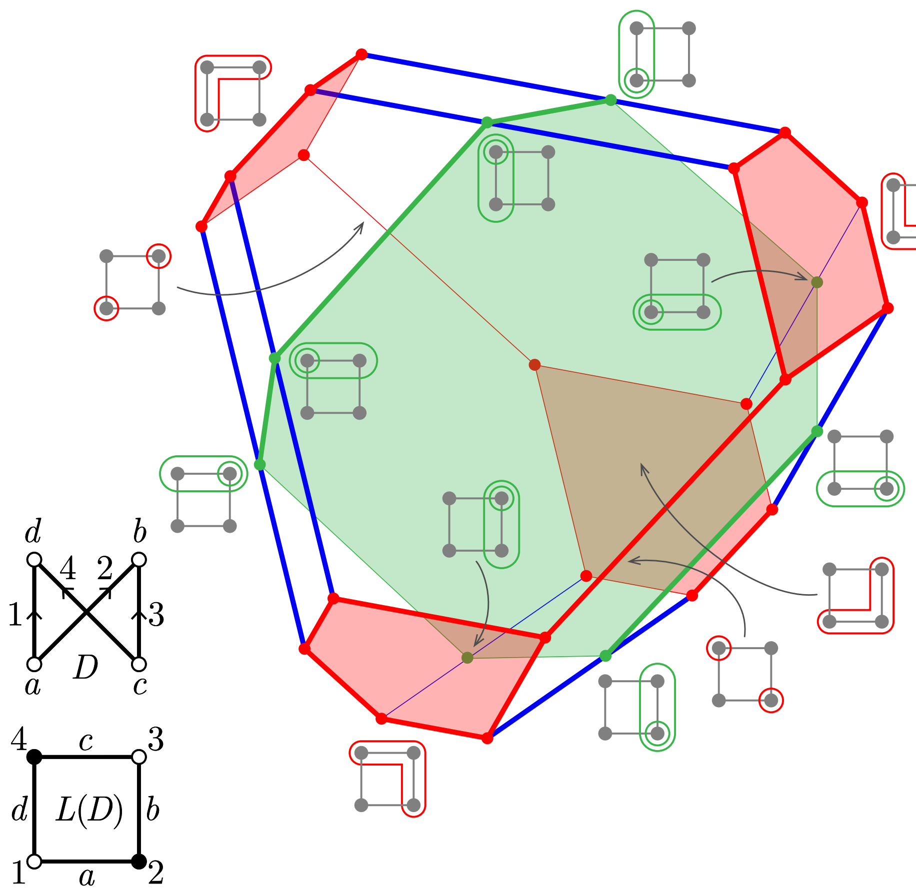
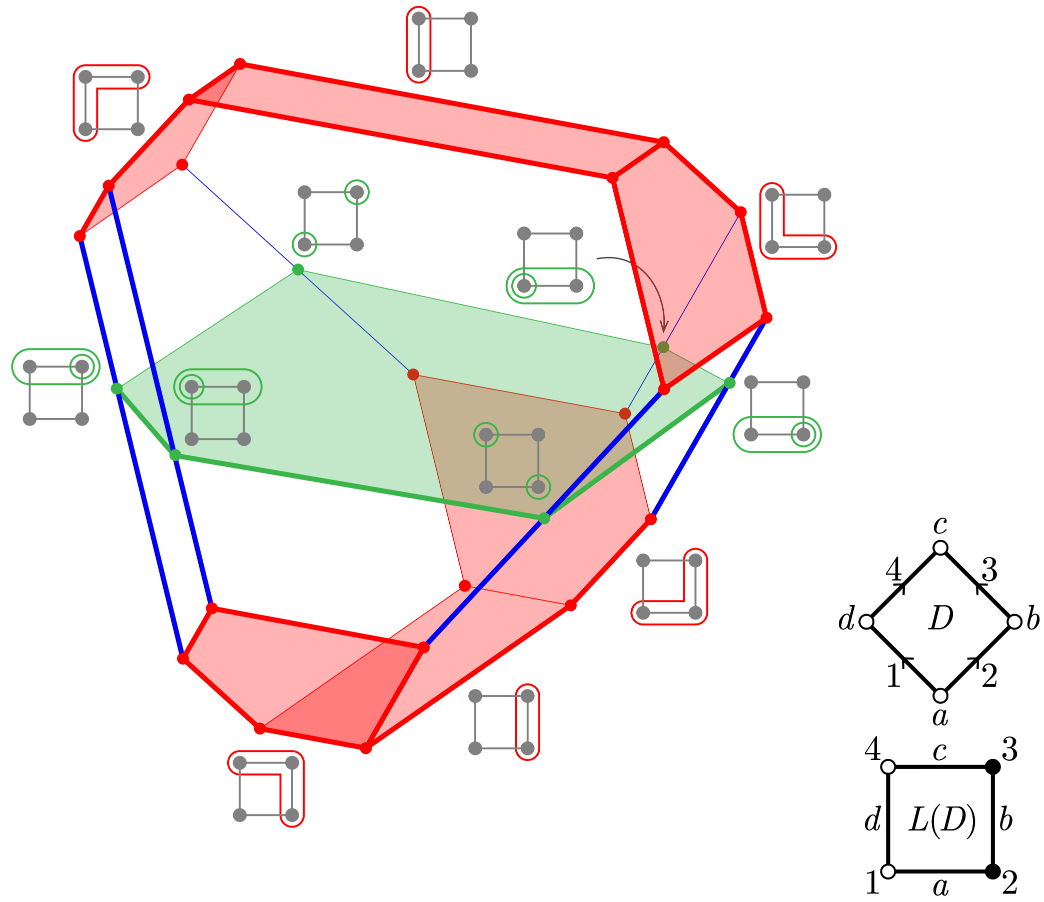
ACYCLONESTOHEDRA

Chiara MANTOVANI (École Polytechnique)
Arnau PADROL (Universitat de Barcelona)
Vincent PILAUD (Universitat de Barcelona)

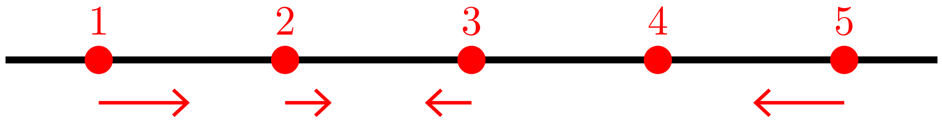




FPSAC'24, Tuesday July 23, 2024

POSET ASSOCIAHEDRA

COLLAPSING LINE



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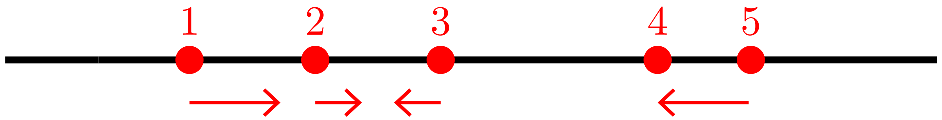

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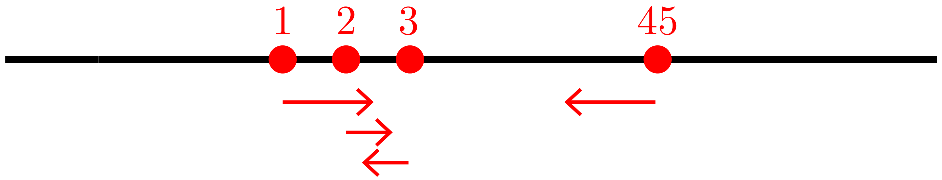

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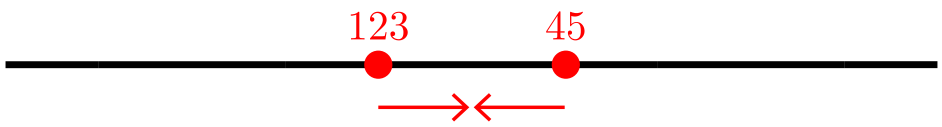

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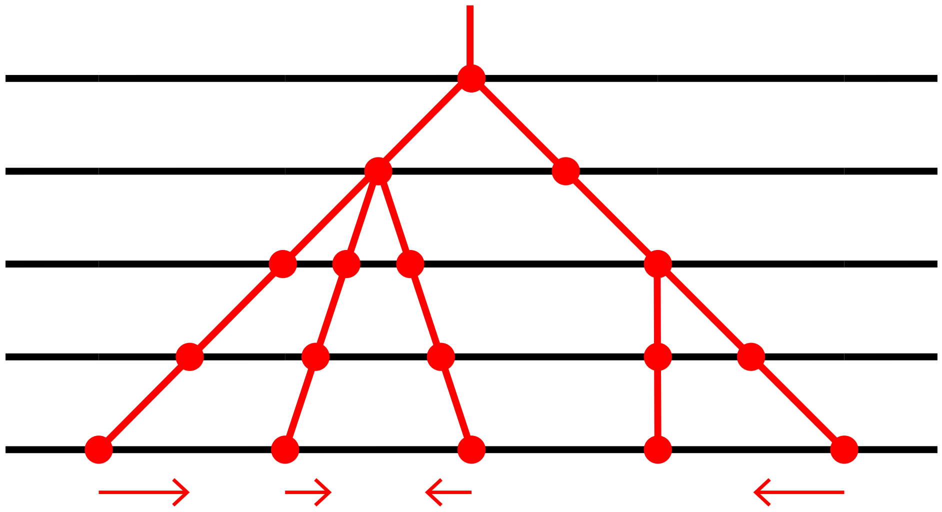
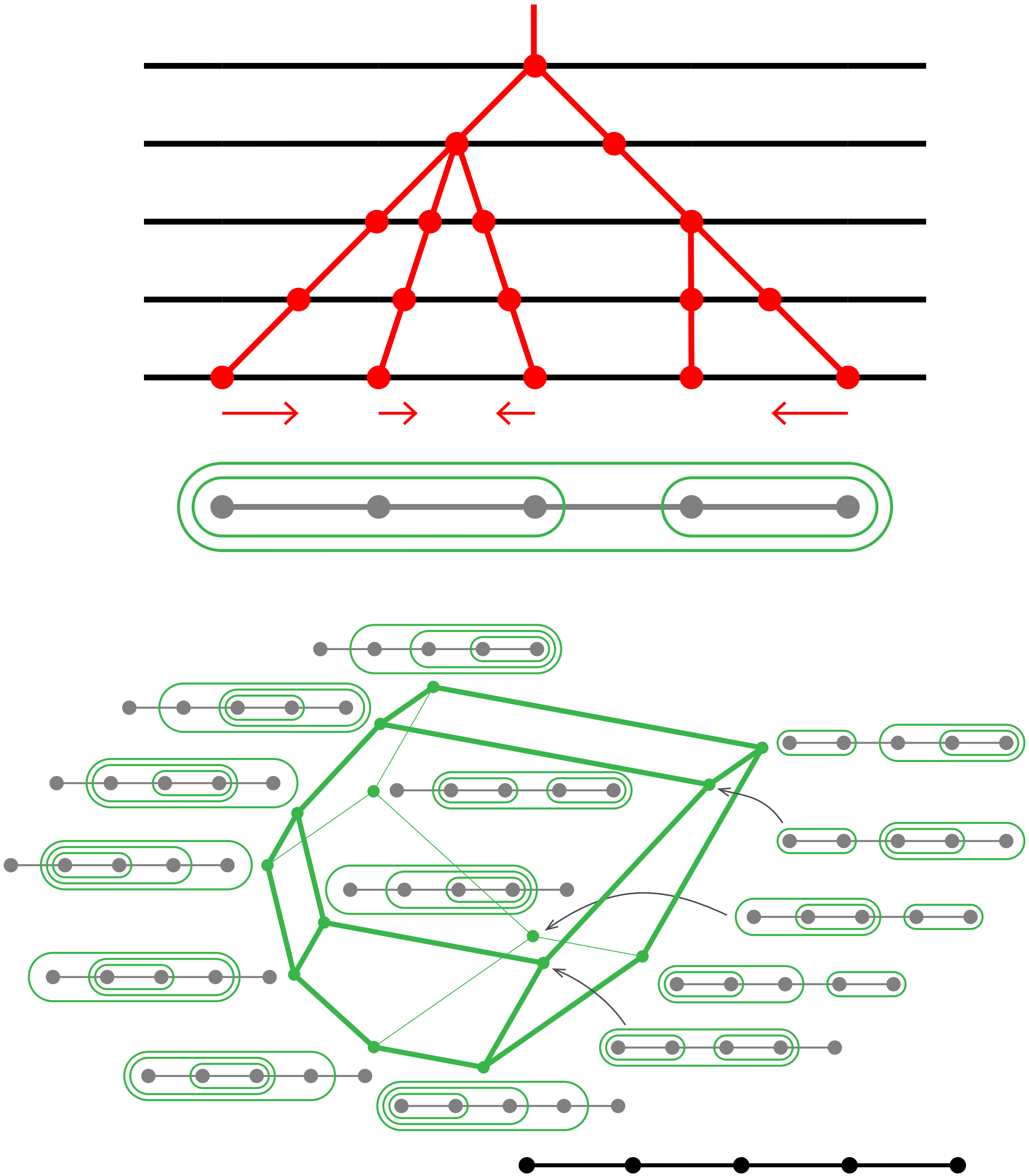
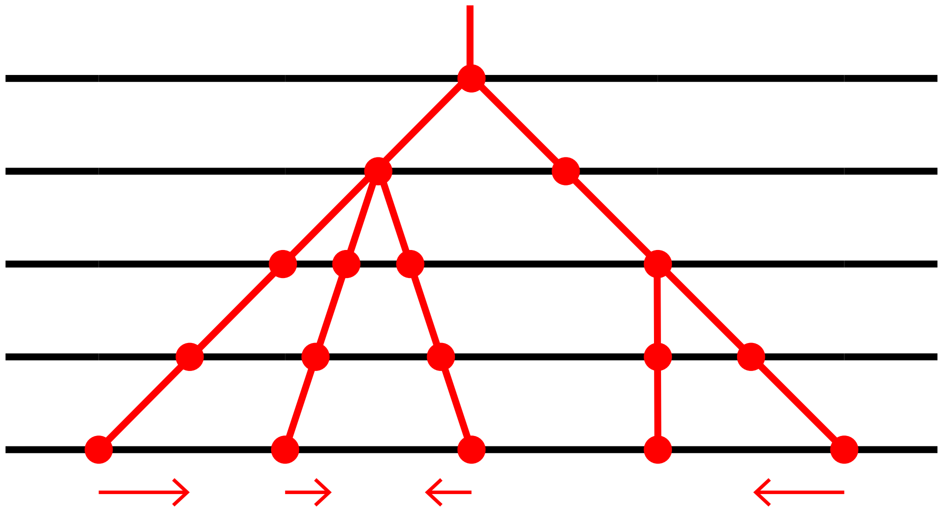
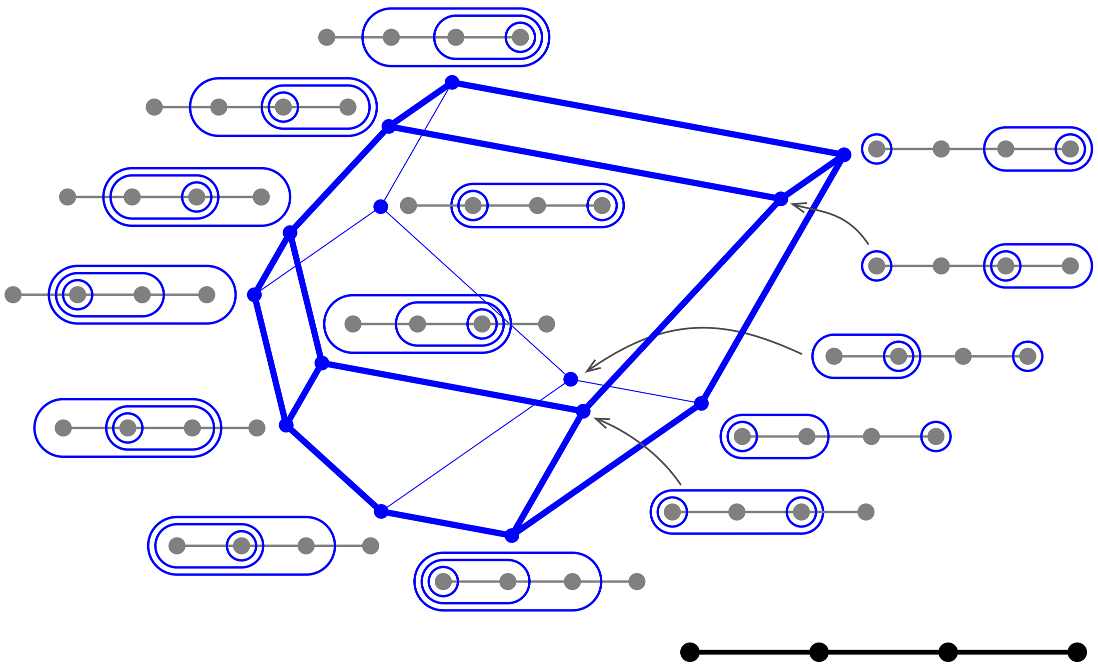


PIPINGS



TUBINGS





COLLAPSING POSET

P poset

$f : P \times [0,1] \to \mathbb{R}$ with $f(p,-)$ continuous, $f(-,t)$ order preserving, and $|f(P,1)| = 1$

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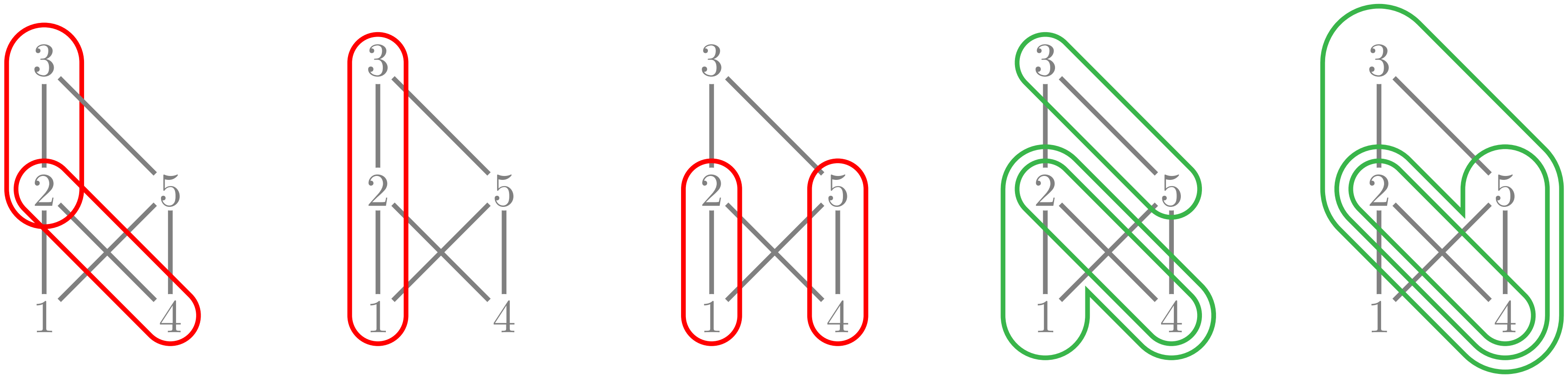

PIPING COMPLEX

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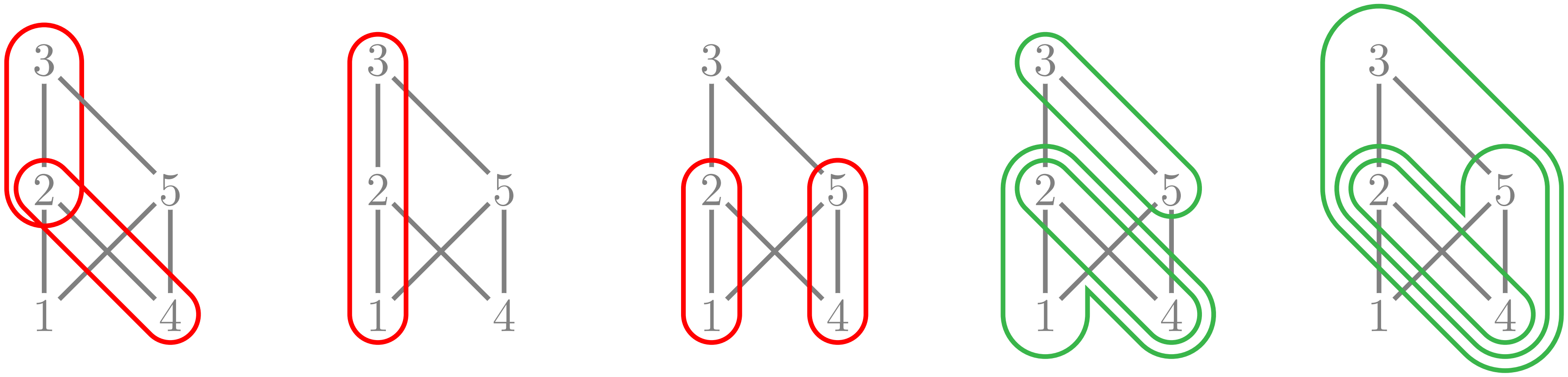

DEF. pipe of P = connected subset of P of size ≥ 2

piping of P = collection Y of pipes of P such that

- pipes are pairwise disjoints or nested
- $P_{/\bigcup X}$ acyclic for any $X \subseteq Y$

piping complex of P = simplicial complex of pipings of P

Galashin '21

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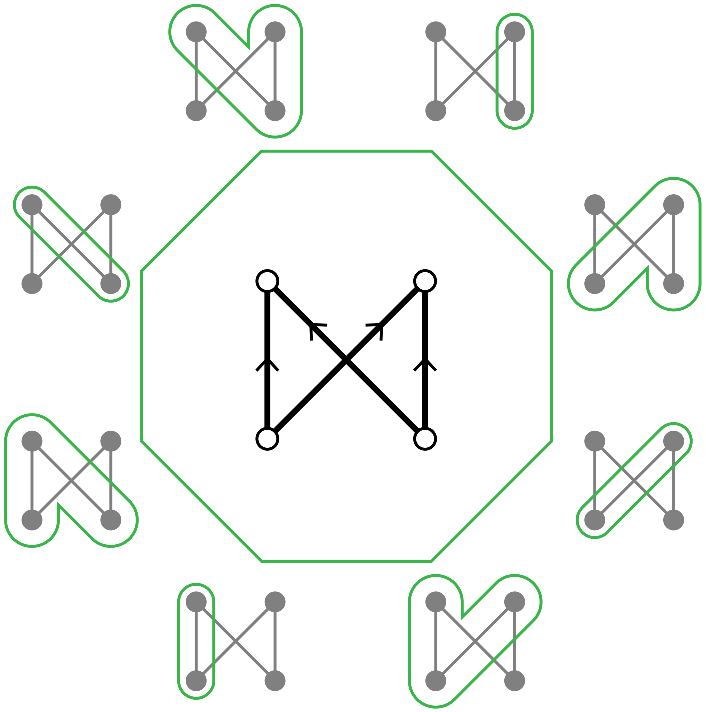

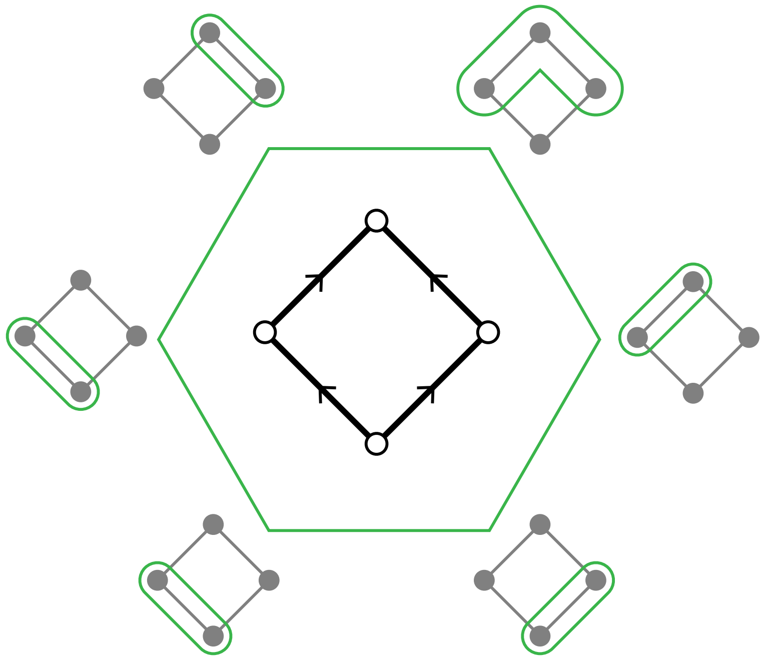

Galashin '21

POSET ASSOCIAHEDRON

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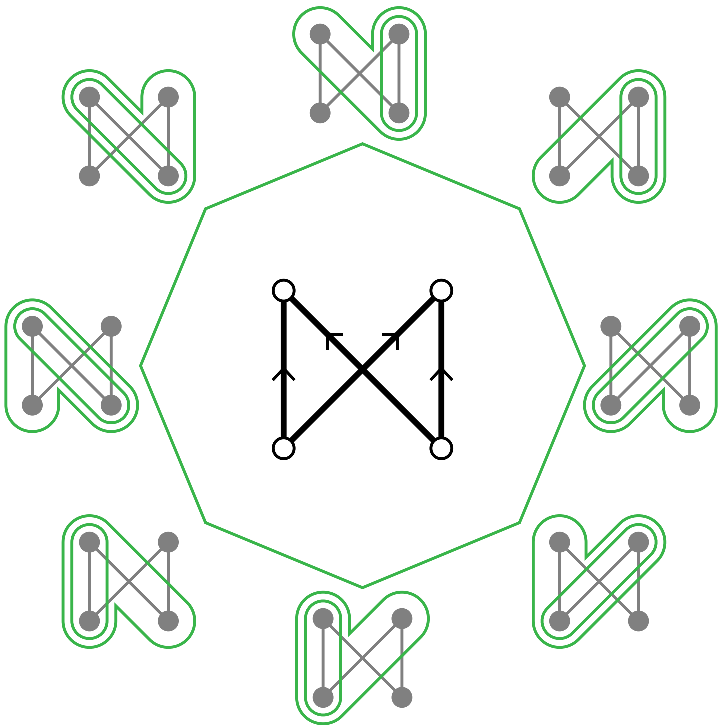
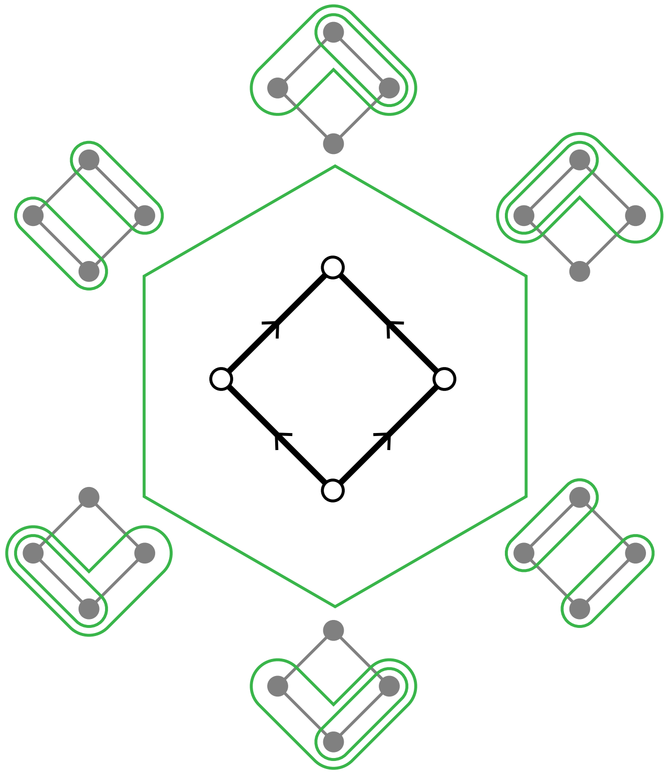
P-associahedron = simple polytope whose polar is the piping complex of P



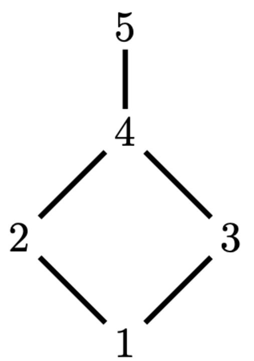


Galashin '21

POSET ASSOCIAHEDRON

THM. P-associahedra can be obtained by <u>truncations</u> of the <u>order polytope</u> of P

Galashin '21



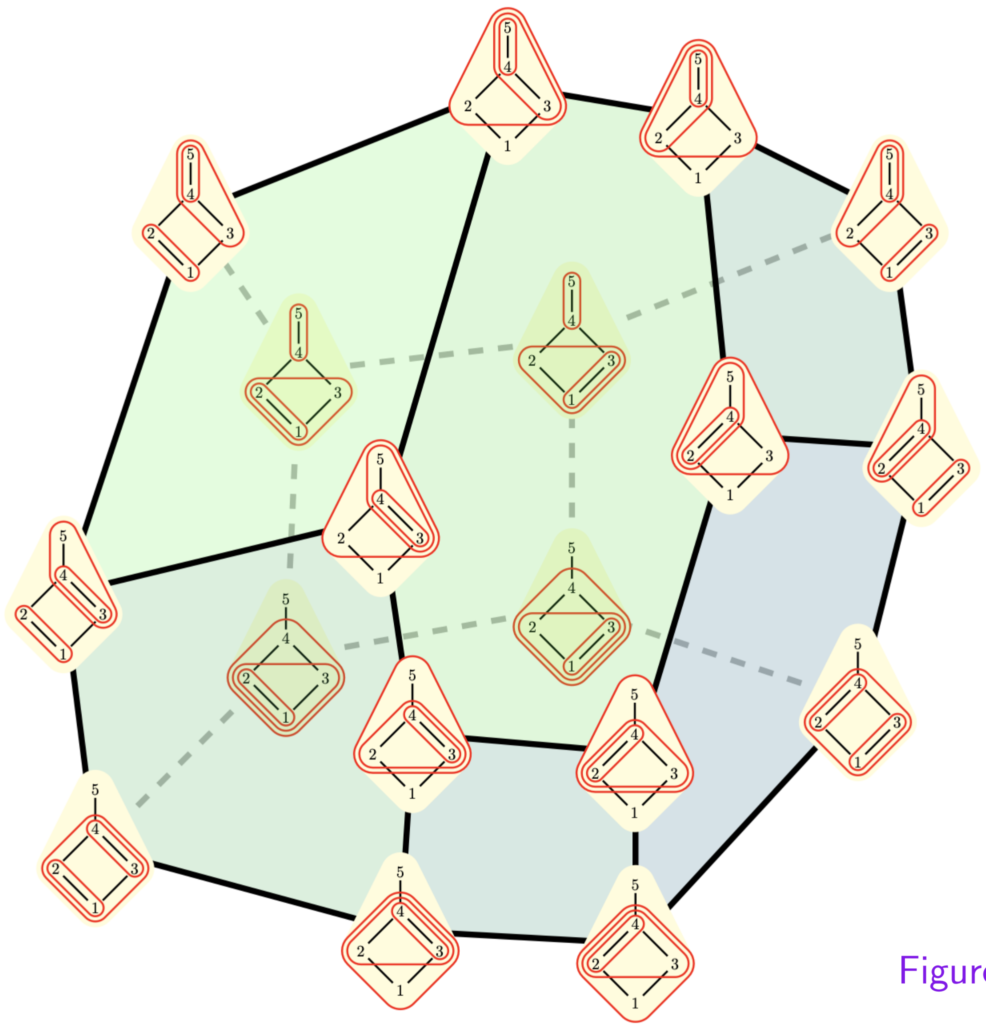


Figure from Galashin '21

QU. Find nice realizations

POSET ASSOCIAHEDRON

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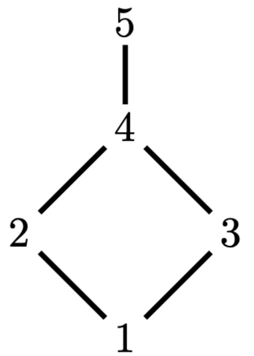


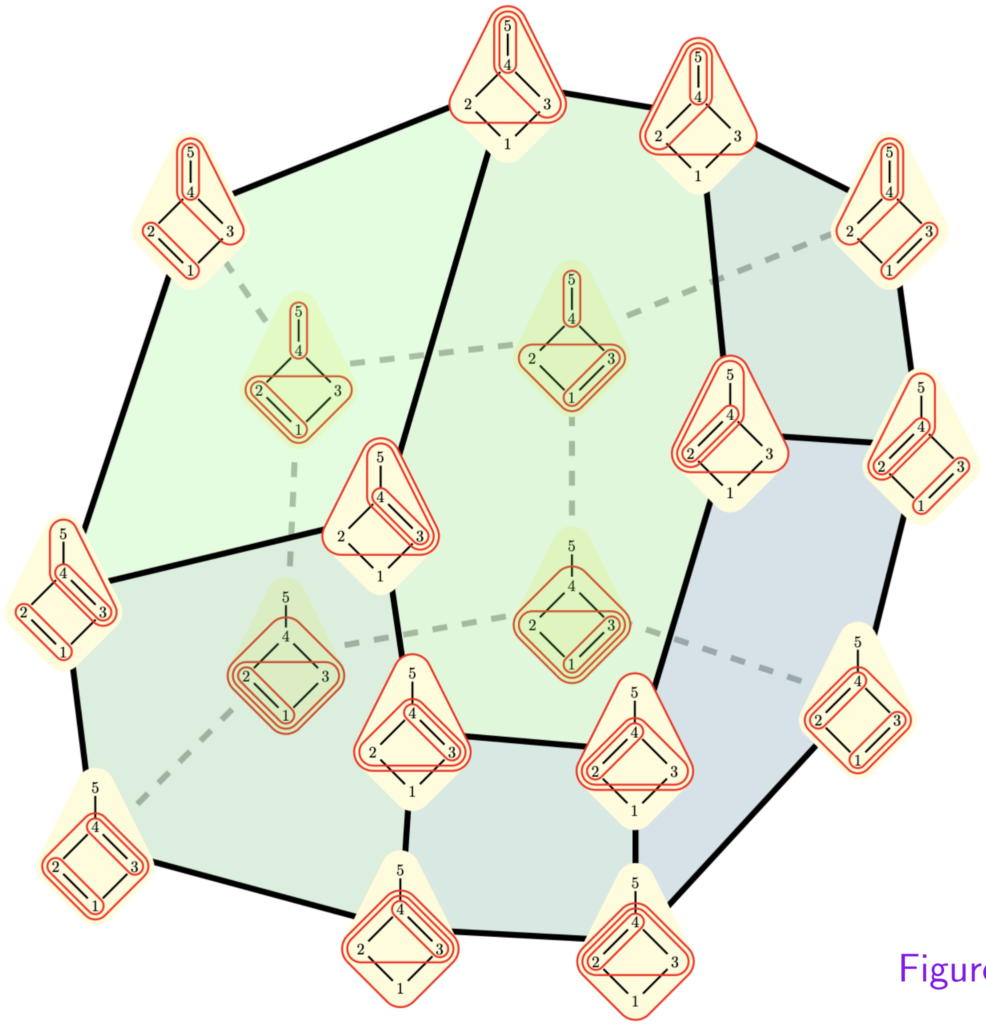
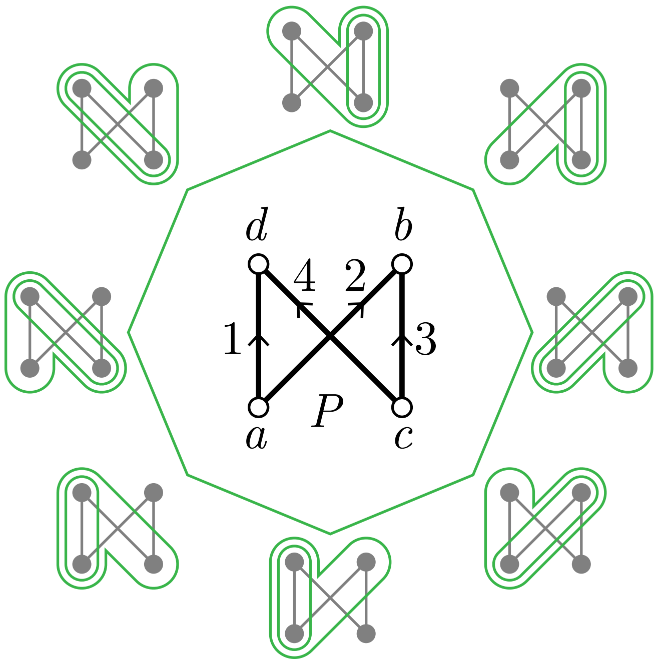


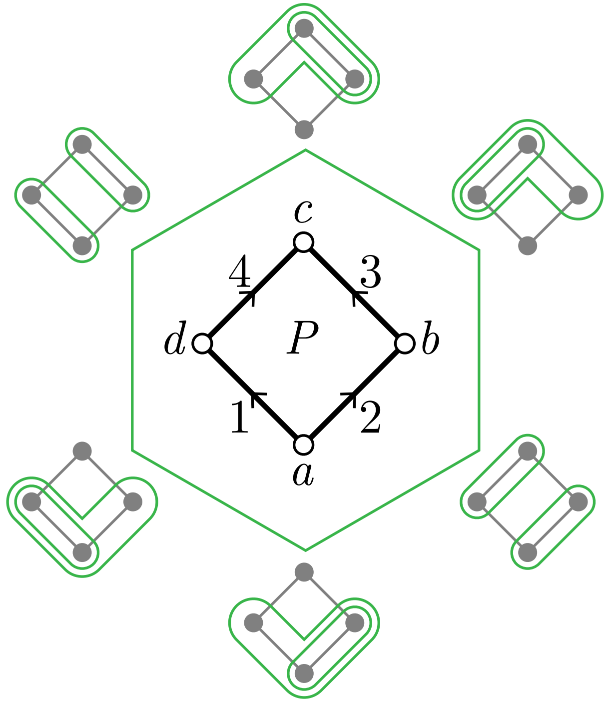
Figure from Galashin '21

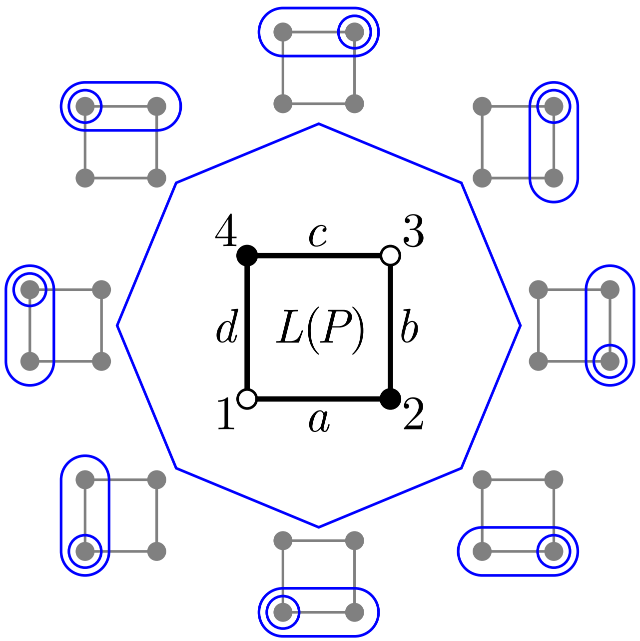
QU. Find nice realizations

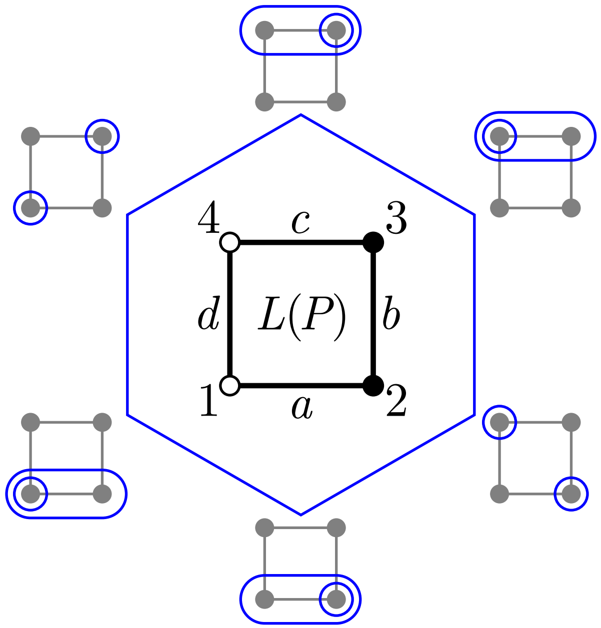
Sack '23 Mantovani–Padrol–P. '23^{+}

POSET ASSOCIAHEDRON



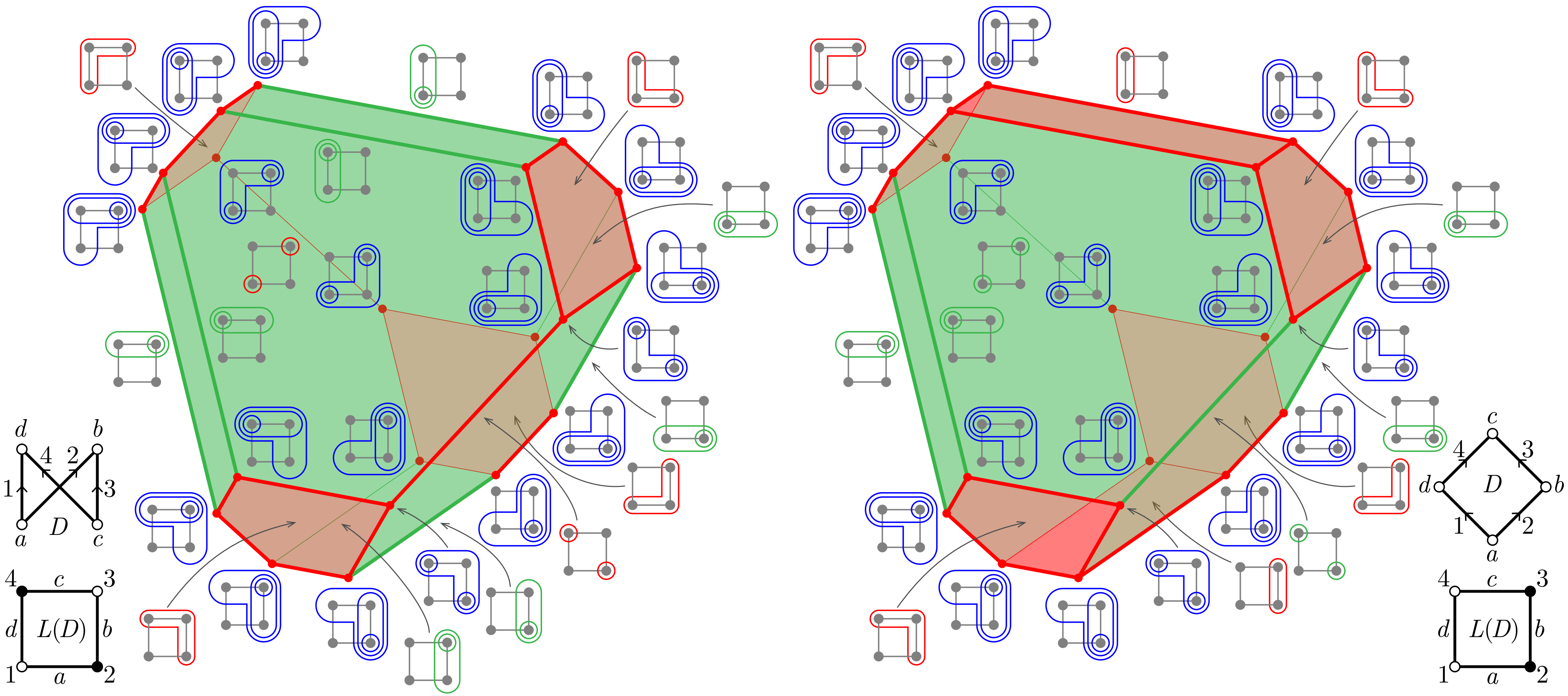






POSET ASSOCIAHEDRON

OBS. The acyclic part of the nested complex of $L(P)$ is the piping complex of P



Mantovani–Padrol–P. '23^{+}

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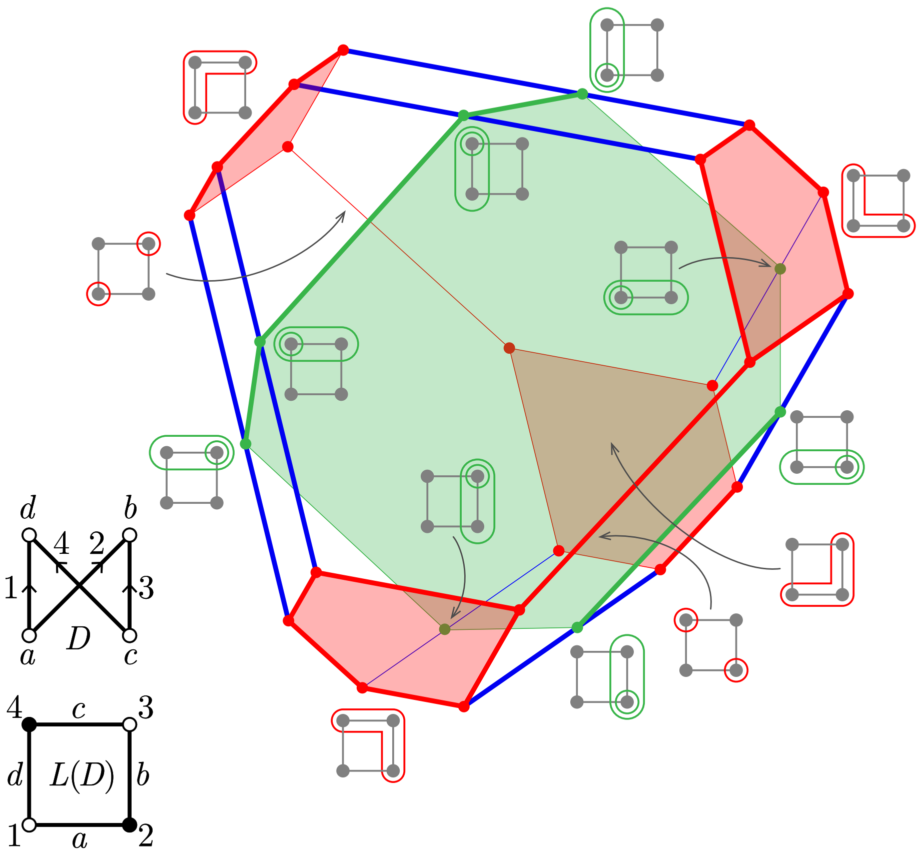
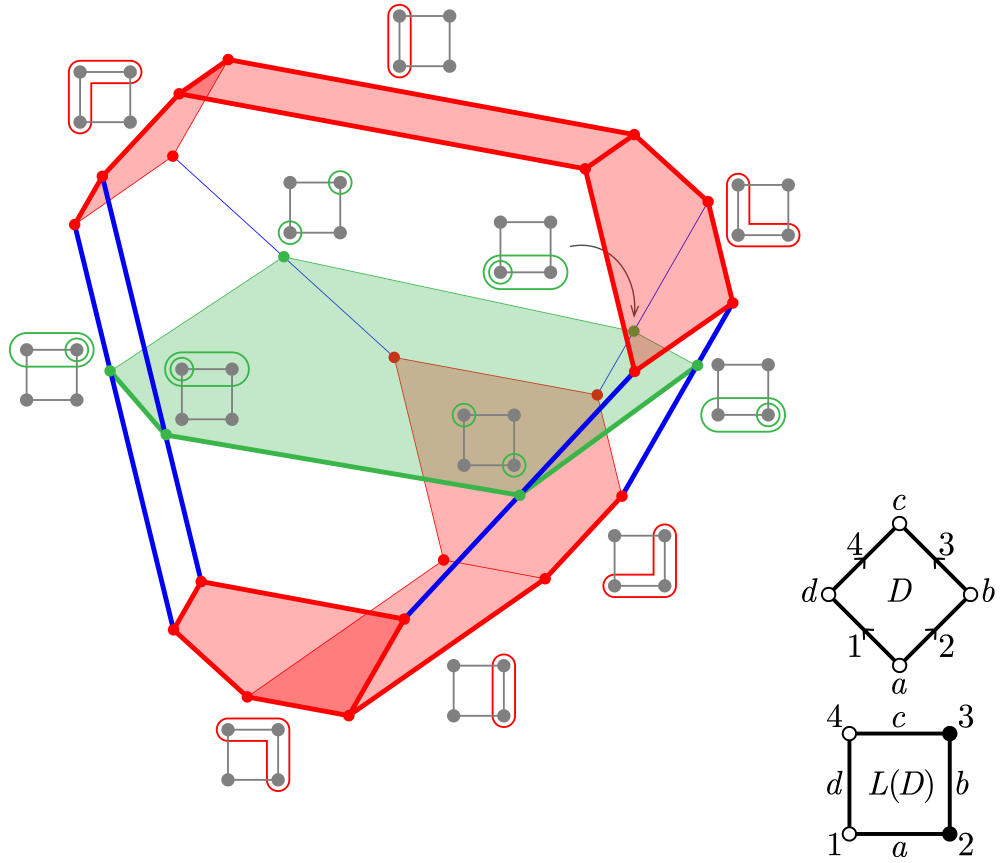
THM. A section of an $L(P)$-associahedron is a P-associahedron





Mantovani–Padrol–P. ’23^{+}

NESTOHEDRA

NESTOHEDRA

DEF. <u>building set</u> on S = collection $\mathcal{B}$ of non-empty subsets of S such that

- $\mathcal{B}$ contains all singletons $\{s\}$ for $s \in S$
- if $B, B' \in \mathcal{B}$ and $B \cap B' \neq \varnothing$, then $B \cup B' \in \mathcal{B}$

$\kappa(\mathcal{B})$ = connected components of $\mathcal{B}$ = inclusion maximal elements of $\mathcal{B}$

Feichtner–Koslov '04, Feichtner–Sturmfels '05, Postnikov '09, Zelevinski '06

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DEF. nested set on $\mathcal{B}$ = subset $\mathcal{N}$ of $\mathcal{B} \smallsetminus \kappa(\mathcal{B})$ such that

- for any $B, B' \in \mathcal{N}$, either $B \subseteq B'$ or $B' \subseteq B$ or $B \cap B' = \varnothing$
- for any $k \geq 2$ pairwise disjoint $B_1, \dots, B_k \in \mathcal{N}$, the union $B_1 \cup \dots \cup B_k$ is not in $\mathcal{B}$

nested complex of $\mathcal{B}$ = simplicial complex of nested sets on $\mathcal{B}$

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THM. The nested complex of $\mathcal{B}$ is isomorphic to the boundary complex of the polar of the nestohedron $\sum_{B \in \mathcal{B}} \lambda_B \triangle_B$ where

- $\triangle_B := \operatorname{conv}\{\boldsymbol{e}_b \mid b \in B\}$ face of the standard simplex $\triangle_S = \operatorname{conv}\{\boldsymbol{e}_s \mid s \in S\}$
- λ_B arbitrary strictly positive coefficients

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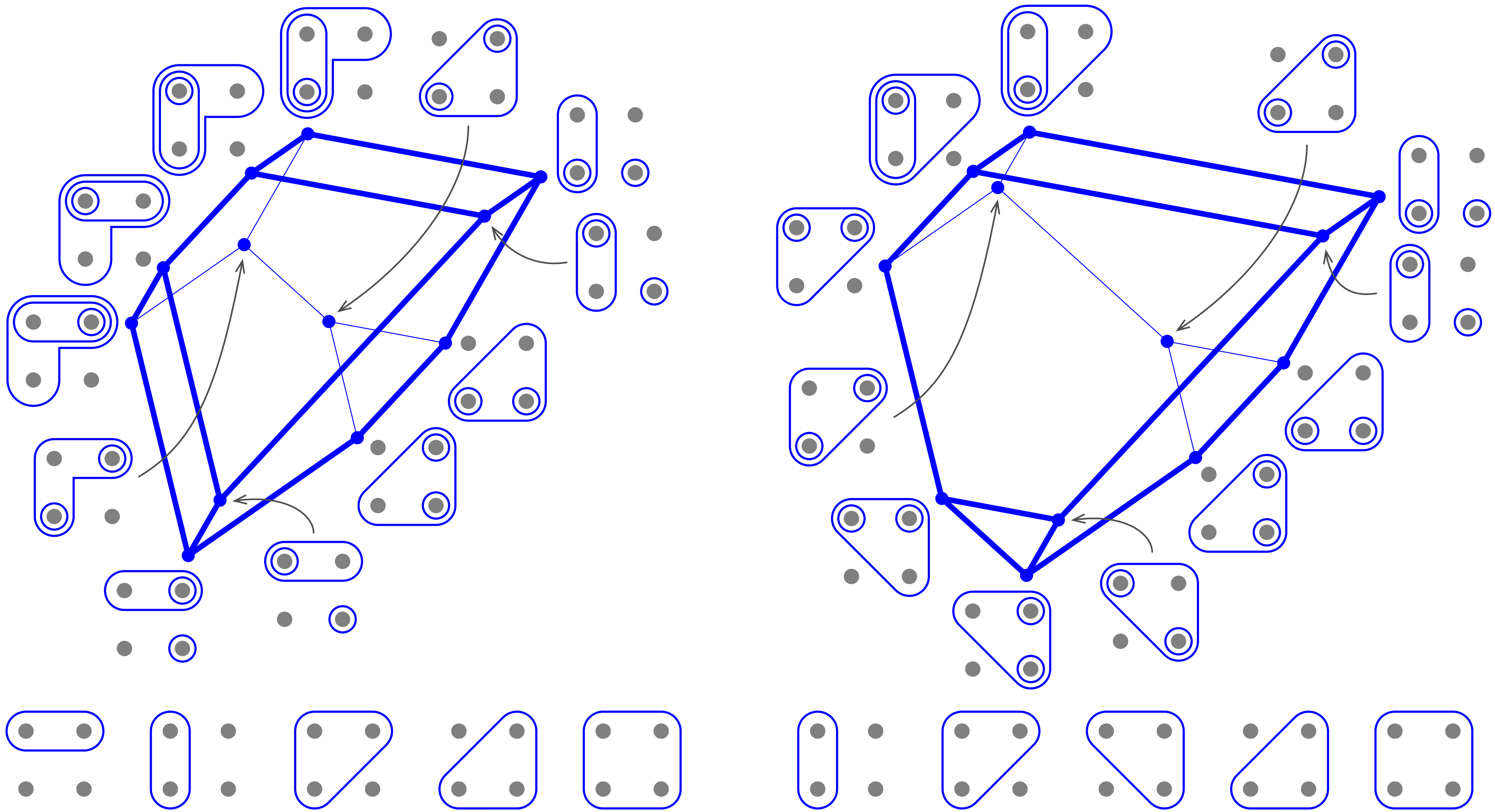

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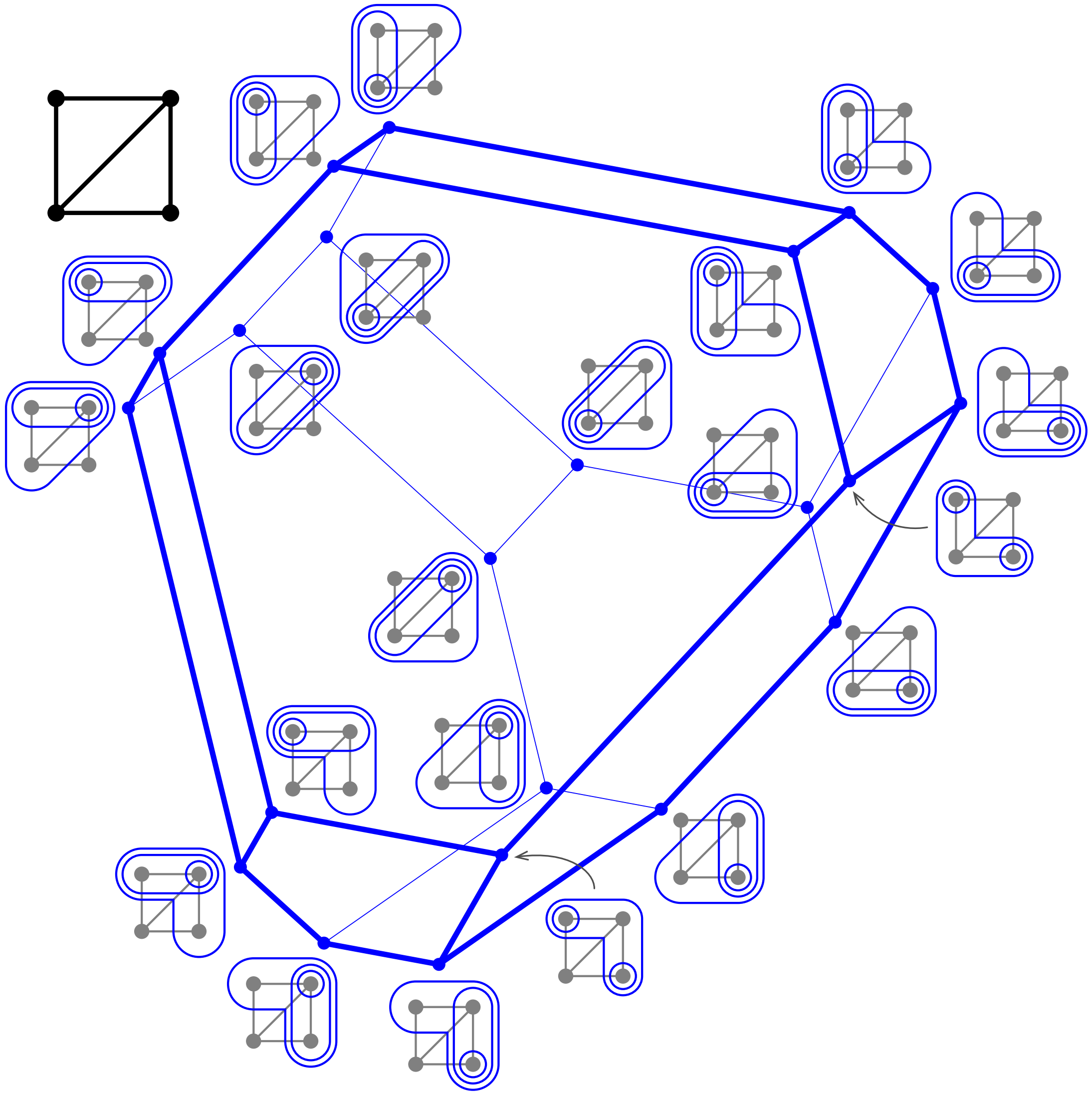
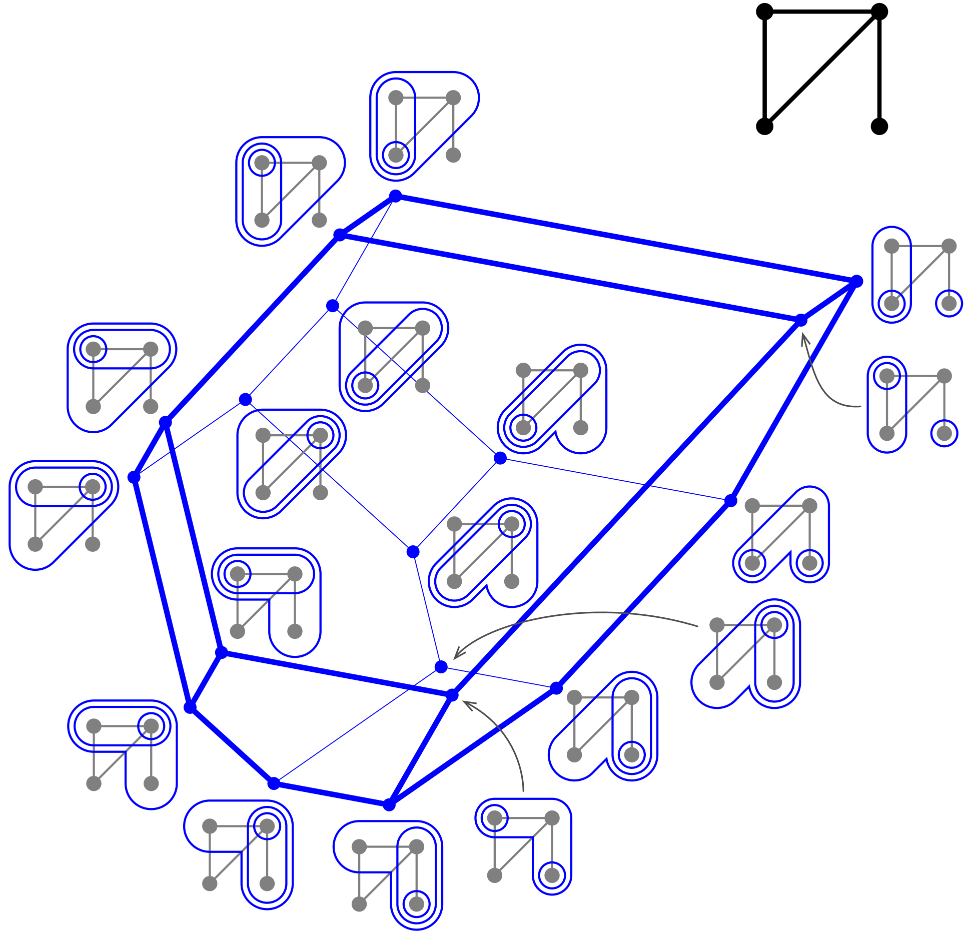
GRAPHICAL NESTOHEDRA

EXM. graphical building set of G = collection of all tubes of G
graphical nested set of G = simplicial complex of tubings on G





Carr–Devadoss '06

ORIENTED MATROIDS

DEF. vector configuration $\boldsymbol{A} = (\boldsymbol{a}_s)_{s \in S}$ with $\boldsymbol{a}_s \in \mathbb{R}^d$

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dependence space $\mathcal{D}(\boldsymbol{A}) = \left\{ \boldsymbol{\delta} \in \mathbb{R}^S \;\middle|\; \sum_{s \in S} \delta_s \boldsymbol{a}_s = \boldsymbol{0} \right\}$

evaluation space $\mathcal{D}^*(\boldsymbol{A}) = \left\{ (f(\boldsymbol{a}_s))_{s \in S} \in \mathbb{R}^S \;\middle|\; f \in (\mathbb{R}^d)^* \right\}$

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oriented matroid $\mathcal{M}(\boldsymbol{A})$ = combinatorial data given by any of the following

- vectors $\mathcal{V}(\boldsymbol{A})$ = signatures of linear dependences of $\boldsymbol{A}$
- covectors $\mathcal{V}^*(\boldsymbol{A})$ = signatures of linear evaluations on $\boldsymbol{A}$
- circuits $\mathcal{C}(\boldsymbol{A})$ = support minimal signatures of linear dependences of $\boldsymbol{A}$
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$$\boldsymbol{A}_\circ = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^4.$$

has 13 vectors, 153 covectors, 6 circuits, and 14 cocircuits
$\mathcal{C}(\boldsymbol{A}_\circ) = \{(1,2), (16,45), (26,45),$ and their opposites$\}$
$\mathcal{C}^*(\boldsymbol{A}_\circ) = \{(12,6), (124,\varnothing), (125,\varnothing), (3,\varnothing), (46,\varnothing), (4,5), (56,\varnothing),$ and their opposites$\}$

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DEF. For $R \subseteq S$

- restriction $\mathcal{M}_{|R}$ = oriented matroid on R with circuits $\{c \in \mathcal{C}(\mathcal{M}) \mid c^+ \cup c^- \subseteq R\}$
- contraction $\mathcal{M}_{/R}$ = oriented matroid on $S \smallsetminus R$ with vectors $\{v \smallsetminus R \mid v \in \mathcal{V}(\mathcal{M})\}$

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DEF. $\mathcal{M}$ acyclic = no positive circuit

GRAPHICAL ORIENTED MATROIDS

DEF. D directed graph with vertices V and arcs S

incidence configuration $\boldsymbol{A}(D) = (\boldsymbol{e}_i - \boldsymbol{e}_j)_{(i,j)\in S}$

graphical oriented matroid $\mathcal{M}(D)$ = oriented matroid of $\boldsymbol{A}(G)$

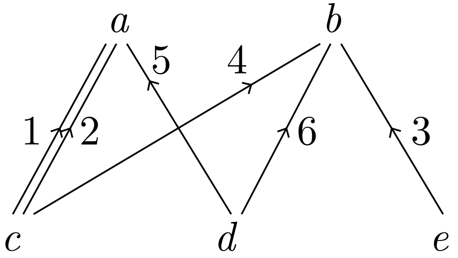
REM. $\mathcal{M}(D)$ has

- a vector v for each set of cycles, with cw arcs v_+ and ccw arcs v_-
- a covector v^* for each edge cut, with fwd arcs v^*_+ and bwd arcs v^*_-
- a circuit c for each simple cycle, with cw arcs c_+ and ccw arcs c_-
- a cocircuit c^* for each support minimal cut, with fwd arcs c^*_+ and bwd arcs c^*_-

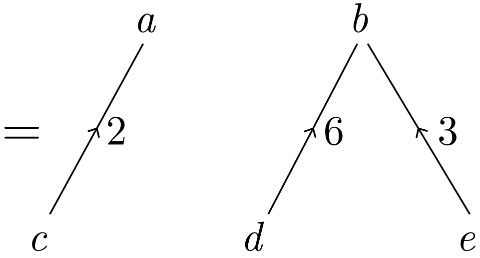
REM. For $R \subseteq S$,

- $\mathcal{M}(D)_{|R} = \mathcal{M}(D_{|R})$, where $D_{|R}$ = subgraph of D formed by the arcs in R
- $\mathcal{M}(D)_{/R} = \mathcal{M}(D_{/R})$, where $D_{/R}$ = contraction of the arcs of R in D

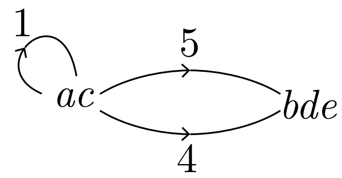
$D =$



$D_{|236} =$



$D_{/236} =$



ACYCLONESTOHEDRA

DEF. oriented building set on S = pair $(\mathcal{B}, \mathcal{M})$ where

- $\mathcal{B}$ = building set on S
- $\mathcal{M}$ = oriented matroid on S

such that $c^+ \sqcup c^-$ belongs to $\mathcal{B}$ for all circuits $c \in \mathcal{C}(\mathcal{M})$

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ACYCLONESTOHEDRA

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DEF. acyclic nested set = nested set $\mathcal{N}$ of $\mathcal{B}$ such that $\mathcal{M}_{/\bigcup \mathcal{N}'}$ is acyclic for any $\mathcal{N}' \subseteq \mathcal{N}$
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THM. For any oriented building set $(\mathcal{B}, \mathcal{M})$, the acyclic nested complex is the face lattice of an oriented matroid obtained by stellar subdivisions of $\mathcal{M}$

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ACYCLONESTOHEDRA

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THM. For any realizable oriented building set $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$, the acyclic nested complex is the boundary complex of the polar of the acyclonestohedron, defined as the section of the nestohedron $\sum_{B \in \mathcal{B}} \lambda_B \triangle_B$ with the evaluation space $\mathcal{D}^*(\boldsymbol{A})$

$\lambda_B = \left(|\mathcal{B}| \cdot \max_{c \in \mathcal{C}(\boldsymbol{A})} \frac{\max \boldsymbol{\delta}^{\neq 0}}{\min \boldsymbol{\delta}^{\neq 0}} \right)^{|B|}$ with $\boldsymbol{\delta}^{\neq 0} := \{|\delta_s| \mid s \in S\} \smallsetminus \{0\}$

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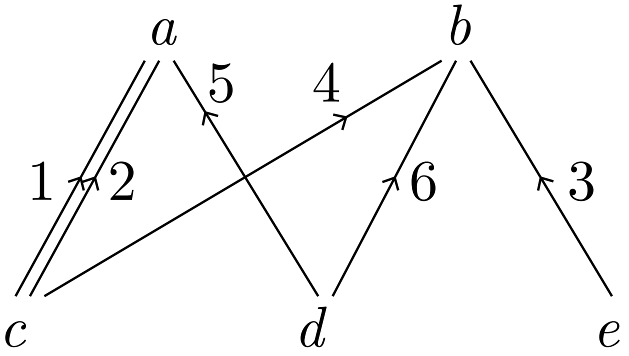
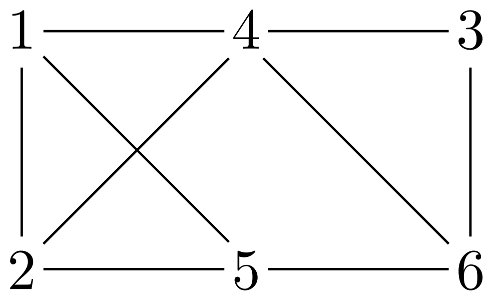
EXM. D directed graph with vertices V and arcs S

graphical oriented building set $= \big(\mathcal{B}(L(D)), \mathcal{M}(D)\big)$ where

- $\mathcal{B}(L(D))$ is the graphical building set of the line graph of D
- $\mathcal{M}(D)$ is the graphical oriented matroid of D



D



$L(D)$

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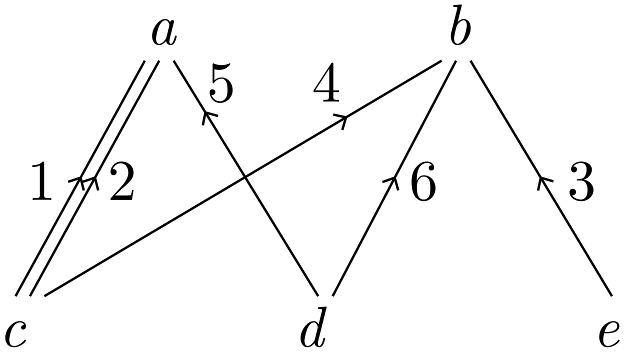
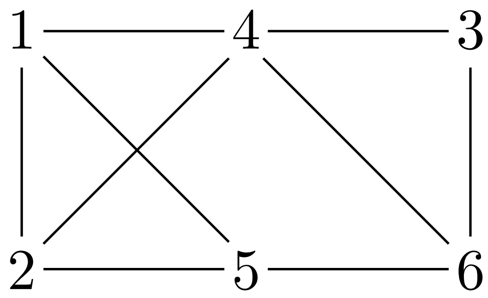


D

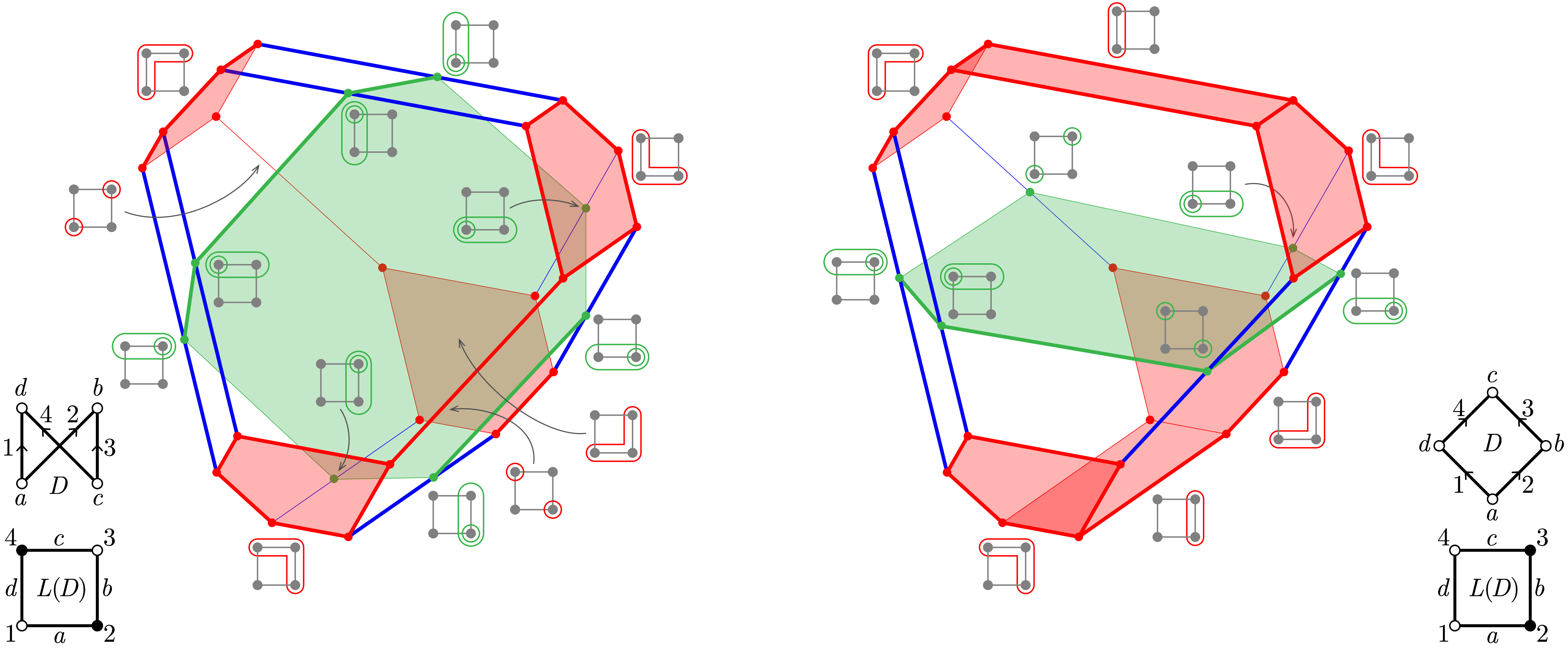


$L(D)$

PROP. The acyclic nested complex of the graphical oriented building set of D is the piping complex of the transitive closure of D

APPLICATION 1: GRAPHICAL ACYCLONESTOHEDRA

THM. The piping complex of P is the boundary complex of the polar of the graphical acyclonestohedron, defined as the section of a graph associahedron of the line graph $L(P)$ with the linear hyperplanes normal to $\mathbb{1}_{c_+} - \mathbb{1}_{c_-}$ for all circuits $c = (c_+, c_-)$ of P



Mantovani–Padrol–P. '23⁺

WHAT WE ACTUALLY DO

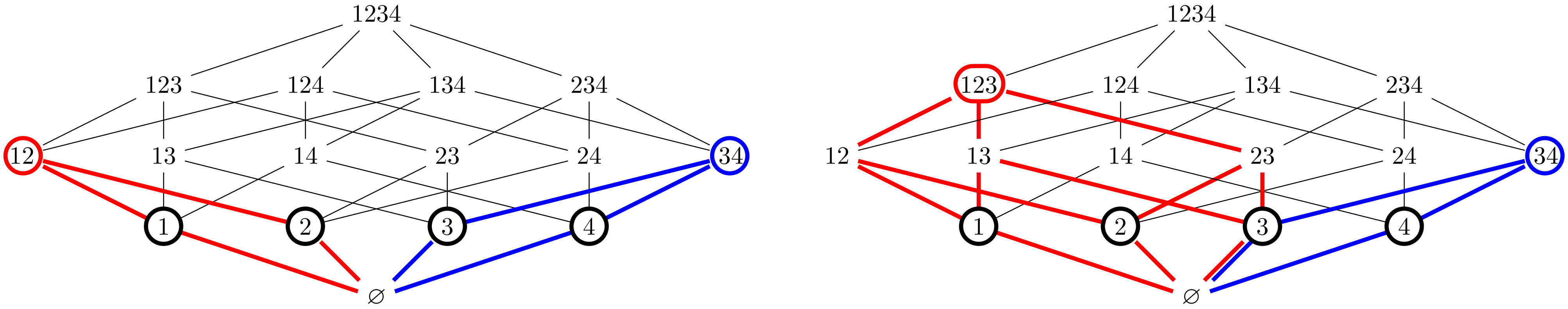
LATTICE NESTED COMPLEXES

DEF. $\mathcal{L} = (L, \leq, \vee, \wedge)$ finite lattice

$\mathcal{L}$-building set = subset $\mathcal{B}$ of $\mathcal{L}$ such that the lower interval of any element $x \in \mathcal{L}$ is the direct product of the lower intervals of the maximal elements of $\mathcal{B}$ below x

$\kappa(\mathcal{B})$ = connected components of $\mathcal{B}$ = $\mathcal{L}$ maximal elements of $\mathcal{B}$

EXM. If $\mathcal{L}$ is the boolean lattice, $\mathcal{L}$-building set $\longleftrightarrow$ classical building set



Feichtner–Kozlov '04

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DEF. $\mathcal{L}$-nested set on $\mathcal{B}$ = subset $\mathcal{N}$ of $\mathcal{B} \smallsetminus \kappa(\mathcal{B})$ such that for any $k \geq 2$ pairwise incomparable elements $B_1, \dots, B_k \in \mathcal{N}$, the join $B_1 \vee \cdots \vee B_k$ does not belong to $\mathcal{B}$

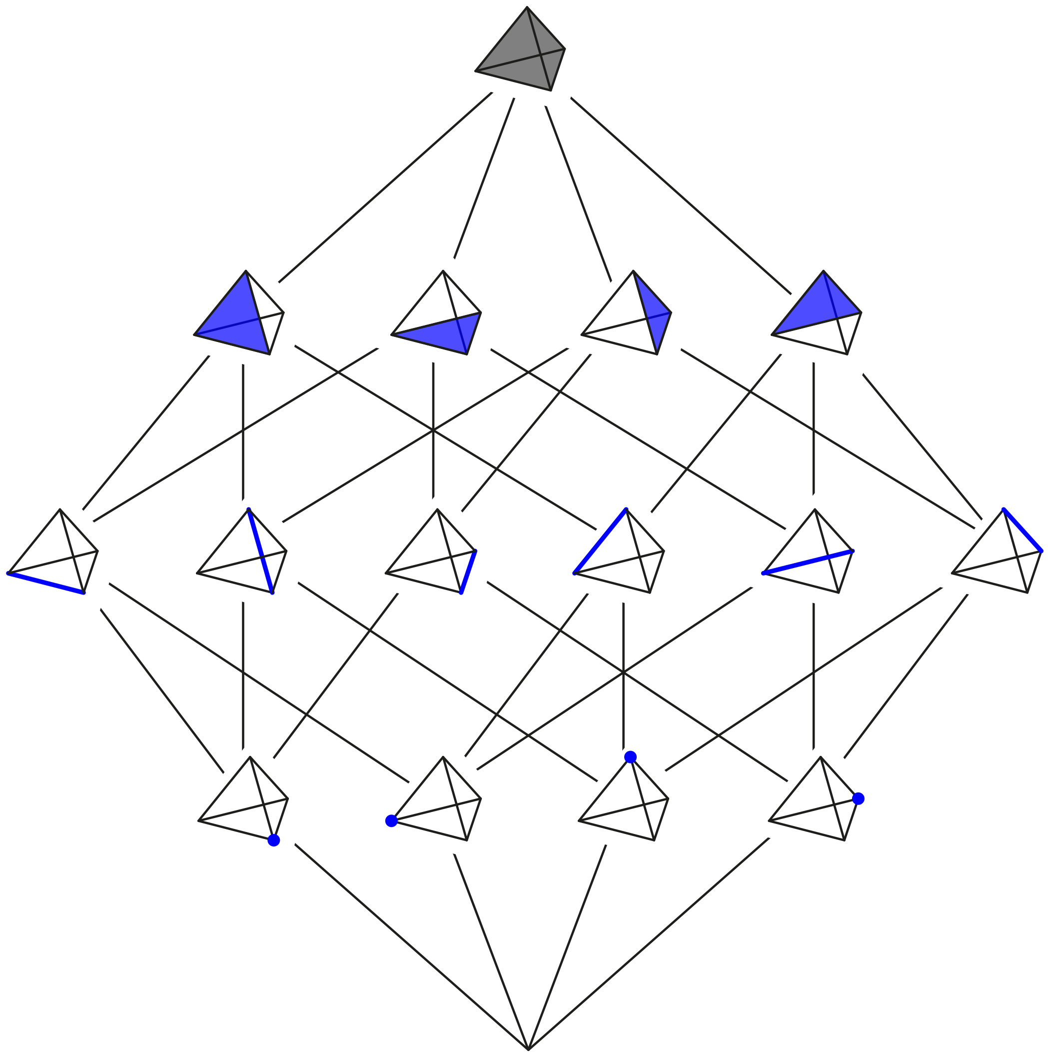
$\mathcal{L}$-nested complex of $\mathcal{B}$ = simplicial complex of $\mathcal{L}$-nested sets on $\mathcal{B}$

EXM. If $\mathcal{L}$ is the boolean lattice, $\mathcal{L}$-nested sets $\longleftrightarrow$ classical nested sets

Feichtner–Kozlov '04

LAS VERGNAS FACE LATTICE

DEF. face of $\mathcal{M}$ = subset F of S such that $(S \smallsetminus F, \varnothing) \in \mathcal{V}^*(\mathcal{M})$
(Las Vergnas) face lattice of $\mathcal{M}$ = inclusion poset on faces



FACIAL BUILDING SETS AND NESTED COMPLEXES

DEF. $(\mathcal{B}, \mathcal{M})$ oriented building set
facial building set $\widehat{\mathcal{B}}$ = set of blocks $B \in \mathcal{B}$ that are also faces of $\mathcal{M}$

THM. facial building sets of $\mathcal{M}$ $=$ $\mathcal{F}(\mathcal{M})$-building sets

THM. acyclic nested complex $(\mathcal{B}, \mathcal{M})$ $=$ $\mathcal{F}(\mathcal{M})$-nested complex of $\widehat{\mathcal{B}}$

CORO. The $\mathcal{F}(\mathcal{M})$-nested complex of any $\mathcal{F}(\mathcal{M})$-building set is the face lattice of

- an oriented matroid obtained by stellar subdivisions of $\mathcal{M}$
- a polytope, obtained either by realizing these stellar subdivisions polytopaly, or as the polar of a section of a nestohedron, when $\mathcal{M} = \mathcal{M}(\boldsymbol{A})$ is realizable

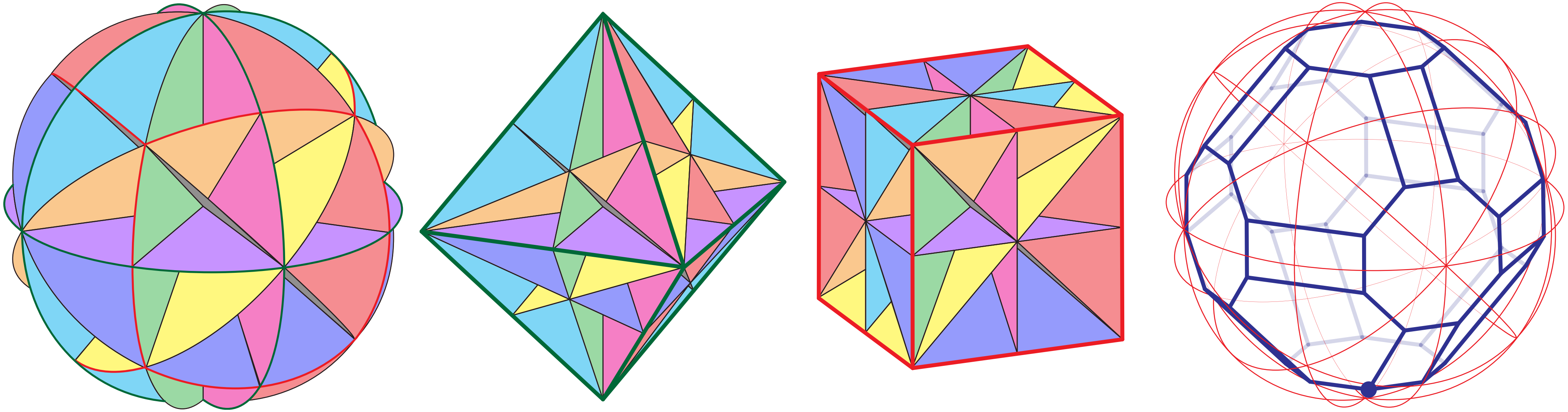
Mantovani–Padrol–P. '23^{+}

APPLICATION 2: TYPE B NESTOHEDRA

$$\boldsymbol{A}_n^\diamond = \left\{ \begin{bmatrix} \pm \boldsymbol{e}_i \\ 1 \end{bmatrix} \;\middle|\; i \in [n] \right\}$$ = homogenized vertices of n-dimensional cross-polytope

Observe that

- for the full $\mathcal{F}(\mathcal{M}(\boldsymbol{A}_n^\diamond))$-building set, the acyclonestohedron is the type B_n permutahedron, obtained as a section of the A_{2n+1} permutahedron



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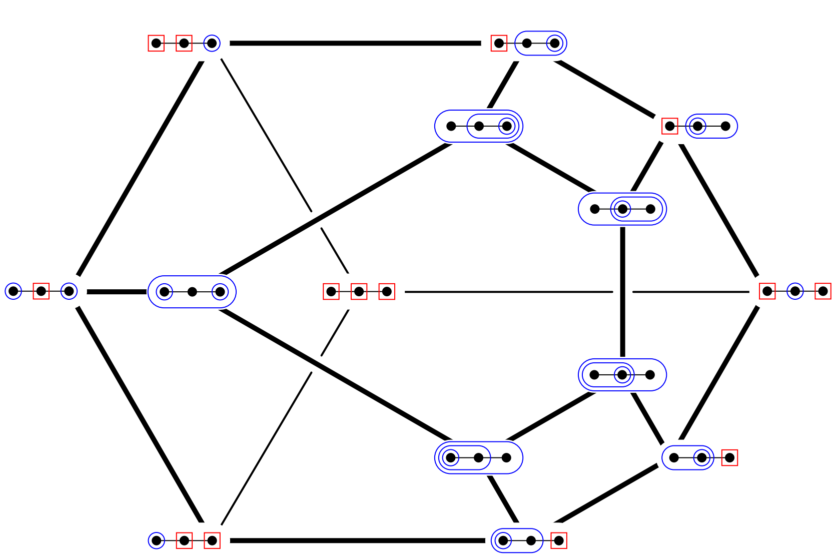
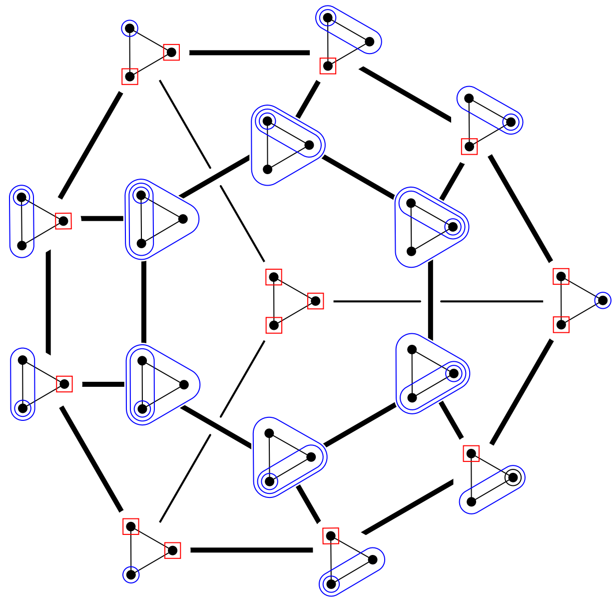
Observe that

- for the full $\mathcal{F}(\mathcal{M}(\boldsymbol{A}_n^\diamond))$-building set, the acyclonestohedron is the type B_n permutahedron, obtained as a section of the A_{2n+1} permutahedron
- any pair of classical building sets $(\mathcal{B}^+, \mathcal{B}^-)$ defines a $\mathcal{F}(\mathcal{M}(\boldsymbol{A}_n^\diamond))$-building set $\{+B^+ \mid B^+ \in \mathcal{B}^+\} \cup \{-B^- \mid B^- \in \mathcal{B}^-\} \cup \{-[n] \cup +[n]\}$

  if $\mathcal{B}^- = \{\text{singletons}\}$ we obtain design nestohedra — Devadoss–Heath–Vipismakul '11



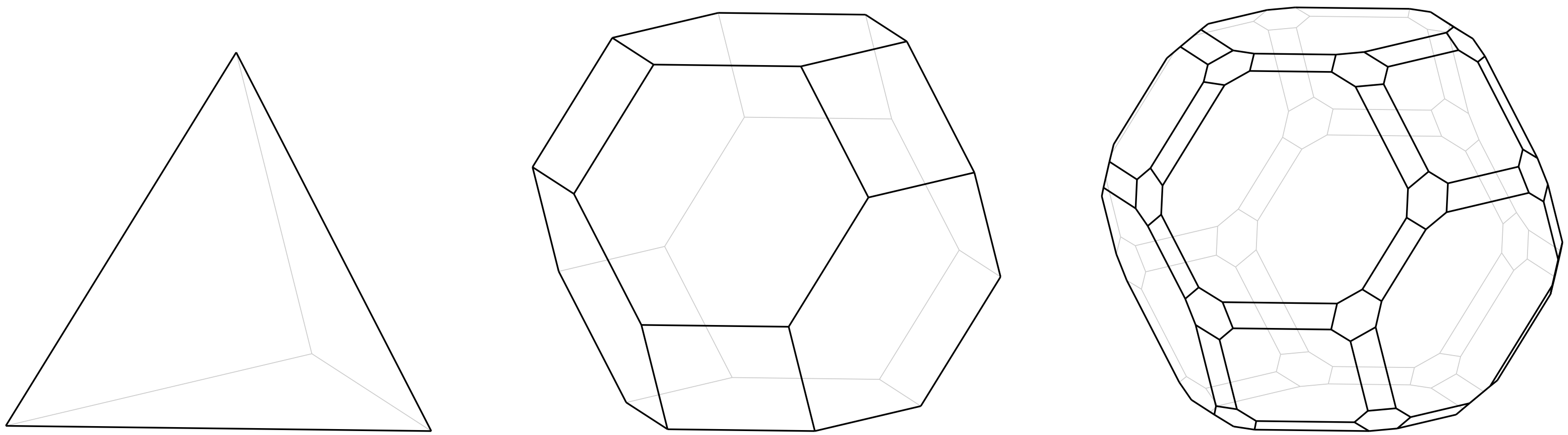


APPLICATION 3: ITERATED NESTOHEDRA

start from a polytope P_1
homogenize to an oriented matroid $\mathcal{M}_1$
choose an oriented building set $(\mathcal{B}_1, \mathcal{M}_1)$
get a polytope $P_2 =$ acyclonestohedron of $(\mathcal{B}_1, \mathcal{M}_1)$
homogenize to an oriented matroid $\mathcal{M}_2$
choose an oriented building set $(\mathcal{B}_2, \mathcal{M}_2)$
get a polytope $P_3 =$ acyclonestohedron of $(\mathcal{B}_2, \mathcal{M}_2)$
...

When starting from a simplex:

- taking the full building set at each steps leads to permuto-permuto-...-permutahedra



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Castillo–Liu '23

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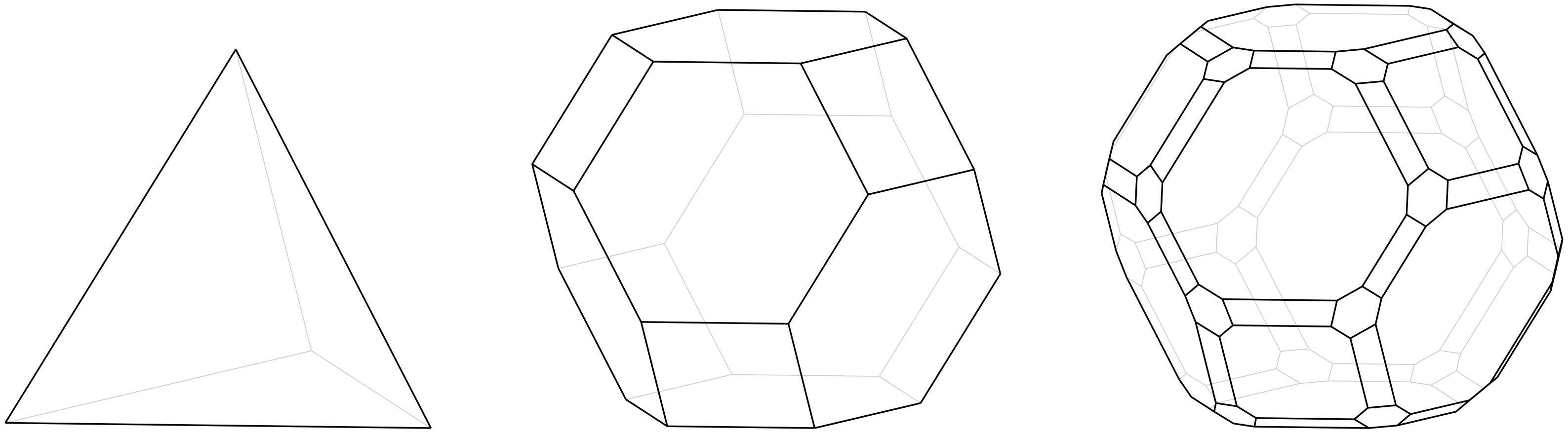
homogenize to an oriented matroid $\mathcal{M}_2$
choose an oriented building set $(\mathcal{B}_2, \mathcal{M}_2)$
get a polytope $P_3 =$ acyclonestohedron of $(\mathcal{B}_2, \mathcal{M}_2)$
...

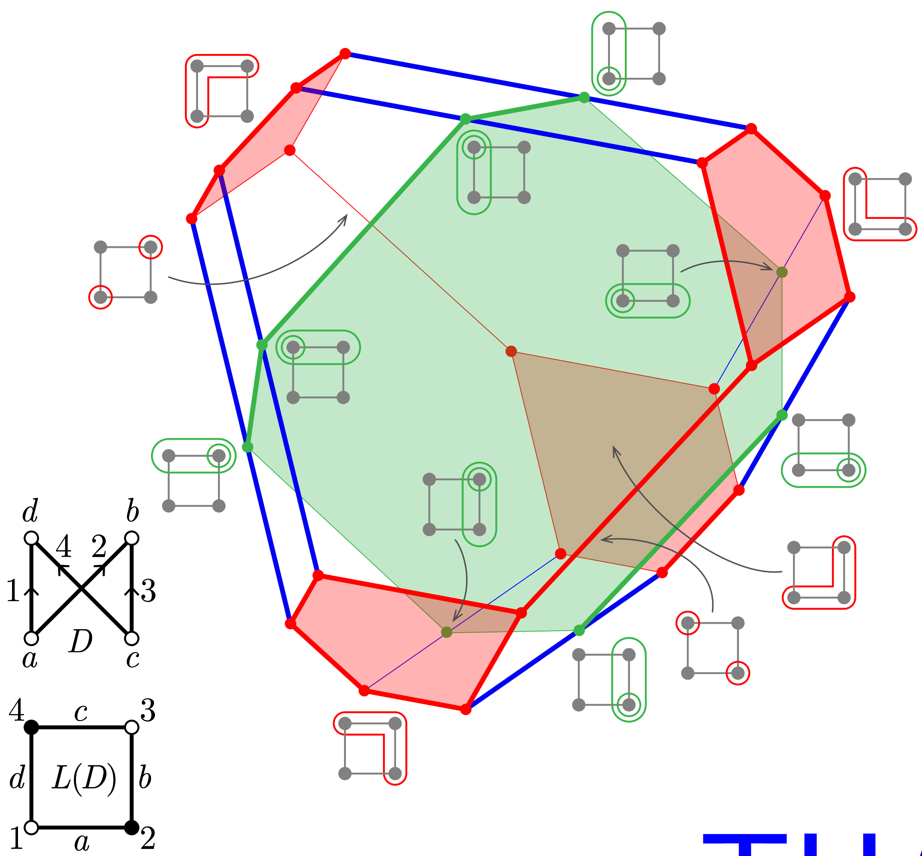
When starting from a simplex:

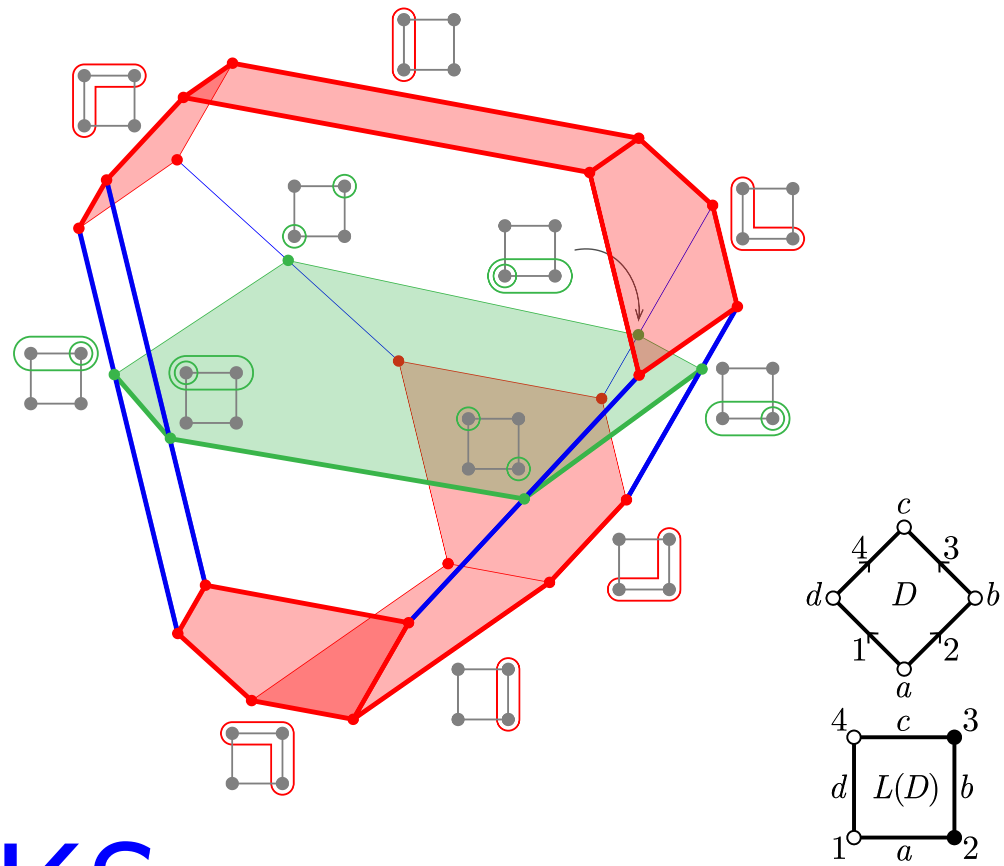
- taking the full building set at each steps leads to permuto-permuto-...-permutahedra
- in two steps, we obtain nesto-nestohedra



...

Gaiffi '15





THANKS