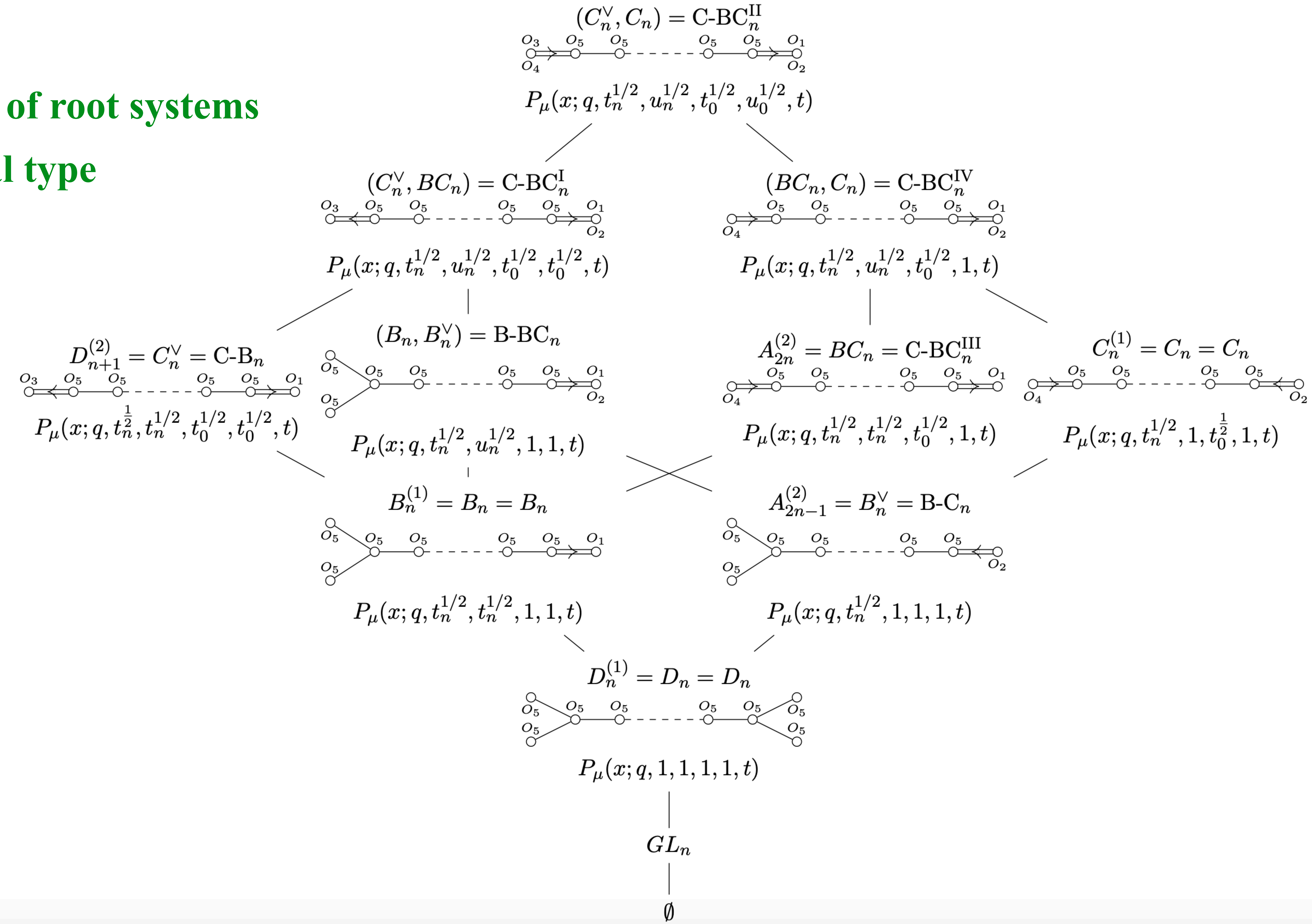
Which type B?

Because, as Macdonald worked out in his 1972 paper on affine root systems,

there are 9 different type Bs.

A diagram showing these is given on the next slide.

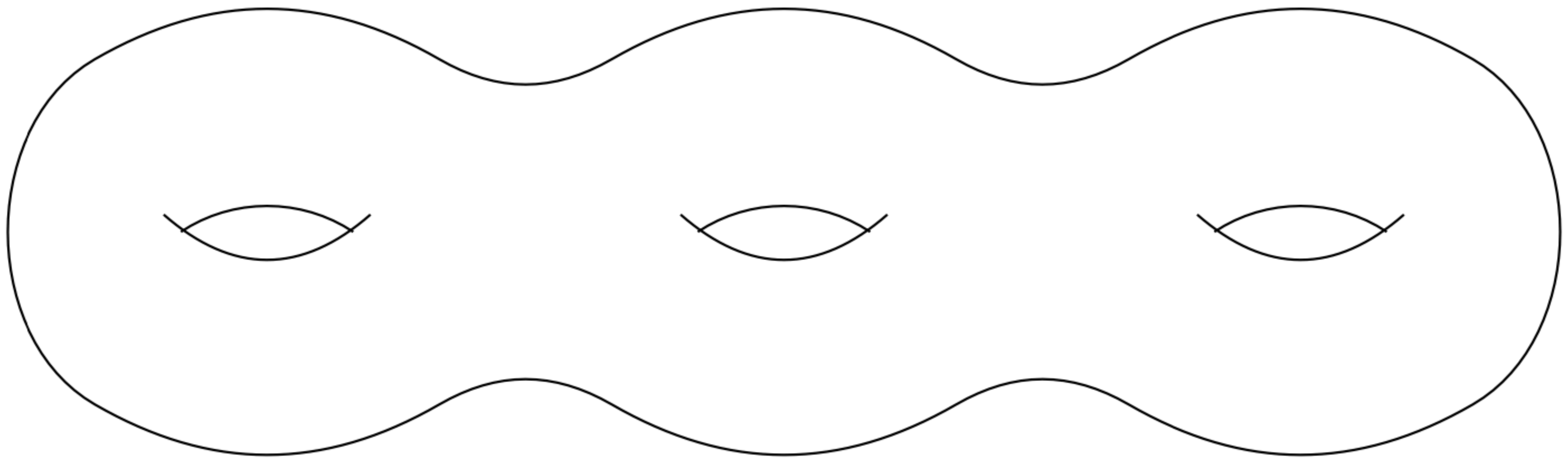
The poset of root systems of classical type



Cohomology

The cohomology of a symmetric product of a curve Σ of genus *g*

$$\Sigma(n) = \Sigma^n / S_n, \qquad \text{where } w \cdot (p_1, \ldots, p_n) = (p_{w^{-1}(1)}, \ldots, p_{w^{-1}(n)}),$$

$\Sigma =$ 

$\Sigma(n) = \Sigma^n / S_n$ is Σ^n folded up symmetrically

The cohomology of a symmetric product of a curve Σ of genus *g*

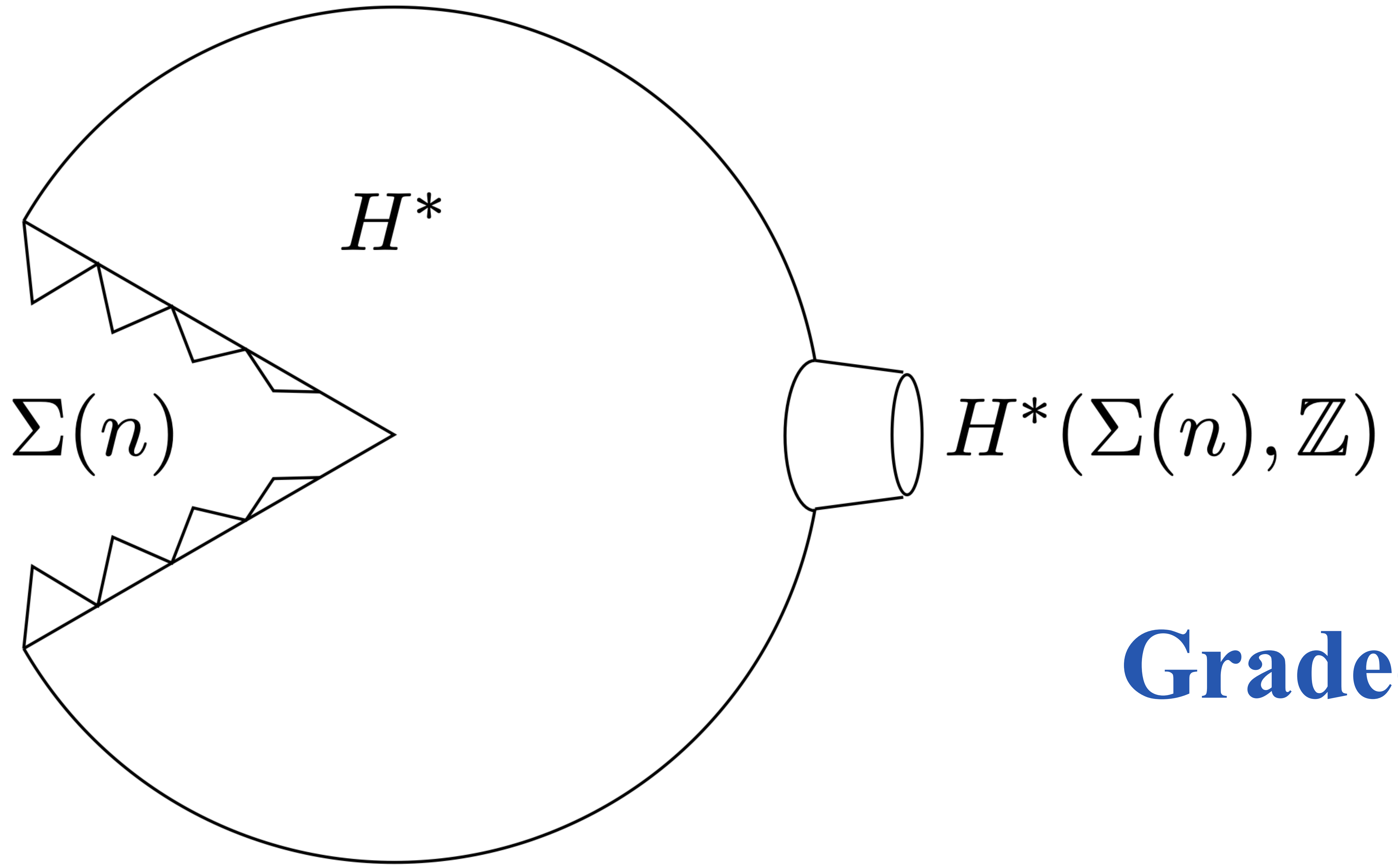
Cohomology

Spaces

Graded rings

The cohomology of a symmetric product of a curve Σ of genus g

Cohomology

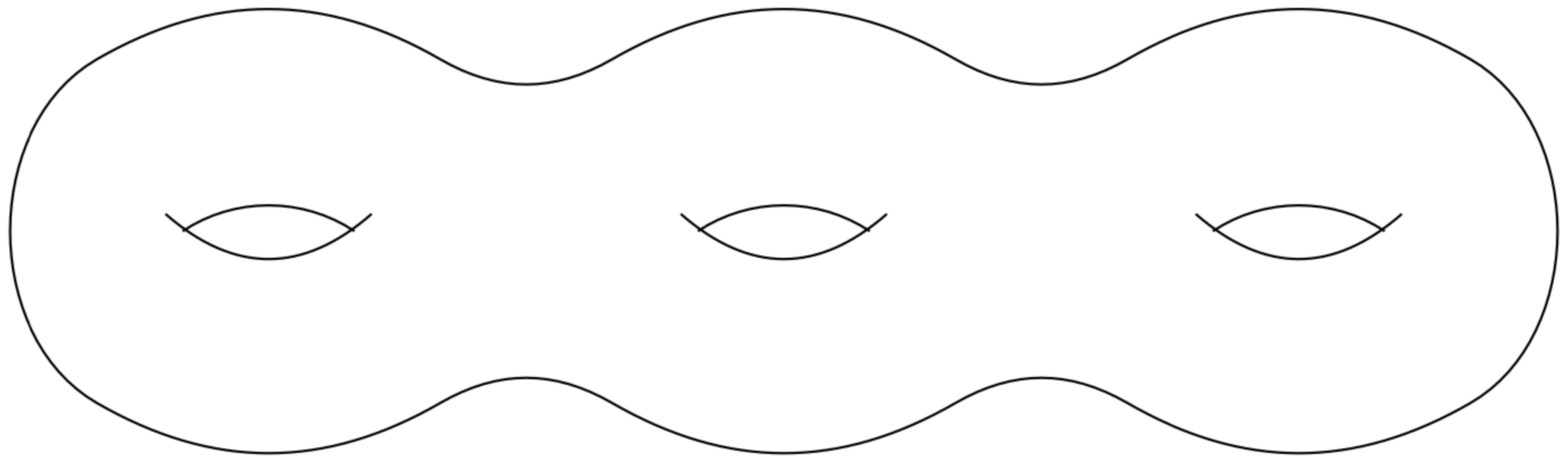


Spaces

Graded rings

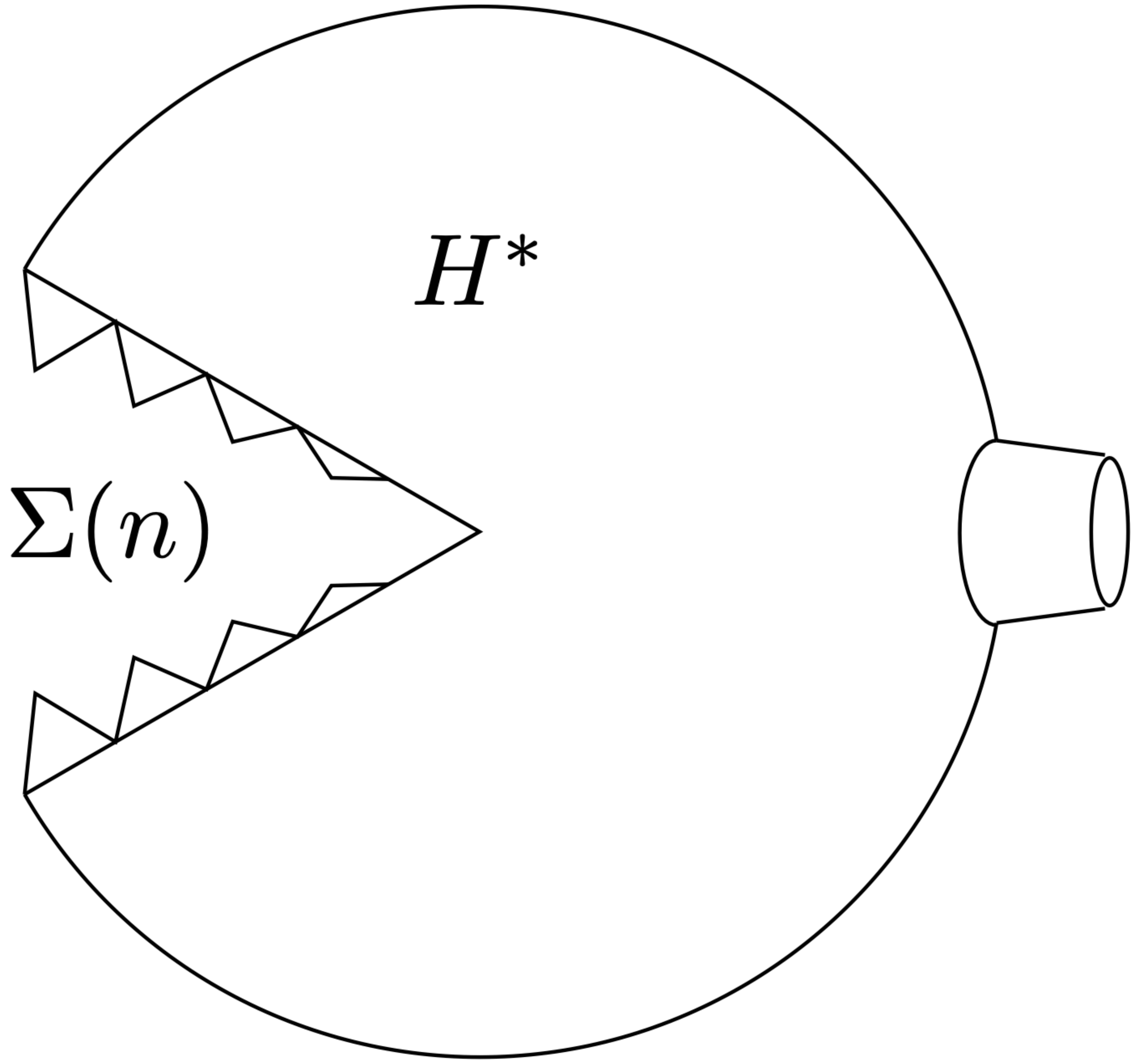
The cohomology of a symmetric product of a curve Σ of genus *g*

$\Sigma =$



$\Sigma(n) = \Sigma^n / S_n$ is Σ^n
folded up symmetrically

Cohomology

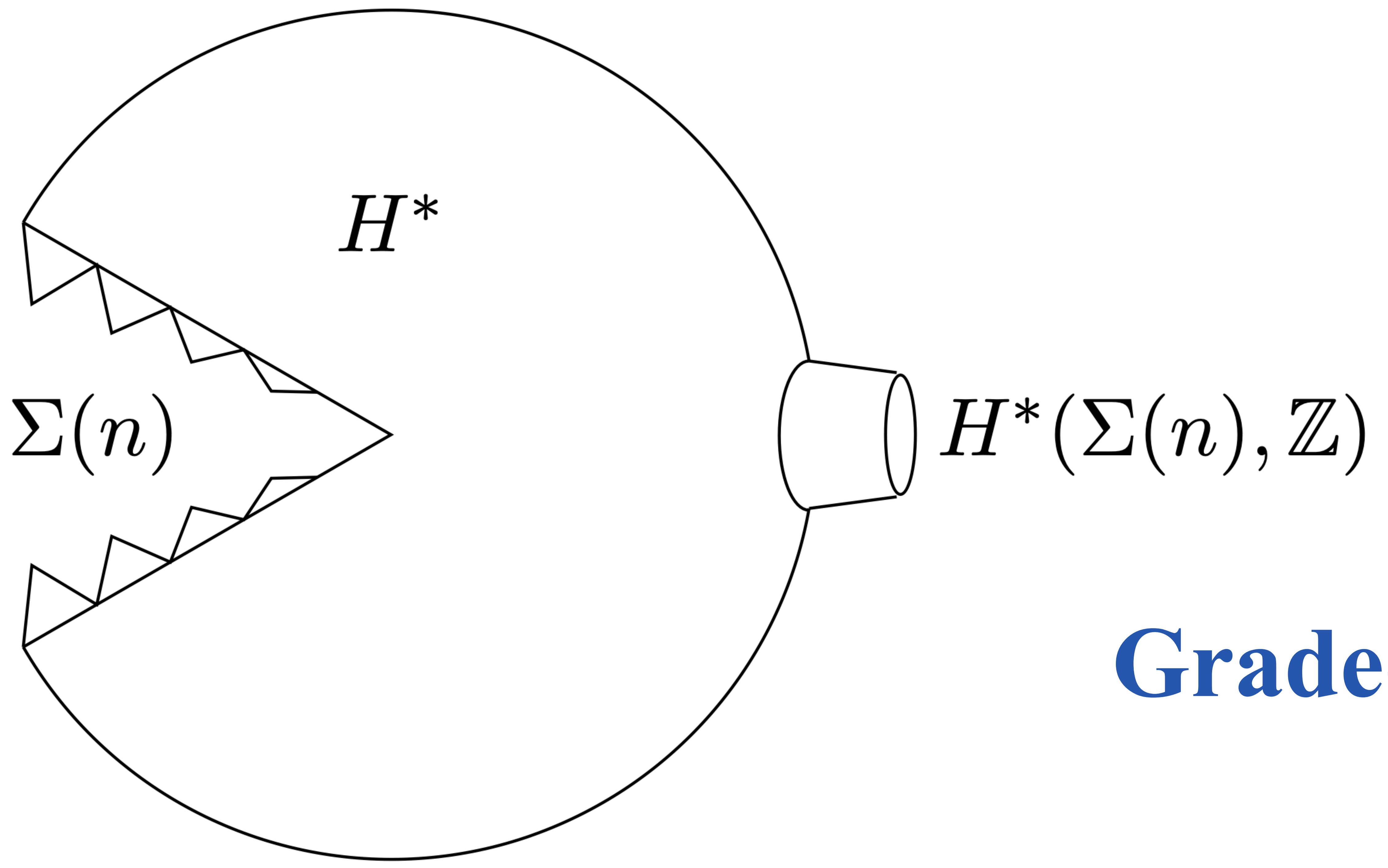


Spaces

Graded rings

The cohomology of a symmetric product of a curve Σ of genus g

Cohomology

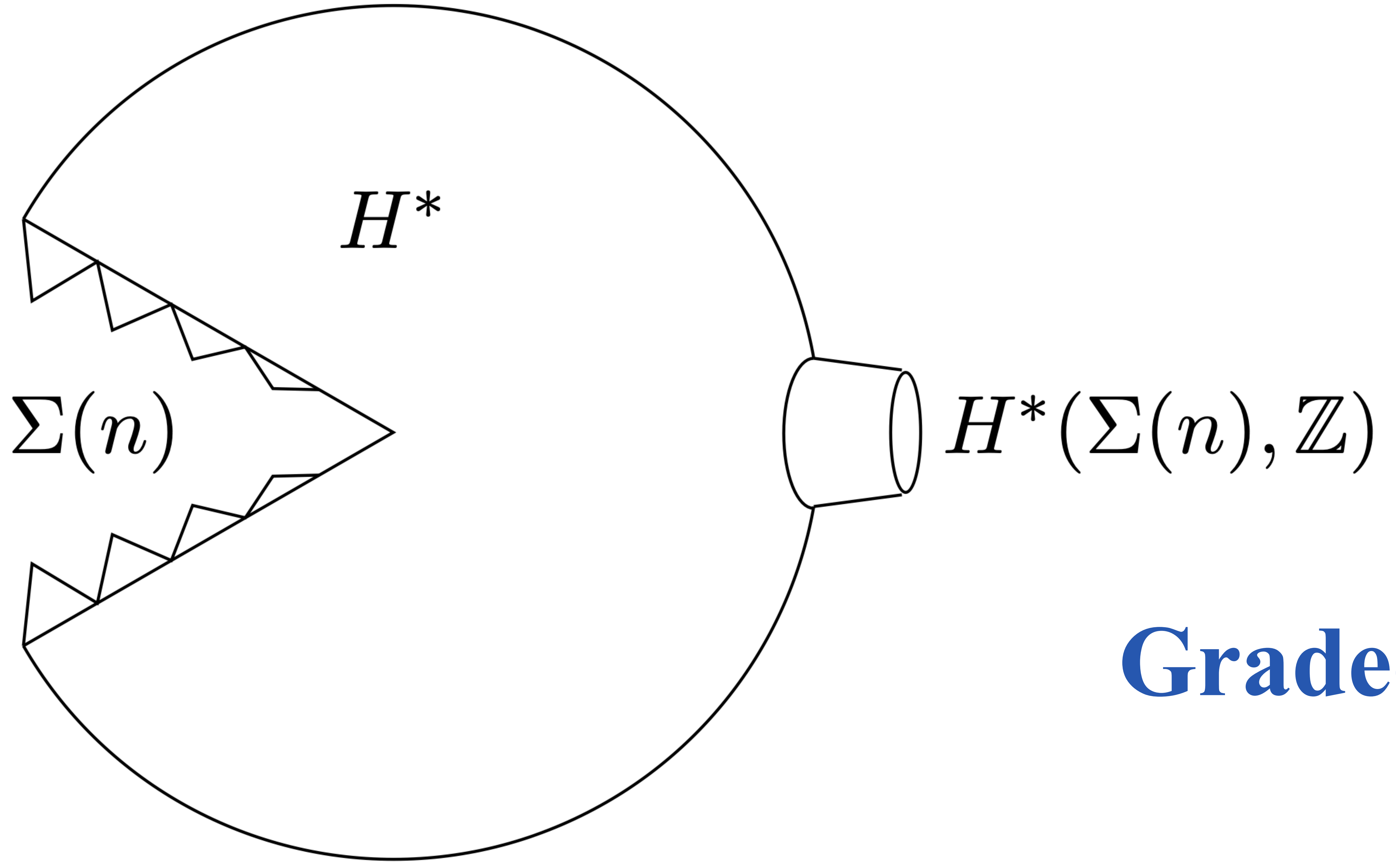


Spaces

Graded rings

The cohomology of a symmetric product of a curve Σ of genus g

Cohomology



Spaces

Graded rings

The cohomology ring $H^*(\Sigma(n), \mathbb{Z})$ is the $\mathbb{Z}$-algebra presented by generators

$\xi_1, \ldots, \xi_g, \quad \xi'_1, \ldots, \xi'_g, \quad \eta$ and relations

The cohomology of a symmetric product of a curve Σ of genus g

The cohomology ring $H^*(\Sigma(n), \mathbb{Z})$ is the $\mathbb{Z}$-algebra presented by generators

$\xi_1, \ldots, \xi_g, \quad \xi'_1, \ldots, \xi'_g, \quad \eta$ and relations

The cohomology of a symmetric product of a curve Σ of genus g

The cohomology ring $H^*(\Sigma(n), \mathbb{Z})$ is the $\mathbb{Z}$-algebra presented by generators

$$\xi_1, \ldots, \xi_g, \quad \xi'_1, \ldots, \xi'_g, \quad \eta$$

and relations

$$\xi_j \xi_j = -\xi_j \xi_i, \qquad \xi'_i \xi'_j = -\xi'_j \xi'_i, \qquad \xi_i \xi'_j = -\xi'_j \xi_i, \qquad \xi_i \eta = \eta \xi_i, \qquad \xi'_i \eta = \eta \xi'_i,$$

if $a, b, c, q \in \mathbb{Z}_{\geq 0}$ and $a + b + 2c + q = n + 1$ and $i_1, \ldots, i_a, j_1, \ldots, j_b, k_1, \ldots, k_c$ are distinct elements of $\{1, \ldots, g\}$

$$\xi_{i_1} \cdots \xi_{i_a} \xi'_{j_1} \cdots \xi'_{j_b} (\xi_{k_1} \xi'_{k_1} - \eta) \cdots (\xi_{k_c} \xi'_{k_c} - \eta) \eta^q = 0.$$

Weil conjectures

The Weil conjectures for a symmetric product of a curve Σ of genus g

The zeta function $Z(t)$ of an algebraic variety X

is an exponential generating function for the number of points of X over $\mathbb{F}_{q^n}$

The Weil conjectures for a symmetric product of a curve Σ of genus g

The zeta function $Z(t)$ of an algebraic variety X

is an exponential generating function for the number of points of X over $\mathbb{F}_{q^n}$

$$\frac{d}{dt}\log Z(t) = \sum_{n \in \mathbb{Z}_{>0}} \mathrm{Card}(X(\mathbb{F}_{q^n}))\, t^{n-1}\,.$$

The Weil conjectures for a symmetric product of a curve Σ of genus g

The zeta function $Z(t)$ of an algebraic variety X

is an exponential generating function for the number of points of X over $\mathbb{F}_{q^n}$

$$\frac{d}{dt} \log Z(t) = \sum_{n \in \mathbb{Z}_{>0}} \mathrm{Card}(X(\mathbb{F}_{q^n}))\, t^{n-1} .$$

Let Σ be a curve of genus g and assume that $\rho_1, \ldots, \rho_{2g} \in \mathbb{C}$ are such that

$$Z_1(t) = \frac{(1-\rho_1 t)\cdots(1-\rho_{2g} t)}{(1-t)(1-qt)}$$ is the zeta function of Σ.

The Weil conjectures for a symmetric product of a curve Σ of genus g

Assume that $\rho_1, \ldots, \rho_{2g} \in \mathbb{C}$ are such that

$$Z_1(t) = \frac{(1-\rho_1 t)\cdots(1-\rho_{2g}t)}{(1-t)(1-qt)}$$ is the zeta function of Σ.

Let $\phi_0(t) = 1 - t$ and

$$\phi_k(t) = \prod_{1 \le i_1 < \cdots < i_k \le 2g} (1 - \rho_{i_1} \cdots \rho_{i_k} t), \qquad \text{for } k \in \{1, \ldots, g\}.$$

The Weil conjectures for a symmetric product of a curve Σ of genus g

Assume that $\rho_1, \ldots, \rho_{2g} \in \mathbb{C}$ are such that

$$Z_1(t) = \frac{(1-\rho_1 t)\cdots(1-\rho_{2g}t)}{(1-t)(1-qt)} \qquad \text{is the zeta function of } \Sigma.$$

Let $\phi_0(t) = 1 - t$ and

$$\phi_k(t) = \prod_{1 \le i_1 < \cdots < i_k \le 2g} (1 - \rho_{i_1} \cdots \rho_{i_k} t), \qquad \text{for } k \in \{1, \ldots, g\}.$$

Then let

$$F_k(t) = \begin{cases} \phi_k(t)\phi_{k-2}(t)\phi_{k-4}(t)\cdots, & \text{if } k \in \{1, \ldots, n\}, \\ F_{2n-k}(q^{k-n}t), & \text{if } k \in \{n+1, \ldots, 2n\}, \end{cases}$$

The Weil conjectures for a symmetric product of a curve Σ of genus g

$$Z_1(t) = \frac{(1-\rho_1 t)\cdots(1-\rho_{2g}t)}{(1-t)(1-qt)} \qquad \phi_k(t) = \prod_{1\le i_1<\cdots<i_k\le 2g} (1-\rho_{i_1}\cdots\rho_{i_k}t),$$

$$F_k(t) = \begin{cases} \phi_k(t)\phi_{k-2}(t)\phi_{k-4}(t)\cdots, & \text{if } k \in \{0,1,\ldots,n\}, \\ F_{2n-k}(q^{k-n}t), & \text{if } k \in \{n+1,\ldots,2n\}, \end{cases}$$

The Weil conjectures hold for $\Sigma(n)$. More specifically

(a) The Zeta function for $\Sigma(n)$ is

$$Z_n(t) = \frac{F_1(t)F_3(t)\cdots F_{2n-1}(t)}{F_0(t)F_2(t)\cdots F_{2n}(t)}$$

The Weil conjectures for a symmetric product of a curve Σ of genus g

The Weil conjectures hold for $\Sigma(n)$. More specifically

(a) The Zeta function for $\Sigma(n)$ is $$Z_n(t) = \frac{F_1(t)F_3(t)\cdots F_{2n-1}(t)}{F_0(t)F_2(t)\cdots F_{2n}(t)}$$

(b) The Riemann hypothesis for $\Sigma(n)$ holds:

all roots of $Z_n(t)$ have absolute value in $$\{q^{-\frac{1}{2}\cdot 0}, q^{-\frac{1}{2}\cdot 1}, q^{-\frac{1}{2}\cdot 2}, \ldots, q^{-\frac{1}{2}\cdot 2n}\}\,.$$

(c) The functional equation for $Z_n(t)$ is:

$$Z_n\Big(\frac{1}{q^n t}\Big) = (-q^{-\frac{1}{2}n}t)^{(-1)^n\binom{2g-2}{n}} Z_n(t)\,.$$

Circles and lines

Circles and lines

Supplementary Chapter

The Four-Line, Five-Line, and n-Line

This chapter, which has nothing to do with the triangle, is put in partly to connect up & put in perspective various isolated results that have arisen incidentally in the preceding chapters (refs.), and partly for its intrinsic interest.

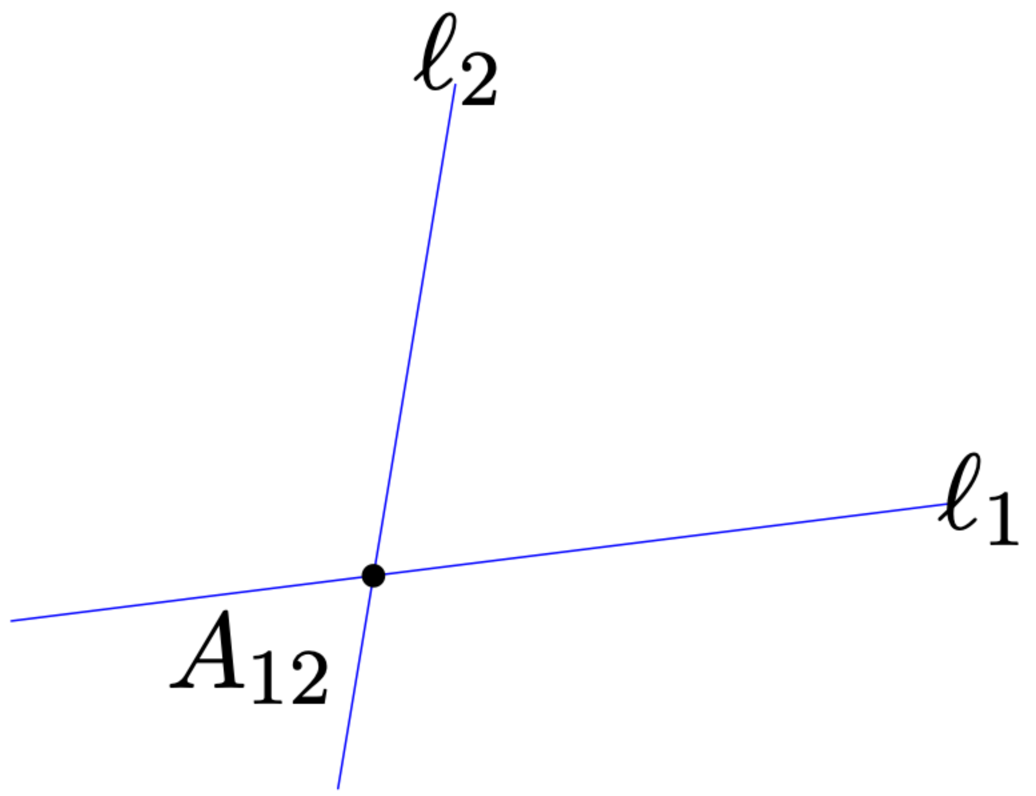
(1) We begin with the four-line. Take any four lines l_i in a plane: each three of them form a triangle, so that the four lines form four triangles in this way. These are called the triangles of the four-line.

Let l_i, l_j meet at the point A_{ij}; let c_i be the circle through A_{kl}, A_{jl}, A_{jk}; C_i the centre of this circle. Our first theorem is that due to Wallace (Leybourn's Math. Repos., 1806):

<u>The four circles c_i meet in a point, called the Wallace point of the four-line.</u> [1]

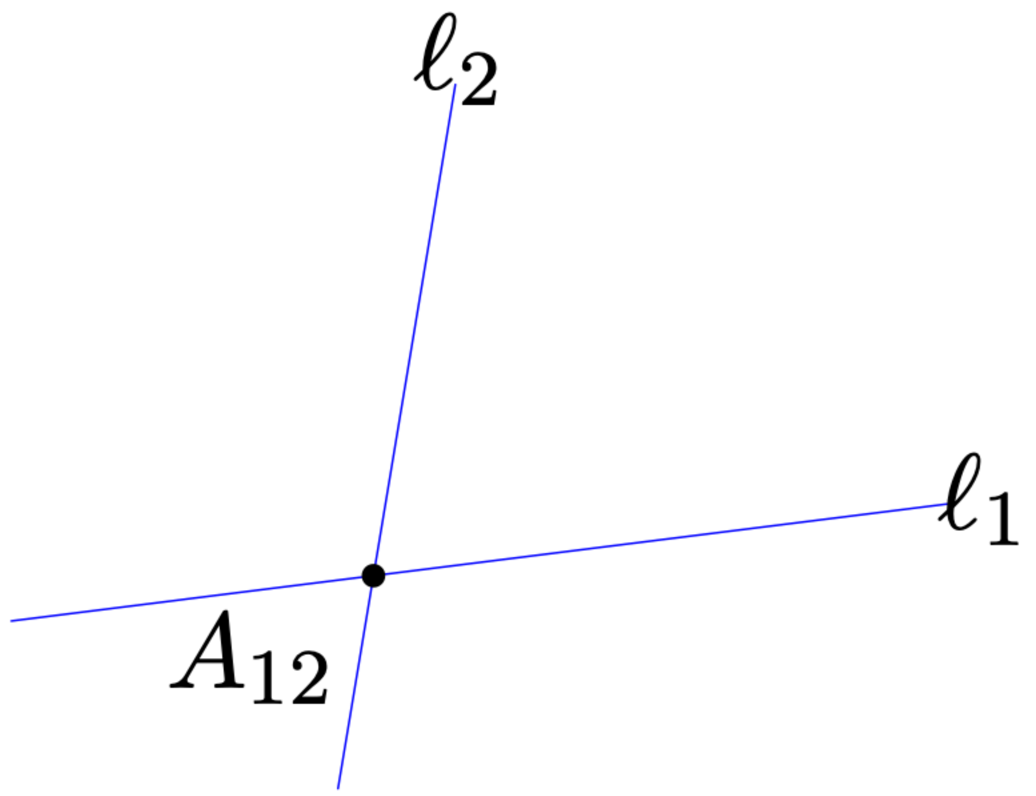
Circles and lines

Two generic lines ℓ_1 and ℓ_2 intersect in a point A_{12}.



Circles and lines

Two generic lines ℓ_1 and ℓ_2 intersect in a point A_{12}.

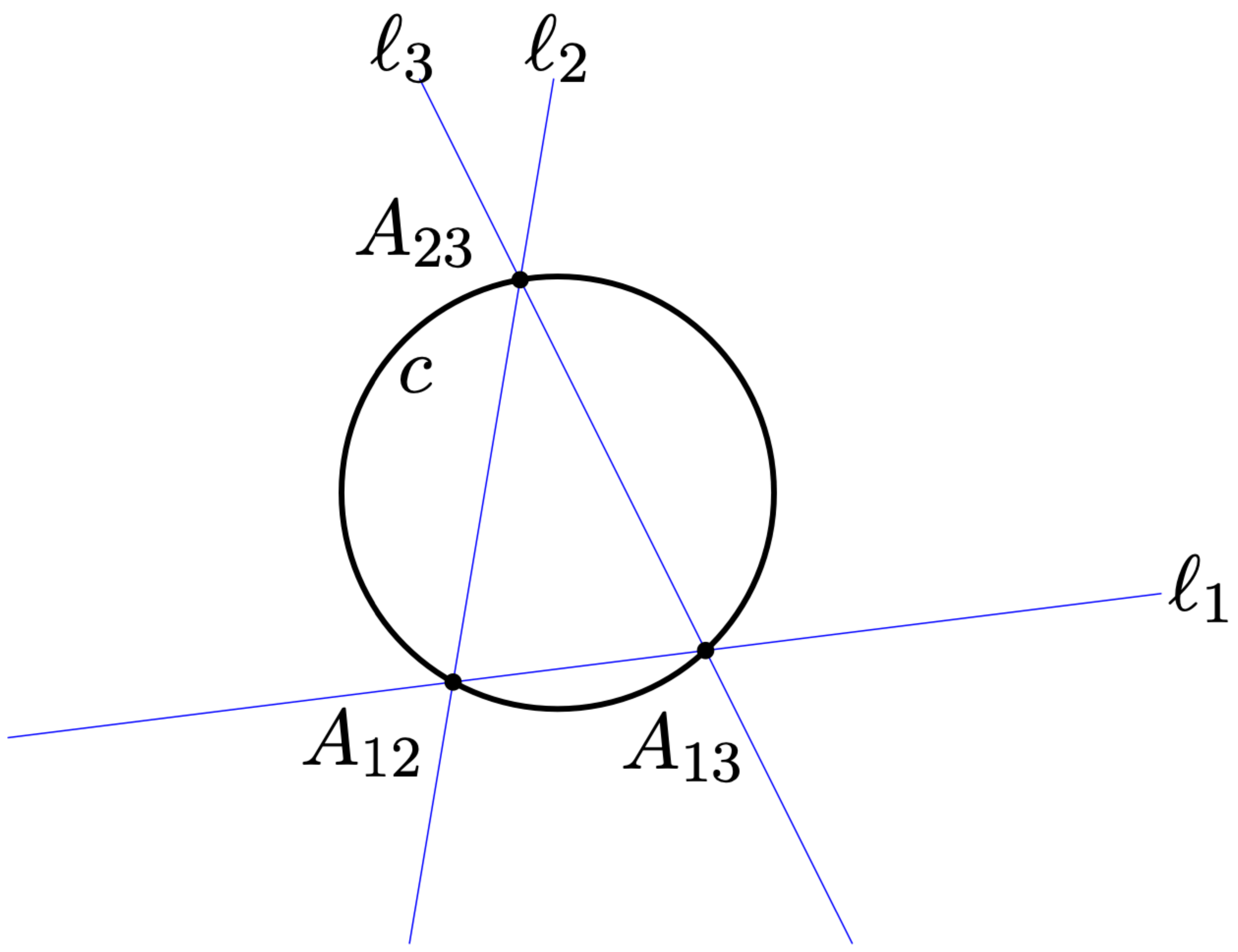


The point A_{12} is the *Clifford point of the 2-line*.

Circles and lines

Each pair of lines in a 3-line $\{\ell_1, \ell_2, \ell_3\}$ intersect in a point.

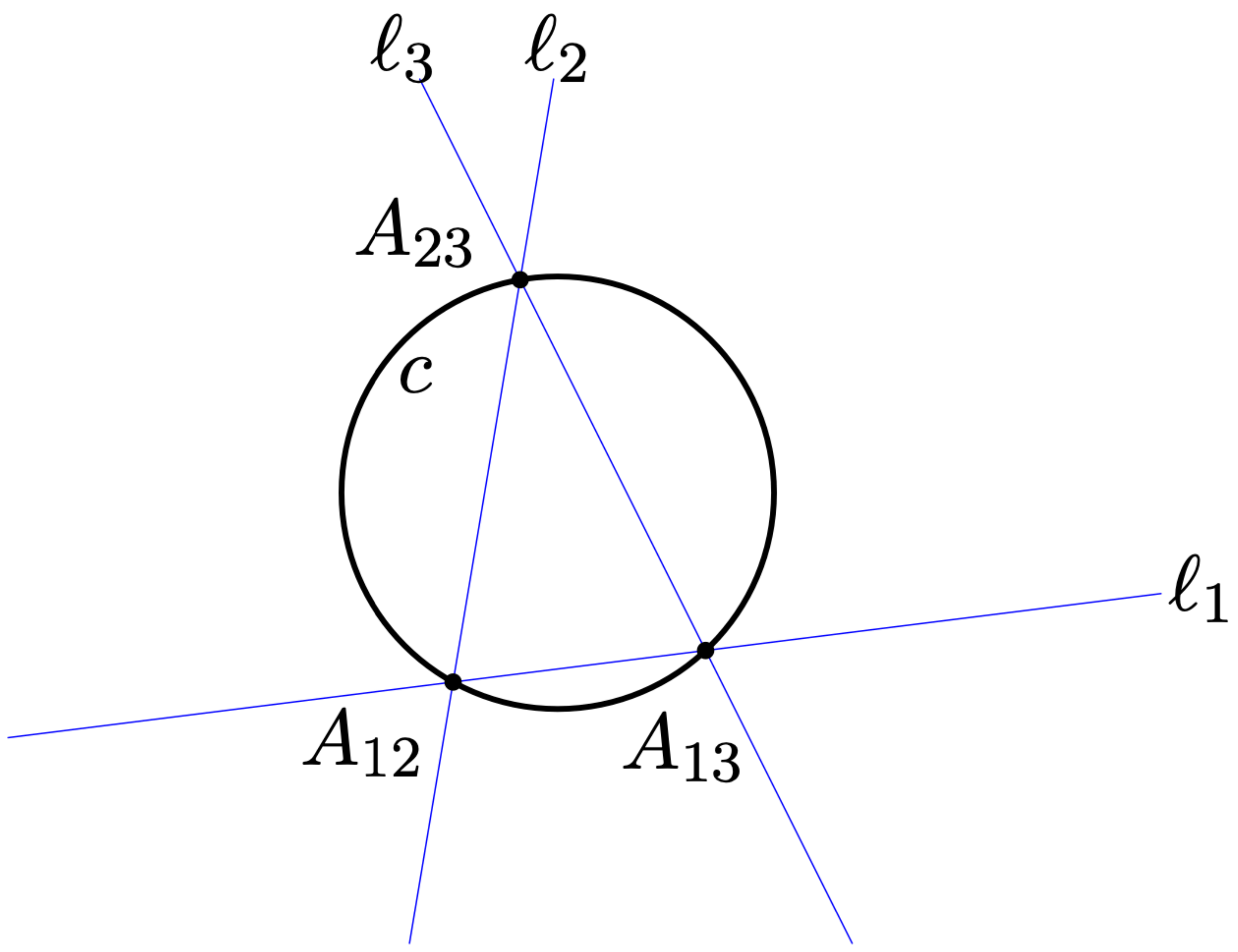
These three points determine a circle c.



Circles and lines

Each pair of lines in a 3-line $\{\ell_1, \ell_2, \ell_3\}$ intersect in a point.

These three points determine a circle c.

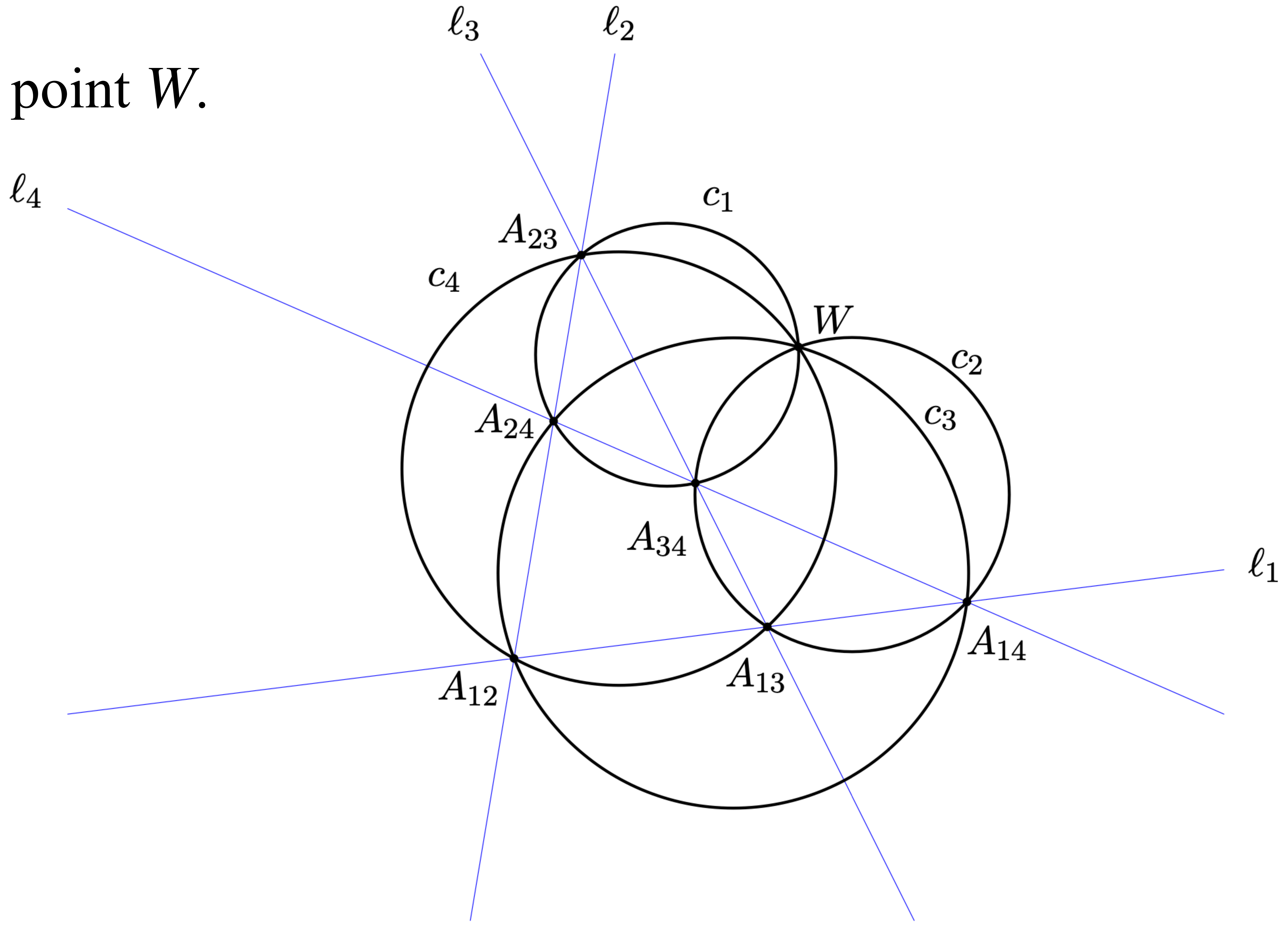


The circle c is the *Clifford circle of the 3-line*.

Circles and lines

Each triple of lines in a 4-line $\{\ell_1, \ell_2, \ell_3, \ell_4\}$ determines a Clifford circle, giving the circles c_1, c_2, c_3, c_4.

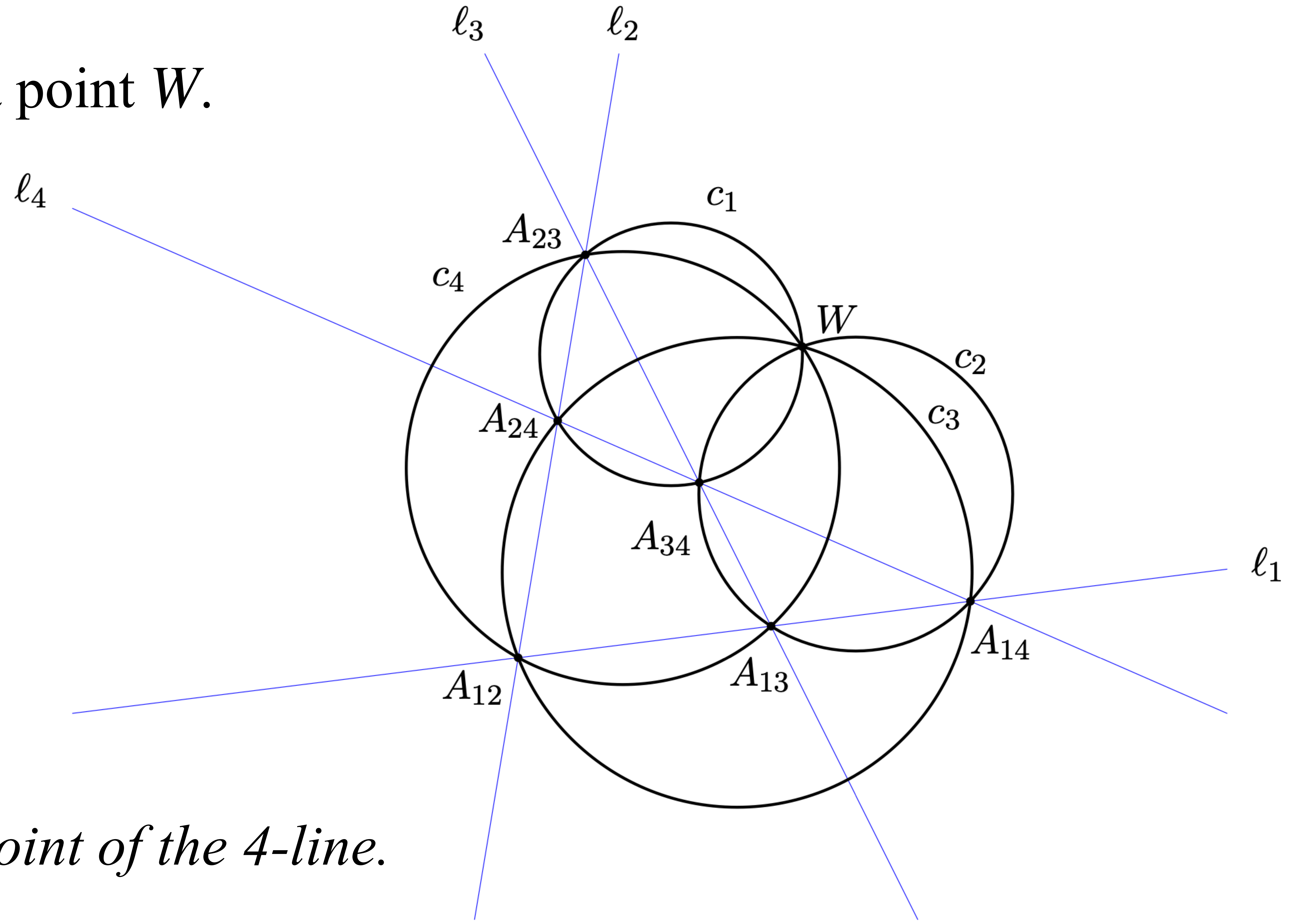
These four circles intersect in a point W.



Circles and lines

Each triple of lines in a 4-line $\{\ell_1, \ell_2, \ell_3, \ell_4\}$ determines a Clifford circle, giving the circles c_1, c_2, c_3, c_4.

These four circles intersect in a point W.

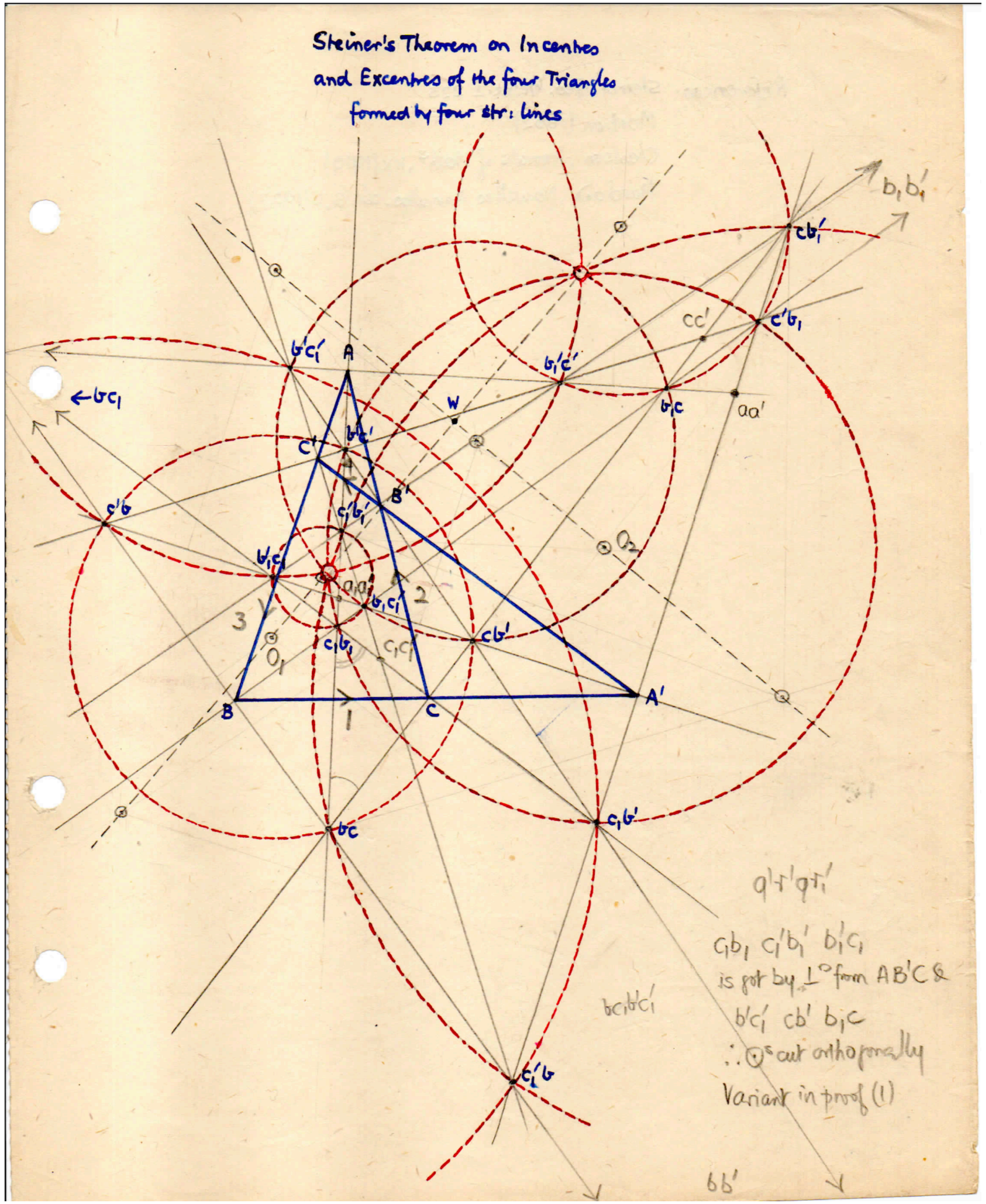


The point W is the *Clifford point of the 4-line.*

Circles and lines

Each 4-tuple of lines in a 5-line $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$ determines a Clifford point, giving the points p_1, p_2, p_3, p_4, p_5.
These five points lie on a circle C.

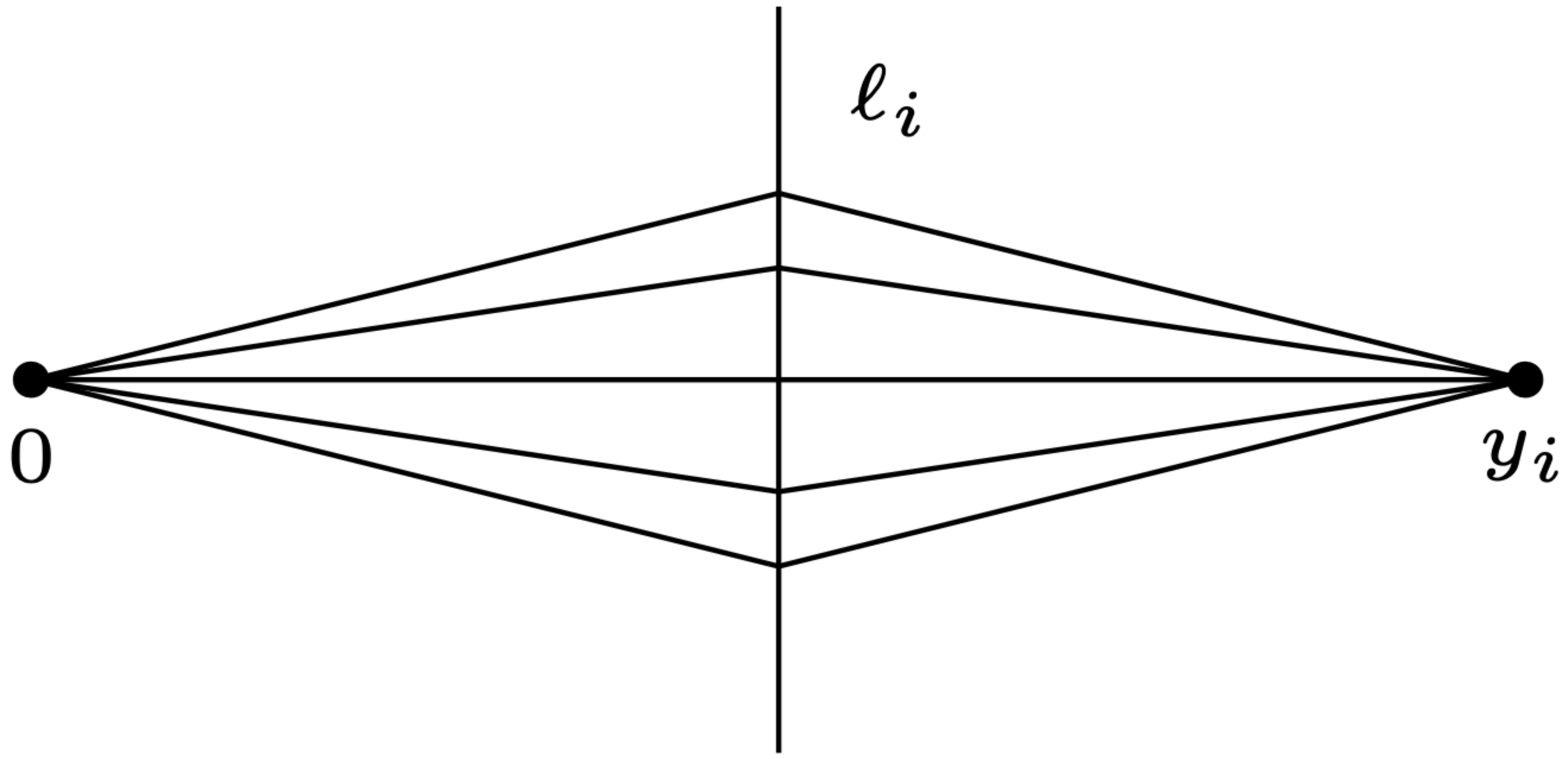
The circle C is the Clifford circle of the 5-line.



Circles and lines: Macdonald's general formulation

Let $y_1, \ldots, y_n \in \mathbb{C}^\times$.

Let ℓ_i be the line consisting of the points in $\mathbb{C}$ that are equidistant from 0 and y_i.



$$\ell_i = \{z \in \mathbb{C} \mid \bar{z} = t_i(z - y_i)\}, \qquad \text{where} \qquad t_i = \frac{-\overline{y_i}}{y_i}$$

Let $\mathscr{L} = \{\ell_1, \ldots, \ell_n\}$ be an n-line.

Circles and lines: Macdonald's general formulation

Let $\mathscr{L} = \{\ell_1, \ldots, \ell_n\}$ be an n-line.

$$\ell_i = \{z \in \mathbb{C} \mid \bar{z} = t_i(z - y_i)\}, \qquad \text{where} \qquad t_i = \frac{-\overline{y_i}}{y_i}.$$

Let

$$c_k(\mathscr{L}) = \frac{p_1 t_1^{n-1-k}}{g_1(\mathscr{L})} + \frac{p_2 t_2^{n-1-k}}{g_2(\mathscr{L})} + \cdots + \frac{p_n t_n^{n-1-k}}{g_n(\mathscr{L})}, \qquad \text{where}$$

$$g_j(\mathscr{L}) = (t_j - t_1)(t_j - t_2)\cdots(t_j - t_{j-1})(t_j - t_{j+1})(t_j - t_{j+2})\cdots(t_j - t_n).$$

Circles and lines: Macdonald's general formulation

Case n even: $n = 2k$. Each $(n-1)$-subset of the n-line determines a Clifford circle, and these n Clifford circles intersect in a unique point $p(\mathscr{L})$.

$$\mathscr{L} = \{\ell_1, \ldots, \ell_n\} \qquad \text{with} \qquad \ell_i = \{z \in \mathbb{C} \mid \bar{z} = t_i(z - y_i)\}, \qquad \text{where} \qquad t_i = \frac{-\overline{y_i}}{y_i}.$$

Let $a_1, \ldots, a_{k-1} \in \mathbb{C}$ be given by

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} c_2(\mathscr{L}) & \cdots & c_k(\mathscr{L}) \\ \vdots & & \vdots \\ c_k(\mathscr{L}) & \cdots & c_{2k-2}(\mathscr{L}) \end{pmatrix}^{-1} \begin{pmatrix} -c_1(\mathscr{L}) \\ -c_2(\mathscr{L}) \\ \vdots \\ -c_{k-1}(\mathscr{L}) \end{pmatrix}.$$

Then

$$p(\mathscr{L}) = c_0(\mathscr{L}) + a_1 c_1(\mathscr{L}) + a_2 c_2(\mathscr{L}) + \cdots + a_{k-1} c_{k-1}(\mathscr{L}) \qquad \text{is the Clifford point.}$$

Circles and lines: Macdonald's general formulation

Case n odd: $n = 2k{+}1$. Each $(n-1)$-subset of the n-line determines a Clifford pont, and these n Clifford points lie on a unique circle $C(\mathscr{L})$, given by

$$C(\mathscr{L}) = \{A(\mathscr{L}) - \theta B(\mathscr{L}) \mid \theta \in U_1(\mathbb{C})\}, \qquad \text{where} \qquad U_1(\mathbb{C}) = \{\theta \in \mathbb{C} \mid \theta\bar{\theta} = 1\},$$

$$A(\mathscr{L}) = \frac{\det\begin{pmatrix} c_0(\mathscr{L}) & \cdots & c_{k-1}(\mathscr{L}) \\ c_1(\mathscr{L}) & \cdots & c_k(\mathscr{L}) \\ \vdots & & \vdots \\ c_{k-1}(\mathscr{L}) & \cdots & c_{2k-2}(\mathscr{L}) \end{pmatrix}}{\det\begin{pmatrix} c_2(\mathscr{L}) & \cdots & c_k(\mathscr{L}) \\ \vdots & & \vdots \\ c_k(\mathscr{L}) & \cdots & c_{2k-2}(\mathscr{L}) \end{pmatrix}} \qquad \text{and} \qquad B(\mathscr{L}) = \frac{\det\begin{pmatrix} c_1(\mathscr{L}) & \cdots & c_k(\mathscr{L}) \\ \vdots & & \vdots \\ c_k(\mathscr{L}) & \cdots & c_{2k-1}(\mathscr{L}) \end{pmatrix}}{\det\begin{pmatrix} c_2(\mathscr{L}) & \cdots & c_k(\mathscr{L}) \\ \vdots & & \vdots \\ c_k(\mathscr{L}) & \cdots & c_{2k-2}(\mathscr{L}) \end{pmatrix}}.$$

Influencer

I.G. Macdonald as influencer

J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2012-0093

Journal für die reine und angewandte Mathematik
© De Gruyter 2013

Macdonald formula for curves with planar singularities

By *Davesh Maulik* at Columbia and *Zhiwei Yun* at Stanford

Abstract. We generalize Macdonald's formula for the cohomology of Hilbert schemes of points on a curve from smooth curves to curves with planar singularities: we relate the cohomology of the Hilbert schemes to the cohomology of the compactified Jacobian of the curve. The new formula is a consequence of a stronger identity between certain perverse sheaves defined by a family of curves satisfying mild conditions. The proof makes essential use of Ngô's support theorem for compactified Jacobians and generalizes this theorem to the relative Hilbert scheme of such families. As a consequence, we give a cohomological interpretation of the numerator of the Hilbert-zeta function of curves with planar singularities.

1. Introduction

Let C be a smooth projective connected curve over an algebraically closed field k. Let $\mathrm{Sym}^n(C)$ be the n-th symmetric product of C. Macdonald's formula [21] says there is a canonical isomorphism between graded vector spaces,

$$(1.1)\qquad \mathrm{H}^*(\mathrm{Sym}^n(C)) \cong \mathrm{Sym}^n(\mathrm{H}^*(C)) = \bigoplus_{i+j\le n, i,j\ge 0} \bigwedge^i(\mathrm{H}^1(C))[-i-2j](-j).$$

Here [?] denotes the cohomological shift and (?) denotes the Tate twist. This formula respects

I.G. Macdonald as influencer

V. Kac, *Infinite dimensional Lie algebras*

Cambridge University Press, 1982

At about the same time Macdonald [1972] obtained his remarkable identities. In this work he undertook to generalize the Weyl denominator

identity to the case of affine root systems. He remarked that a straightforward generalization is actually false. To salvage the situation he had to add some "mysterious" factors, which he was able to determine as a result of lengthy calculations. The simplest example of Macdonald's identities is the famous Jacobi triple product identity:

$$\prod_{n\geq 1}(1-u^n v^n)(1-u^{n-1}v^n)(1-u^n v^{n-1})$$
$$= \sum_{m\in\mathbb{Z}}(-1)^m u^{\frac{1}{2}m(m+1)} v^{\frac{1}{2}m(m-1)}.$$

The "mysterious" factors which do not correspond to affine roots are the factors $(1-u^n v^n)$.

After the appearance of the two works mentioned above very little remained to be done: one had to place them on the desk next to one another to understand that Macdonald's result is only the tip of the iceberg—the representation theory of Kac–Moody algebras. Namely, it turned out that a simplified version of Bernstein–Gelfand–Gelfand's proof may be applied to the proof of a formula generalizing Weyl's formula, for the formal character of the representation π_Λ of an arbitrary Kac–Moody algebra $\mathfrak{g}'(A)$ corresponding to a symmetrizable matrix A. In the case of the simplest 1-dimensional representation π_0, this formula becomes the generalization of Weyl's denominator identity. In the case of an affine Lie algebra, the generalized Weyl denominator identity turns out to be equivalent to the Macdonald identities. In the process, the "mysterious" factors receive a

I.G. Macdonald as influencer

Annals of Mathematics, **103** (1976), 103-161

Representations of reductive groups over finite fields

By P. DELIGNE and G. LUSZTIG

Introduction

Let us consider a connected, reductive algebraic group G, defined over a finite field $\mathbf{F}_q$, with Frobenius map F. We shall be concerned with the representation theory of the finite group G^F, over fields of characteristic 0.

In 1968, Macdonald conjectured, on the basis of the character tables known at the time (GL_n, Sp_4), that there should be a well defined correspondence which, to any F-stable maximal torus T of G and a character θ of T^F in general position, associates an irreducible representation of G^F; moreover, if T modulo the centre of G is anisotropic over $\mathbf{F}_q$, the corresponding representation of G^F should be cuspidal (see Seminar on algebraic groups and related finite groups, by A. Borel et al., Lecture Notes in Mathematics, 131, pp. 117 and 101). In this paper we prove Macdonald's conjecture. More precisely, for T as above and θ an arbitrary character of T^F we construct virtual representations R_T^θ which have all the required properties.

I.G. Macdonald as influencer

Remarks on Macdonald's book on p-adic spherical functions

Bill Casselman
University of British Columbia
cass@math.ubc.ca

When Ian Macdonald's book **Spherical functions on a group of *p*-adic type** first appeared, it was one of a very small number of publications concerned with representations of $\mathfrak{p}$-adic groups. At just about that time, however, the subject began to be widely recognized as indispensable in understanding automorphic forms, and the literature on the subject started to grow rapidly. Since it has by now grown so huge, in discussing here the subsequent history of some of Macdonald's themes I shall necessarily restrict myself only to things closely related to them. This will be no serious restriction since some of the most interesting problems in all of representation theory—among others, those connected with Langlands' 'fundamental lemma'—are concerned with $\mathfrak{p}$-adic spherical functions. Along the way I'll reformulate from a few different perspectives what his book contains. I'll begin, in the next section, with a brief sketch of the main points, postponing most technical details until later.

Throughout, suppose k to be what I call a $\mathfrak{p}$-adic field, which is to say that it is either a finite extension of some $\mathbb{Q}_p$ or the field of Laurent polynomials in a single variable with coefficients in a finite field. Further let

- $\mathfrak{o}$ = the ring of integers of k;
- $\mathfrak{p}$ = the maximal ideal of $\mathfrak{o}$;
- ϖ = a generator of $\mathfrak{p}$;
- $q = |\mathfrak{o}/\mathfrak{p}|$, so that $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$.

Let $\mathbb{D}$ be a field of characteristic 0, which will play the role of coefficient field in representations. The minimal requirement on $\mathbb{D}$ is that it contain $\sqrt{q}$, but it will in the long run be convenient to assume that it is algebraically closed. It may usually be taken to be $\mathbb{C}$, but I want to emphasize that special properties of $\mathbb{C}$ are rarely required.

In writing this note I had one major decision to make about what class of groups I would work with. What made it difficult was that there were conflicting goals to take into account. On the one hand, I wanted to be able to explain a few basic ideas without technical complications. For this reason, I did not want to deal with arbitrary reductive groups, because even to state results precisely in this case would have required much distracting effort—effort, moreover, that would have just duplicated things explained very well in Macdonald's book. On the other, I wanted to illustrate some of the complexities that Macdonald's book confronts. In the end, I chose to restrict myself to **unramified** groups. *I will suppose throughout this account that G is a reductive group defined over k arising by base extension from a smooth reductive scheme over $\mathfrak{o}$.* I hope that the arguments I present here are clear enough that generalization to arbitrary reductive groups will be straightforward once one understands their fine structure. I also hope that the way things go with this relatively simple class of groups will motivate the geometric treatment in Macdonald's book, which although extremely elegant is somewhat terse and short of examples. I'll say something later on in the section on root data about their structure.

Upon learning that I was going to be writing this essay, Ian Macdonald asked me to mention that Axiom V in Chapter 2 of his book is somewhat stronger than the corresponding axiom of Bruhat-Tits, and not valid for the type C–B_2 in their classification. Deligne pointed this out to him, and made the correction:

> Axiom V. *The commutator group $[U_\alpha, U_\beta]$ for $\alpha, \beta > 0$ is contained in the group generated by the U_γ with $\gamma > 0$ and not parallel to α or β.*

Translator

I.G. Macdonald as translator: Bourbaki

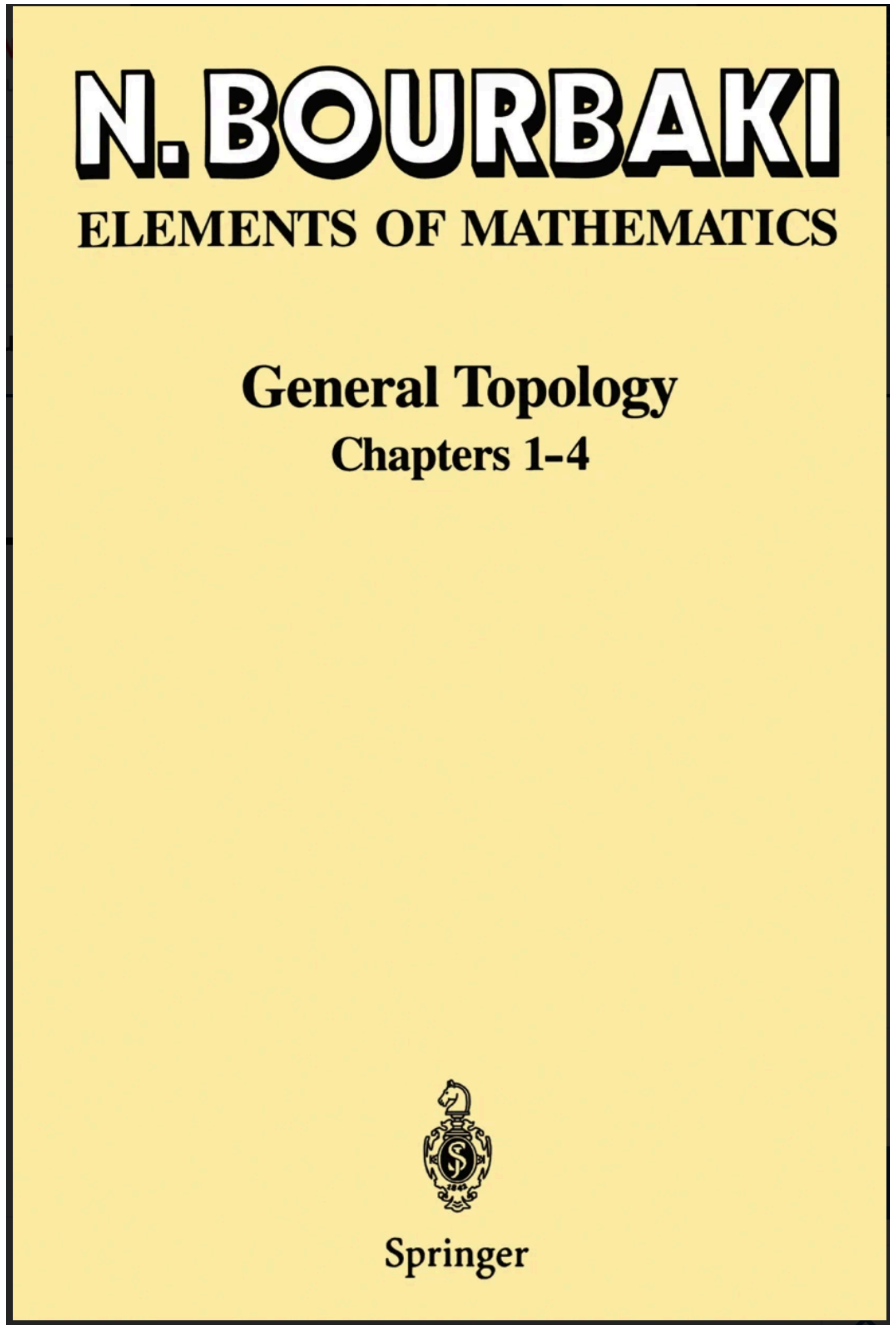
Bourbaki, General Topology Part I 1966

Bourbaki, General Topology Part II 1966

Bourbaki, Theory of Sets 1968

Bourbaki, Commutative Algebra 1972

Bourbaki, Algebra 1974



I.G. Macdonald as translator: Bourbaki

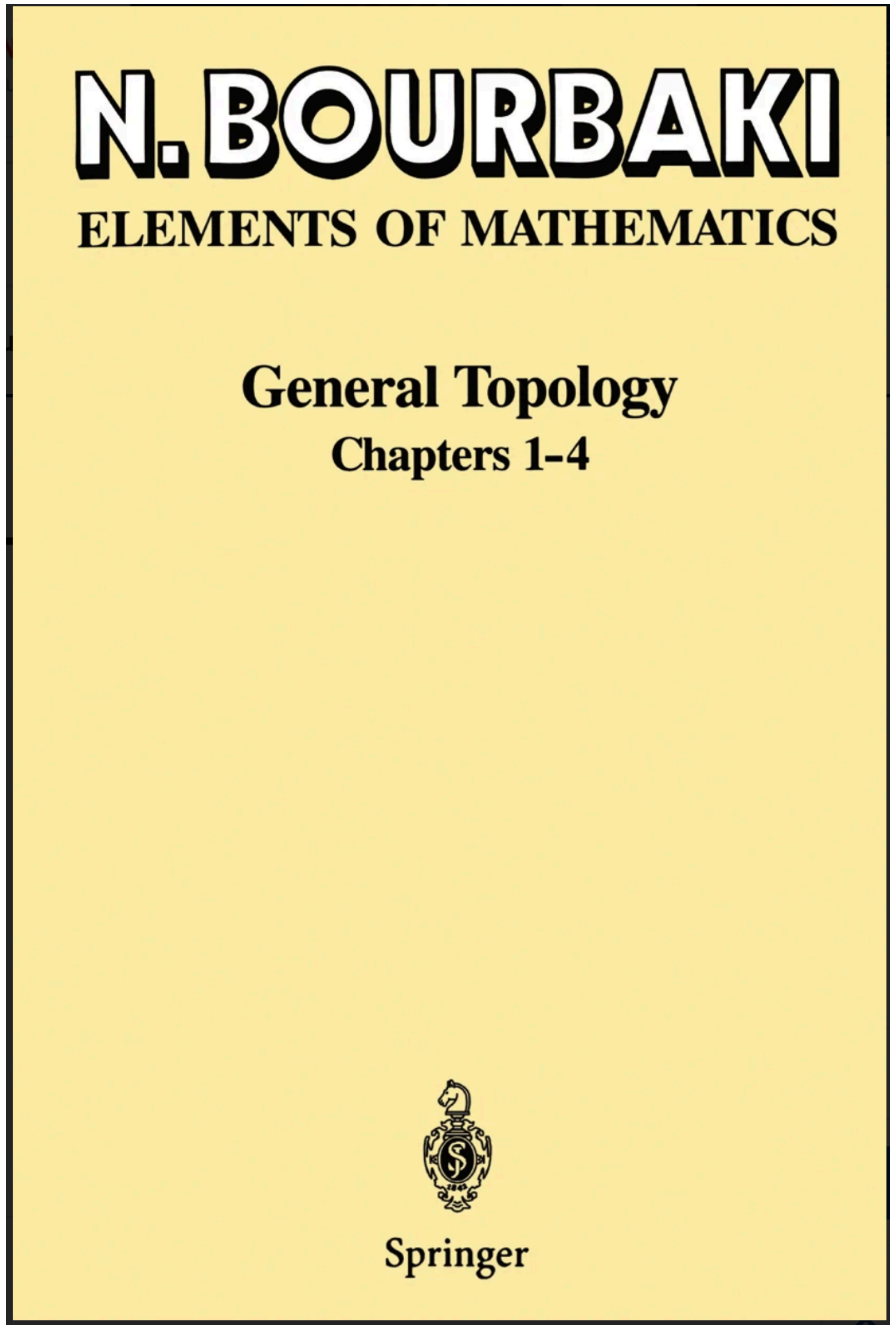
Bourbaki, General Topology Part I 1966, vii+437 pp.

Bourbaki, General Topology Part II 1966, iv+363 pp.

Bourbaki, Theory of Sets 1968, viii+414 pp.

Bourbaki, Commutative Algebra 1972, xxiv+625 pp.

Bourbaki, Algebra 1974, xxiii+709 pp.



I.G. Macdonald as translator: Dieudonné

Dieudonné, Foundations of Modern Analysis 1960 and 1969

Dieudonné, Treatise on Analysis Vol. II 1970 and 1976

Dieudonné, Treatise on Analysis Vol. III 1972

Dieudonné, Treatise on Analysis Vol. IV 1974

Dieudonné, Treatise on Analysis Vol. V 1977

Dieudonné, Treatise on Analysis Vol. VI 1978

Dieudonné, A panorama of pure mathematics 1982

I.G. Macdonald as translator: Dieudonné

Dieudonné, Foundations of Modern Analysis 1960 and 1969, xiv+361 pp.

Dieudonné, Treatise on Analysis Vol. II 1970 and 1976, xviii+387 pp.

Dieudonné, Treatise on Analysis Vol. III 1972, xvii+388 pp.

Dieudonné, Treatise on Analysis Vol. IV 1974, xiv+444 pp.

Dieudonné, Treatise on Analysis Vol. V 1977, xii+243 pp.

Dieudonné, Treatise on Analysis Vol. VI 1978, xi+239 pp.

Dieudonné, A panorama of pure mathematics 1982, x+289 pp.

For students

I.G. Macdonald for my students

Every once in a while, not infrequently, a student comes by my office and says
"I'd like to learn about **Lie groups**,
do you have a reference that you can recommend?"

I usually find myself saying, "How about the notes of Macdonald?"

Algebraic structure of Lie groups, Cambridge University Press, 1980.
https://doi.org/10.1017/CBO9780511662683.005

I.G. Macdonald for my students

Every once in a while, not infrequently, a student comes by my office and says

"I'd like to learn about **algebraic groups**,

do you have a reference that you can recommend?"

I usually find myself saying, "How about the notes of Macdonald?"

Linear algebraic groups, in Lectures on Lie Groups and Lie Algebras,

Cambridge University Press 1995. https://doi.org/10.1017/CBO9781139172882

I.G. Macdonald for my students

Every once in a while, not infrequently, a student comes by my office and says

“I'd like to learn about **reflection groups**,

do you have a reference that you can recommend?”

I usually find myself saying, “How about the notes of Macdonald?”

Reflection groups, unpublished notes 1991.

Available at http://math.soimeme.org/~arunram/resources.html

I.G. Macdonald for my students

Every once in a while, not infrequently, a student comes by my office and says
“I'd like to learn about **algebraic geometry**,
do you have a reference that you can recommend?”

I usually find myself saying, “How about the notes of Macdonald?”

Algebraic Geometry - Introduction to schemes,
published by W.A. Benjamin 1968.
Available at http://math.soimeme.org/~arunram/resources.html

I.G. Macdonald for my students

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **Haar measure, spherical functions and harmonic analysis**, do you have a reference that you can recommend?"

I usually find myself saying, "How about the book of Macdonald?"

Spherical functions on a group of *p*-adic type,
University of Madras 1971.
Available at http://math.soimeme.org/~arunram/resources.html

I.G. Macdonald for my students

Every once in a while, not infrequently, a student comes by my office and says

“I'd like to learn about **Kac-Moody Lie algebras,**

do you have a reference that you can recommend?”

I usually find myself saying, “How about the notes of Macdonald?”

Kac-Moody Lie algebras, unpublished notes 1983.

Available at http://math.soimeme.org/~arunram/resources.html

I.G. Macdonald for my students

Every once in a while, not infrequently, a student comes by my office and says
“I'd like to learn about **flag varieties and Schubert varieties,**
do you have a reference that you can recommend?”

I usually find myself saying, “How about the notes of Macdonald?”

Notes on Schubert polynomials: Appendix: Schubert varieties.
Published by LACIM 1991,
available at http://math.soimeme.org/~arunram/resources.html

Books

I.G. Macdonald as an author of books

"If you see a gap in the literature, write a book to fill it."— I.G. Macdonald

Atiyah-Macdonald, Introduction to commutative algebra 1969

Spherical functions on a group of p-adic type 1971

Symmetric functions and Hall polynomials First Edition 1979

Kac-Moody Lie algebras: unpublished notes 1983

Hypergeometric functions: unpublished notes 1987

Reflection groups: unpublished notes 1991

I.G. Macdonald as an author of books

"*If you see a gap in the literature, write a book to fill it.*"— I.G. Macdonald

Atiyah-Macdonald, Introduction to commutative algebra 1969

Spherical functions on a group of p-adic type 1971

Symmetric functions and Hall polynomials First Edition 1979

Kac-Moody Lie algebras: unpublished notes 1983

Hypergeometric functions: unpublished notes 1987

Reflection groups: unpublished notes 1991

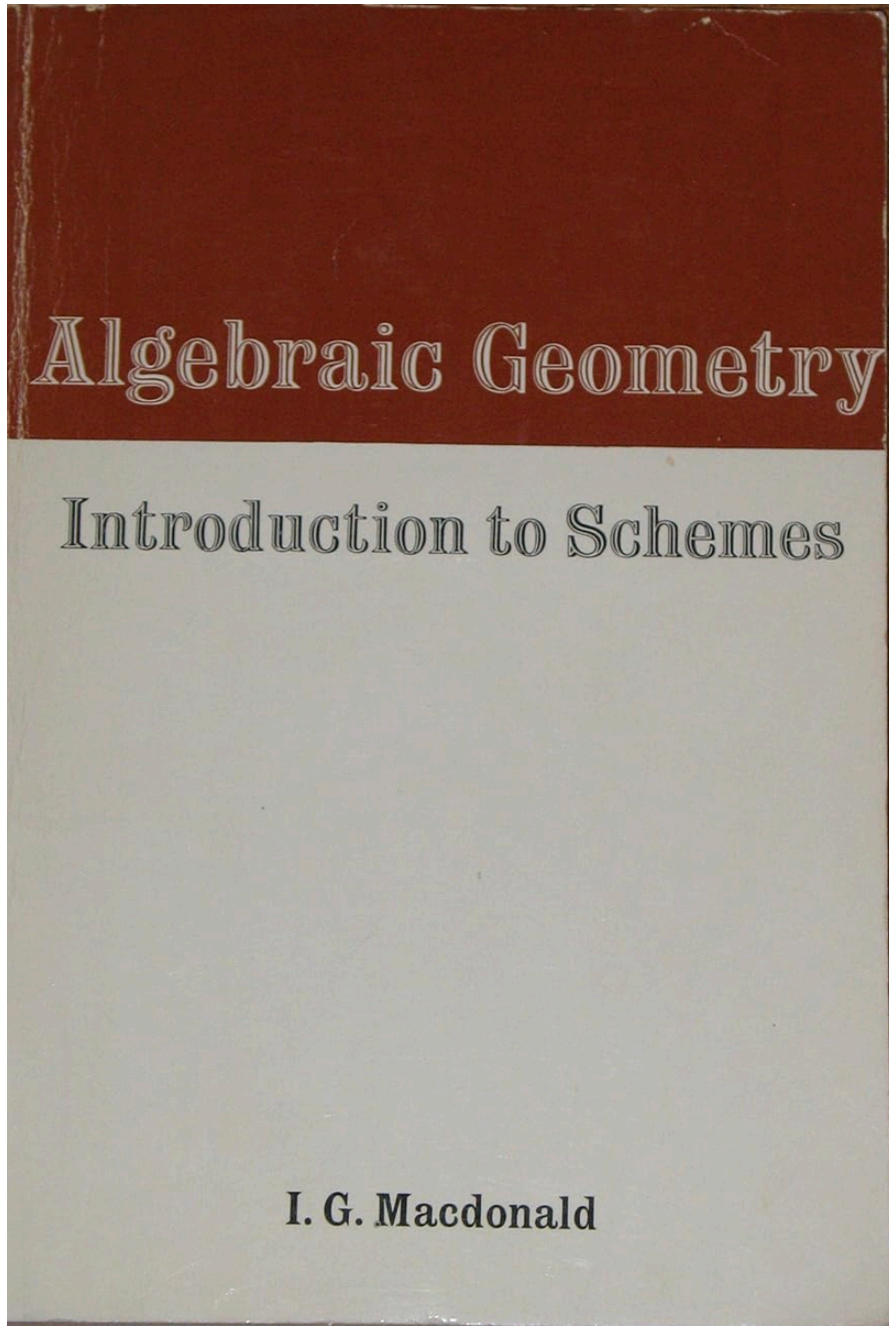
Schubert polynomials 1991

Symmetric functions and Hall polynomials Second Edition 1995

Linear algebraic groups: in Lectures on Lie groups and Lie algebras 1995

Affine Hecke algebras and orthogonal polynomials 2003

The first book: Algebraic Geometry - Introduction to Schemes 1968



“One starts out in life trying to do
something else,
and winds up doing combinatorics.”