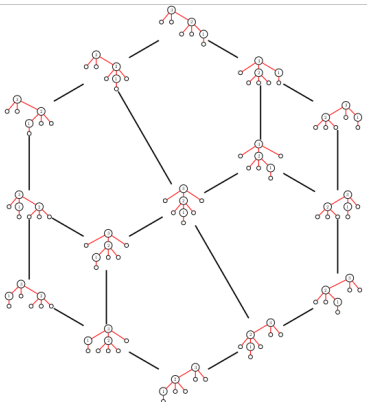
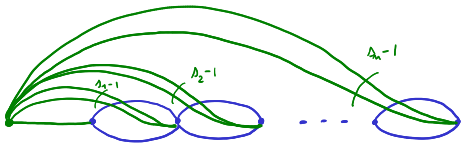
Realizations of the s-permutahedron via flow polytopes



Eva Philippe
Sorbonne Université



Joint work with
Rafael González D'León, Alejandro Morales,
Daniel Tamayo Jiménez, Martha Yip

FPSAC 2024, Bochum



Daniel Tamayo Jiménez

Autobiz,
Paris (France)



Rafael S. Gonzalez D'Leon

Loyola university,
Chicago (USA)



Alejandro H. Morales

Université du Québec,
Montréal (Canada)



Martha Yip

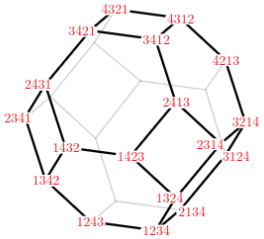
University of Kentucky,
Lexington (USA)

Do you know?

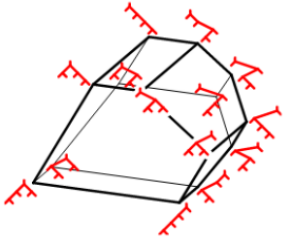
round trip Chicago - Frankfurt $\simeq 2,6\ t\,CO_2$

goal of carbon neutrality by 2050 $\simeq 2\ t\,CO_2$ / year / person

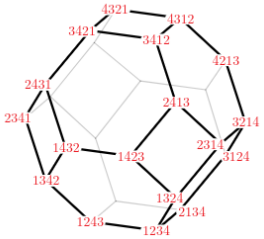
Motivation for the s-permutahedron



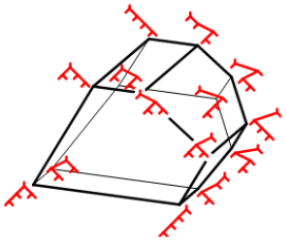
Weak order & Permutahedron



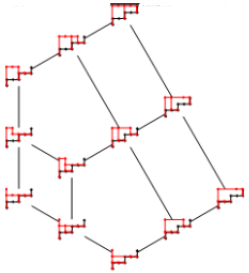
Tamari lattice & Associahedron

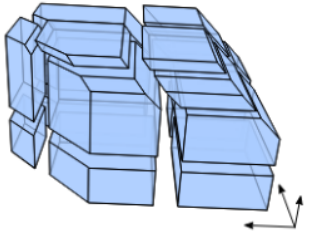


Weak order & Permutahedron



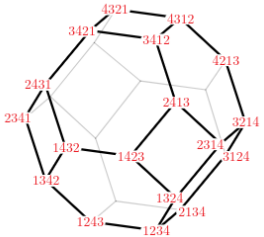
Tamari lattice & Associahedron





ν-Tamari lattice & ν-associahedron

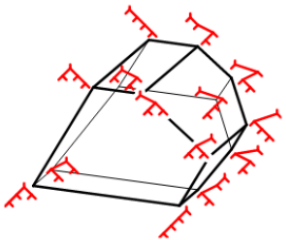
Préville-Ratelle, Viennot '17 Ceballos, Padrol, Sarmiento '19



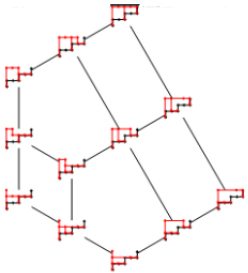
Weak order & Permutahedron

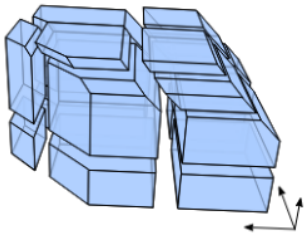
Do you know?

Caracoles = autonomous democratic councils for Zapatistas in Chiapas (Mexico)



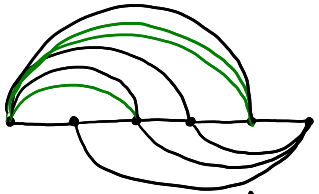
Tamari lattice & Associahedron





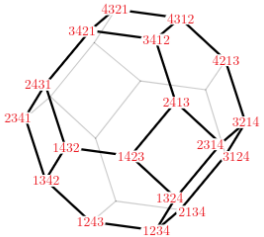
ν-Tamari lattice & ν-associahedron

Préville-Ratelle, Viennot '17 — Ceballos, Padrol, Sarmiento '19



ν-caracol

von Bell, González D'León, + Mayorga Cetina, Ejp '23

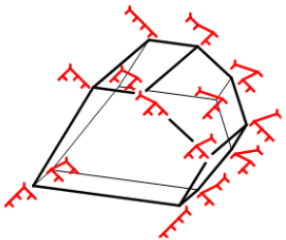


Weak order & Permutahedron

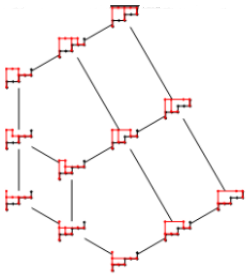
?

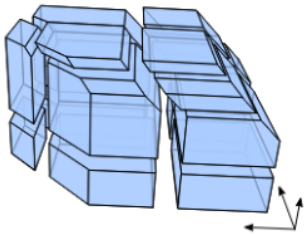
Do you know?

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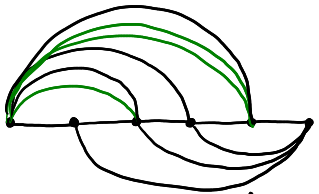
Tamari lattice & Associahedron





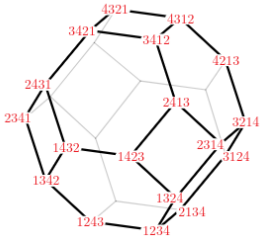
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Préville-Ratelle, Viennot '17 Ceballos, Padrol, Sarmiento '19

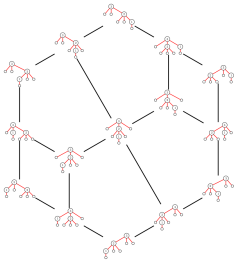


ν-caracol

von Bell, González D'León, + Mayorga Cetina, Ejip '23



Weak order & Permutahedron

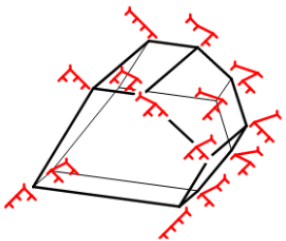


s-weak order & s-permutahedron

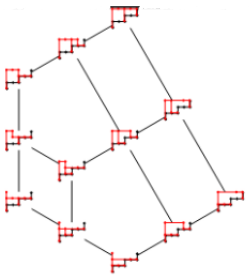
Ceballos, Pons '19

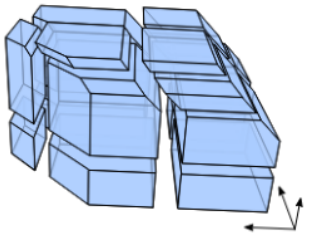
$\in (\mathbb{Z}_{\geq 0})^m$

Conj (Ceballos, Pons '19):
The s-permutahedron can be realized as a polytopal subdivision of a combinatorial permutahedron



Tamari lattice & Associahedron

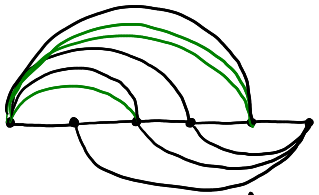




ν-Tamari lattice & ν-associahedron

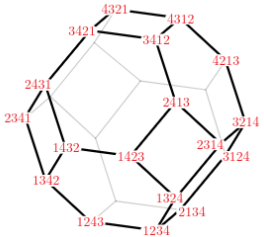
Préville-Ratelle, Viennot '17

Ceballos, Padrol, Sarmiento '19

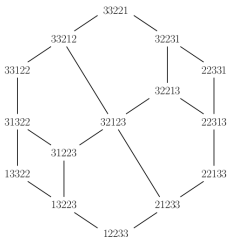


ν-caracol

von Bell, González D'León, + Mayorga Cetina, Ejp '23



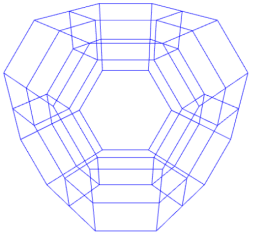
Weak order & Permutahedron



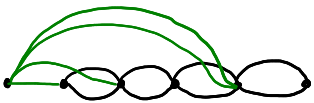
s-weak order & s-permutahedron

Ceballos, Pons '19

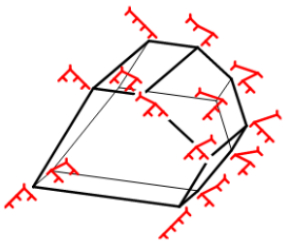
Thm (González D'León, Morales, P., Tamayo, Yip '23+)
The s-permutahedron can be realized as a polytopal subdivision of a combinatorial permutahedron when $s \in (\mathbb{Z}_{>0})^n$



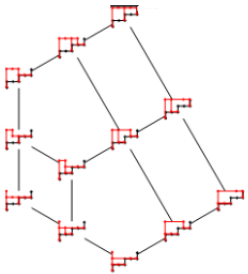
$s \in (\mathbb{Z}_{>0})^m$

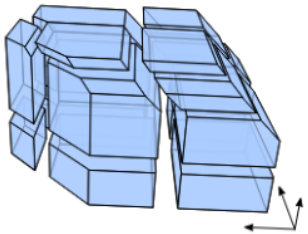


s-oruga



Tamari lattice & Associahedron

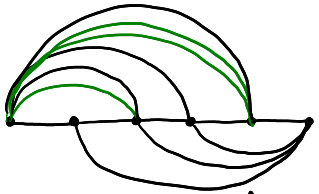




ν-Tamari lattice & ν-associahedron

Préville-Ratelle, Viennot '17

Ceballos, Padrol, Sarmiento '19

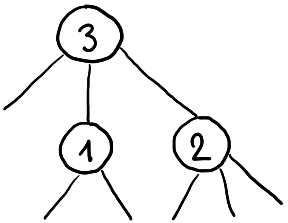


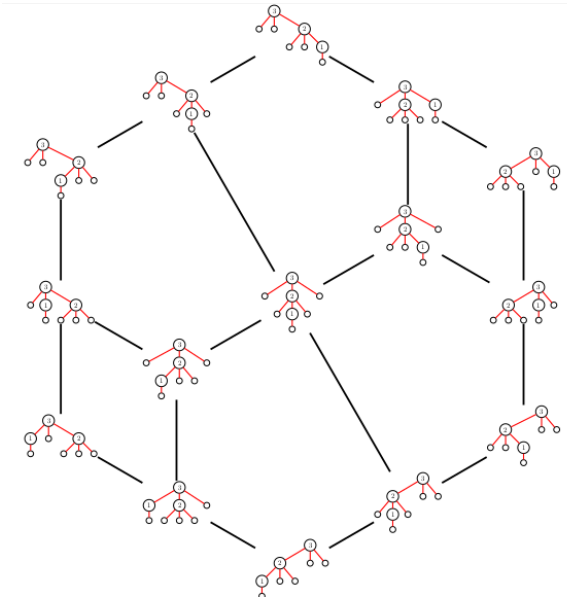
ν-caracol

von Bell, González D'León, + Mayorga Cetina, Yip '23

s-weak order

$s = (1, 2, 2)$



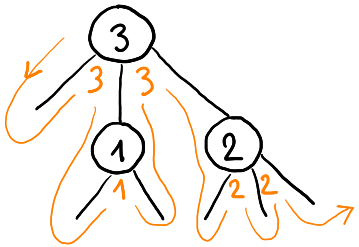


© Ceballos-Pons '19

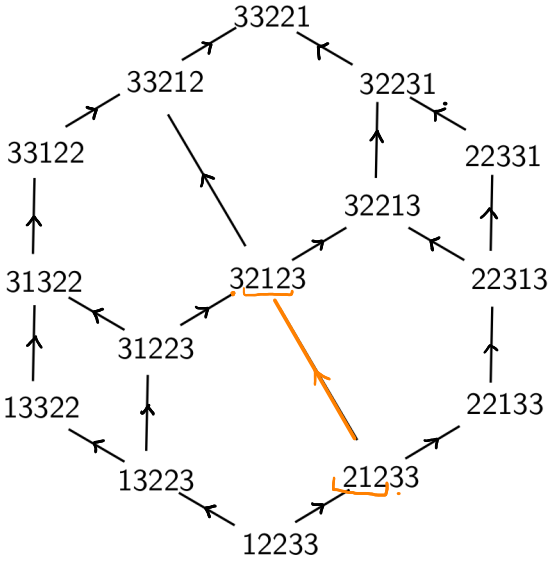
Conj (Ceballos, Pons '19):
The s-permutahedron can be realized as a polytopal subdivision of a combinatorial permutahedron

s-weak order

$s = (1, 2, 2)$



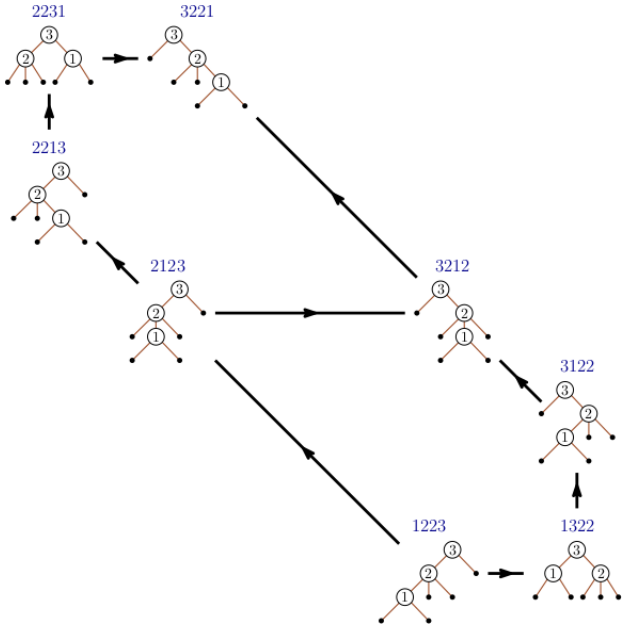
$w = 3\,1\,3\,2\,2$



Conj (Ceballos, Pons '19):
The s-permutahedron can be realized as a polytopal subdivision of a combinatorial permutahedron

s-weak order

$s = (1, 2, 1)$



Combinatorial faces of the s-permutahedron

$*$: bijection that reverses inclusion

$\updownarrow *$

1 Interior cells of a DKK triangulation of a flow polytope of dimension $\sum_{i=1}^{m} s_i$

$\updownarrow$

2 Interior cells of a mixed subdivision of a sum of cubes of dimension n

$\updownarrow *$

3 Bounded cells of an arrangement of tropical hypersurfaces

$\rightsquigarrow$ explicit coordinates! + subdivision of a permutahedron

Combinatorial faces of the s-permutahedron

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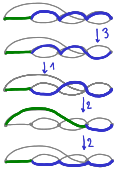
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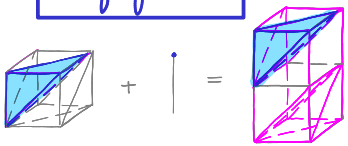
3 Bounded cells of an arrangement of tropical hypersurfaces

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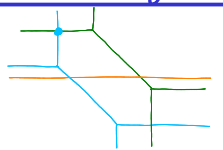
Framed graph G_s



Cayley trick

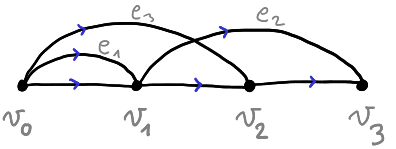


Tropical dualization



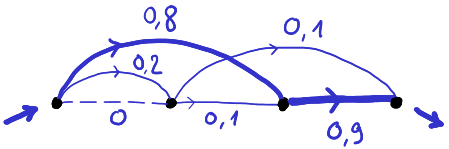
Flow polytopes

Graph G on vertices $v_0, v_1, \dots, v_{n+1}$ (source v_0, sink v_{n+1})
edges E



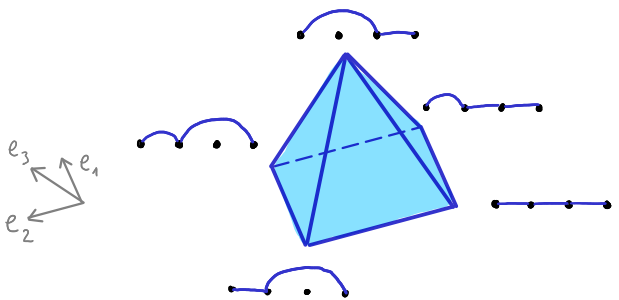
Flow: $(f_e)_{e\in E}$ s.t.

$$\begin{cases} \forall e\in E,\ f_e \geqslant 0 \\ \forall i\in[1,n],\ \sum_{e\in In_i} f_e = \sum_{e\in Out_i} f_e \\ 1 = \sum_{e\in Out_0} f_e = \sum_{e\in In_{n+1}} f_e \end{cases}$$



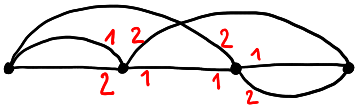
Flow polytope $\mathcal{F}_G := \{\text{flows } (f_e)_{e\in E}\}$

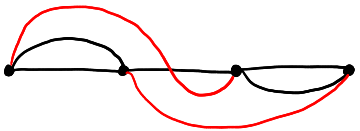
↳ vertices $\leftrightarrow$ routes = paths from v_0 to v_{n+1}



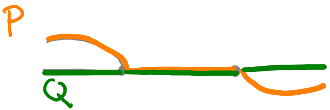
DKK triangulations of flow polytopes

Framing $\leq$





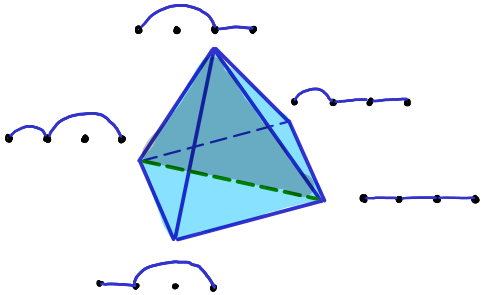
from top to bottom





crossing routes VS coherent routes

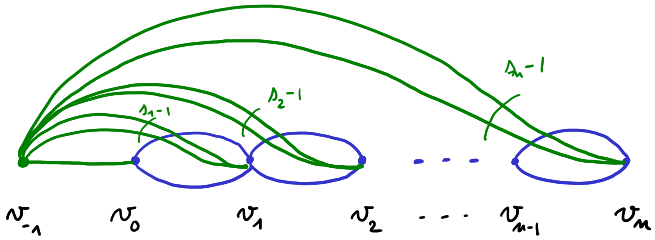
Thm (Danilov, Karzanov, Koshevoy, '12):
The maximal cliques of coherent routes give the maximal cells of a regular triangulation of $\mathcal{F}_G$.



Graph G_s (s-ouga)

Do you know?

≥ 70 % decline in total biomass of flying insects in Germany over 27 years



Framing F : from top to bottom

{Faces of the s-permutahedron} $\overset{*}{\longleftrightarrow}$ {Certain cliques of coherent routes of (G_s, F)} $\overset{DKK}{\longleftrightarrow}$ {Interior cells of the DKK-triangulation $DKK(G_s, F)$ of $\mathcal{F}_{G_s}$}

$\dim \sum_{i=1}^{n} s_i$

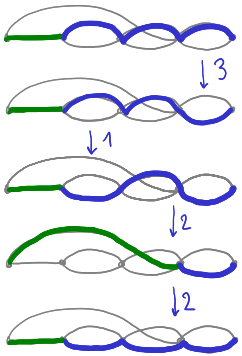
Vertices
↕
212-avoiding s-permutations

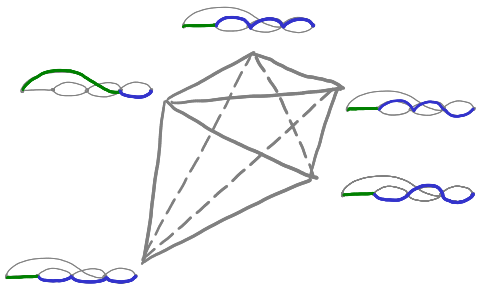
Maximal cliques

Maximal cells

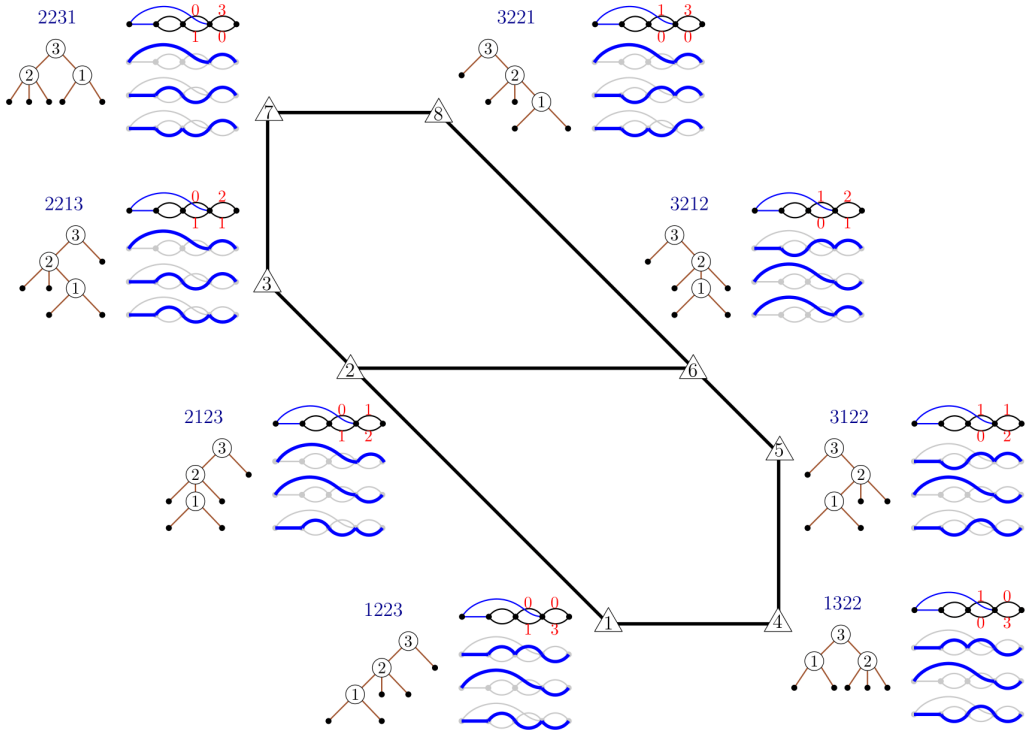
Ex: $s=(1,2,1)$

$w = 3122$





$s = (1, 2, 1)$



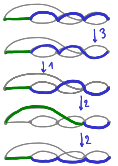
Combinatorial faces of the s-permutahedron

* : bijection that reverses inclusion

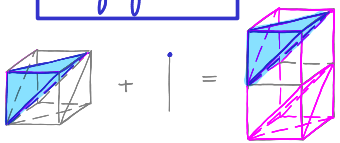
*

1 Interior cells of a DKK triangulation of a flow polytope of dimension $\sum_{i=1}^{n} s_i$

Framed graph G_s



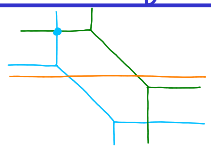
Cayley trick



2 Interior cells of a mixed subdivision of a sum of cubes of dimension n

*

Tropical dualization



3 Bounded cells of an arrangement of tropical hypersurfaces

⇝ explicit coordinates! + subdivision of a permutahedron

Cayley trick

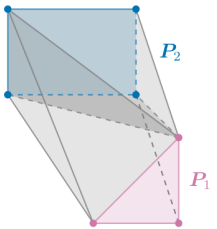
(2) Sum of cubes

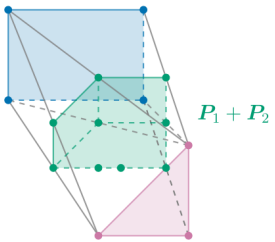
$P_1, \dots, P_k$ = polytopes in $\mathbb{R}^d$

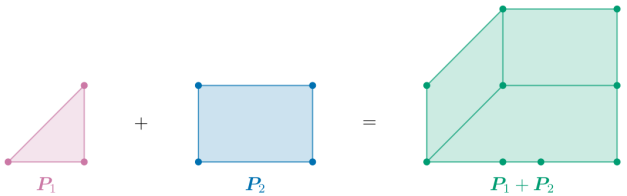
{ (Regular) subdivisions of $Cay(P_1, \dots, P_k)$ } ⟷ (Sturmfels, Huber-Rambau-Santos) ⟷ { (Regular) mixed subdivisions of $P_1 + \dots + P_k$ }

$$conv\left(e_{d+1} \times P_1, \dots, e_{d+k} \times P_k\right) \subset \mathbb{R}^{d+k}$$

$$\left\{ x_1 + \dots + x_k \mid x_1 \in P_1, \dots, x_k \in P_k \right\} \subset \mathbb{R}^d$$

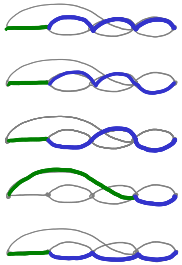


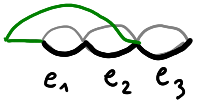


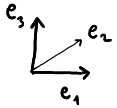


$s = (1,2,1)$

$w = 3122$

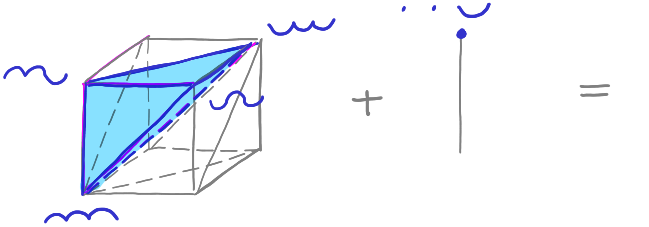




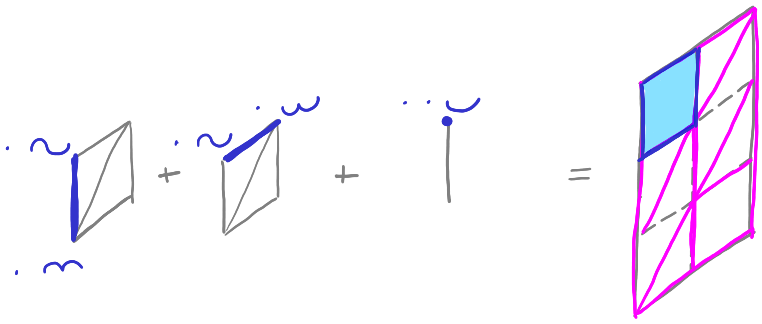


② Sum of cubes

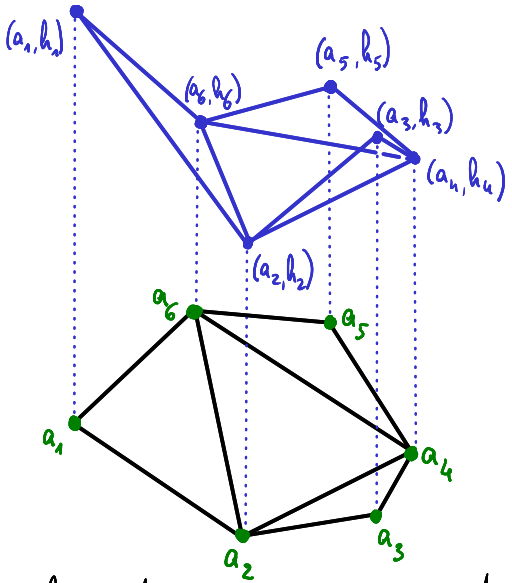
$$\mathcal{F}_{\sim\sim\sim} = \mathrm{Cay}(\mathcal{F}_{\sim\sim\sim}, \mathcal{F}_{\sim})$$



$$\mathcal{F}_{\sim\sim\sim} = \mathrm{Cay}(\mathcal{F}_{\sim\sim\sim}, \mathcal{F}_{\sim\sim\sim}, \mathcal{F}_{\sim})$$



Tropical dualization



Regular subdivision of a point configuration $\mathcal{A} = (a_1, \dots, a_m) \subset \mathbb{R}^d$ with lifting function $h : [m] \to \mathbb{R}$

Tropical semiring

$$(\mathbb{T}, \oplus, \odot) := (\mathbb{R} \cup \{\infty\}, \min, +)$$

Tropical polynomial

$$F(x) := \bigoplus_{i \in [m]} (h_i \odot x^{\odot a_i})$$

$x \in \mathbb{R}^d$

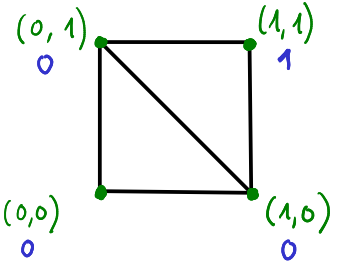
$$:= \min_{i \in [m]} (h_i + \langle x, a_i \rangle)$$

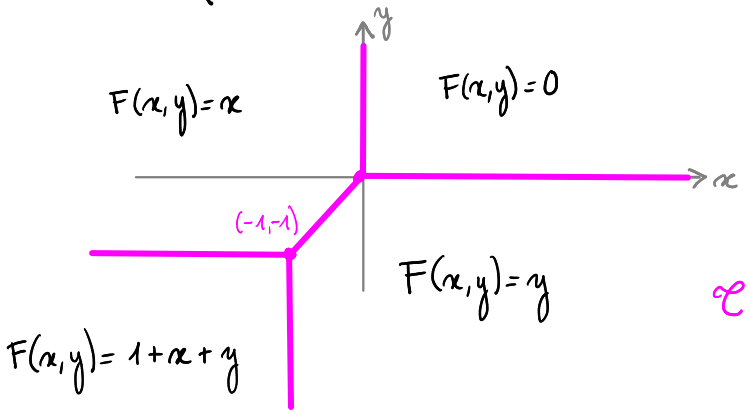
Tropical hypersurface

$$\mathcal{C}(F) := \{ x \in \mathbb{R}^d \mid \text{the minimum of } F(x) \text{ is attained at least twice} \}$$

Tropical dualization

{ (Interior) cells of the regular subdivision of A with lifting h } $\longleftrightarrow^{*}$ { (Bounded) cells of the tropical hypersurface $\mathscr{C}(F)$ }

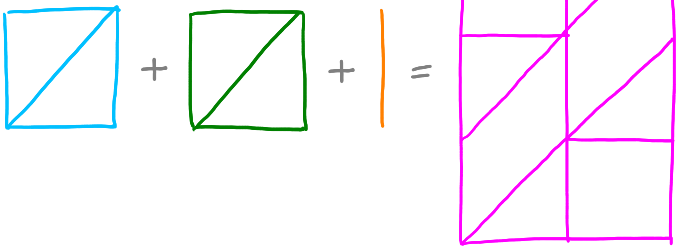


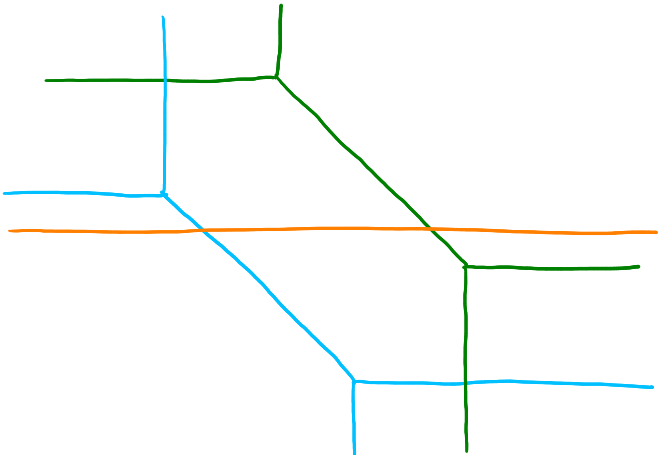


$$F(x,y) = \min(0, x, y, 1+x+y)$$

{ (Interior) cells of the regular subdivision of the sum $A_1 + \dots + A_k$ with lifting functions $h_1, \dots, h_k$ } ↔ { (Bounded) cells of the arrangement of tropical hypersurfaces $\mathcal{C}(F_1), \dots, \mathcal{C}(F_k)$ }

$h_1, \dots, h_k$ → given by DKK





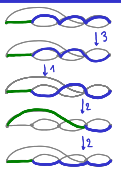
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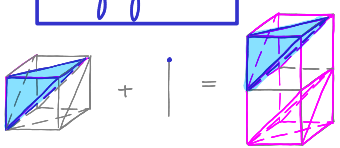
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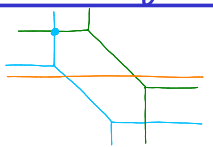
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Tropical dualization



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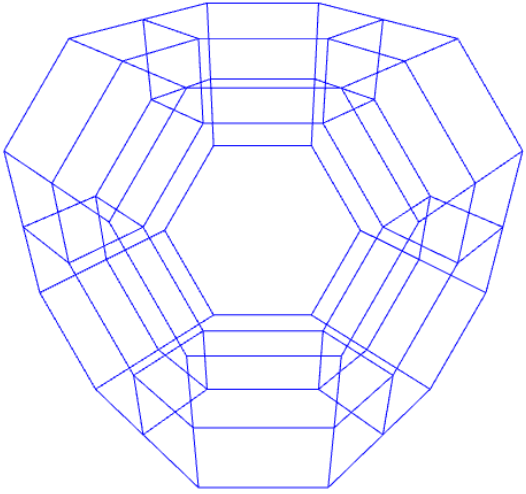
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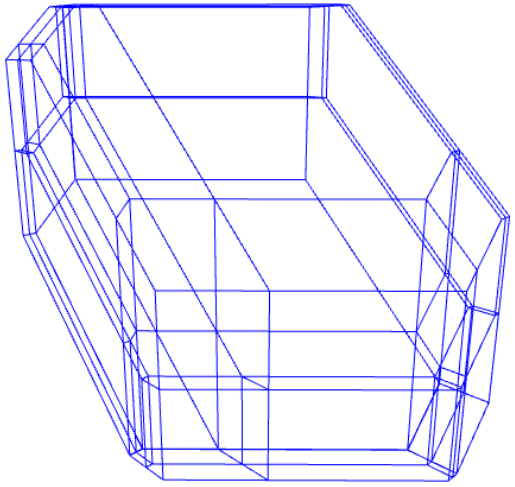
Perspectives

- relation with the s-associahedron and other s-quotientopes
- case where $s \in (\mathbb{Z}_{\geq 0})^m$

solved by B.-Piland '24+ with other techniques (s-shardoplexes)

- study the other framing lattices of the s-ouga graph
- other Coxeter types ?







Thank you for your attention !