All Kronecker coefficients are reduced Kronecker coefficients

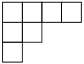
Christian Ikenmeyer and Greta Panova*
arXiv:2305.03003

*University of Southern California

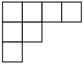
FPSAC Bochum, July 2024

Partitions, SYTs and the Symmetric group

Integer partitions and Young diagrams:

$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots 0$, $\lambda_1 + \lambda_2 + \cdots = n$.  for $\lambda = (4, 2, 1) \vdash 7$.

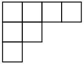
Integer partitions and Young diagrams:

$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots 0$, $\lambda_1 + \lambda_2 + \cdots = n$.  for $\lambda = (4, 2, 1) \vdash 7$.

Standard Young Tableaux of shape λ: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$

Partitions, SYTs and the Symmetric group

Integer partitions and Young diagrams:

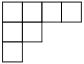
$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots 0$, $\lambda_1 + \lambda_2 + \cdots = n$.  for $\lambda = (4, 2, 1) \vdash 7$.

Standard Young Tableaux of shape λ: $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$

The **irreducible representations** of the **symmetric group** S_n: the *Specht modules* $\mathbb{S}_\lambda$
Basis indexed by SYTs of shape λ,

Partitions, SYTs and the Symmetric group

Integer partitions and Young diagrams:

$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots 0$, $\lambda_1 + \lambda_2 + \cdots = n$.  for $\lambda = (4, 2, 1) \vdash 7$.

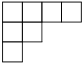
Standard Young Tableaux of shape λ: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$

The **irreducible representations** of the **symmetric group** S_n: the *Specht modules* $\mathbb{S}_\lambda$
Basis indexed by SYTs of shape λ,

Irreducible (polynomial) representations of the **General Linear group** $GL_N(\mathbb{C})$:
Weyl modules V_λ, indexed by highest weights λ, $\ell(\lambda) \leq N$.

Partitions, SYTs and the Symmetric group

Integer partitions and Young diagrams:

$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots 0$, $\lambda_1 + \lambda_2 + \cdots = n$.  for $\lambda = (4,2,1) \vdash 7$.

Standard Young Tableaux of shape λ: $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$

The **irreducible representations** of the **symmetric group** S_n: the *Specht modules* $\mathbb{S}_\lambda$
Basis indexed by SYTs of shape λ,

Irreducible (polynomial) representations of the **General Linear group** $GL_N(\mathbb{C})$:
Weyl modules V_λ, indexed by highest weights λ, $\ell(\lambda) \leq N$.

Schur functions: characters of V_λ

$$s_{(2,2)}(x_1,x_2,x_3) = \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}}{x_1^2x_2^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}}{x_1^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}}{x_2^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}}{x_1^2x_2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}}{x_1x_2^2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}}{x_1x_2x_3^2}.$$

Structure constants

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

Littlewood-Richardson coefficients: $c^\nu_{\lambda\mu}$

Structure constants

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

Littlewood-Richardson coefficients: $c^\nu_{\lambda\mu}$

$$s_\lambda(x)s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x)$$

Structure constants

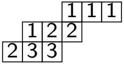
$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

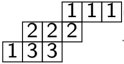
Littlewood-Richardson coefficients: $c^\nu_{\lambda\mu}$

$$s_\lambda(x)s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x)$$

Theorem (Littlewood-Richardson, stated 1934, proven 1970's)

The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.





(LR tableaux of shape $(6,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

Structure constants

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

Littlewood-Richardson coefficients: $c^\nu_{\lambda\mu}$

$$s_\lambda(x)s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x)$$

Theorem (Littlewood-Richardson, stated 1934, proven 1970's)

The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.

			1	1	1
	1	2	2		
2	3	3			

			1	1	1
	2	2	2		
1	3	3			

(LR tableaux of shape $(6,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

Kronecker coefficients: $g(\lambda,\mu,\nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

$$s_\nu(x.y) = \sum_{\mu,\nu} g(\lambda,\mu,\nu) s_\mu(x) s_\lambda(y)$$

Structure constants

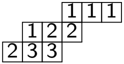
$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^{\nu}_{\lambda\mu}}$$

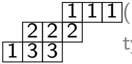
Littlewood-Richardson coefficients: $c^{\nu}_{\lambda\mu}$

$$s_\lambda(x)s_\mu(x) = \sum_\nu c^{\nu}_{\lambda\mu} s_\nu(x)$$

Theorem (Littlewood-Richardson, stated 1934, proven 1970's)

The coefficient $c^{\nu}_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.





(LR tableaux of shape $(6,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

Kronecker coefficients: $g(\lambda,\mu,\nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

$$s_\nu(x.y) = \sum_{\mu,\nu} g(\lambda,\mu,\nu) s_\mu(x) s_\lambda(y)$$

[Murnaghan, 1938]: $c^{\lambda}_{\mu\nu} = g\left((N-|\lambda|,\lambda),(N-|\mu|,\mu),(N-|\nu|,\nu)\right)$ for $|\lambda| = |\mu| + |\nu|$ and N-large.

Open problems

Problem[Murnaghan 1938]: Determine way to compute the Kronecker coefficients.

Open problems

Problem[Murnaghan 1938]: Determine way to compute the Kronecker coefficients.

[Lascoux, Garsia-Remmel 1980s]: some approaches

Open problems

Problem[Murnaghan 1938]: Determine way to compute the Kronecker coefficients.

[Lascoux, Garsia-Remmel 1980s]: some approaches

Positivity Problems and Conjectures in Algebraic Combinatorics

Richard P. Stanley[1]
Department of Mathematics 2-375
Massachusetts Institute of Technology
Cambridge, MA 02139

version of 24 September 1999

Problem 10. *Find a combinatorial interpretation of the "Kronecker product coefficients" $g_{\lambda\mu\nu}$, thereby combinatorially reproving that they are nonnegative.*

Open problems

Problem[Murnaghan 1938]: Determine way to compute the Kronecker coefficients.

[Lascoux, Garsia-Remmel 1980s]: some approaches

Positivity Problems and Conjectures in Algebraic Combinatorics

Richard P. Stanley[1]
Department of Mathematics 2-375
Massachusetts Institute of Technology
Cambridge, MA 02139

version of 24 September 1999

Problem 10. *Find a combinatorial interpretation of the "Kronecker product coefficients" $g_{\lambda\mu\nu}$, thereby combinatorially reproving that they are nonnegative.*

[Mulmuley-Sohoni 2000's]: conjectures of importance to Geometric Complexity Theory (VP vs VNP)

Open problems

Problem[Murnaghan 1938]: Determine way to compute the Kronecker coefficients.

[Lascoux, Garsia-Remmel 1980s]: some approaches

Positivity Problems and Conjectures in Algebraic Combinatorics

Richard P. Stanley[1]
Department of Mathematics 2-375
Massachusetts Institute of Technology
Cambridge, MA 02139

version of 24 September 1999

Problem 10. *Find a combinatorial interpretation of the "Kronecker product coefficients" $g_{\lambda\mu\nu}$, thereby combinatorially reproving that they are nonnegative.*

[Mulmuley-Sohoni 2000's]: conjectures of importance to Geometric Complexity Theory (VP vs VNP)

Any answers: known only for very, very special cases...

The reduced Kronecker coefficients

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^{\nu}_{\lambda\mu}} \qquad \mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_\nu \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

$$\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g\big(\alpha[n],\beta[n],\gamma[n]\big), \qquad \alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots),\ n \geq |\alpha|+\alpha_1,$$

The reduced Kronecker coefficients

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}} \qquad \mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_\nu \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

$$\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g\big(\alpha[n],\beta[n],\gamma[n]\big), \qquad \alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots),\ n \geq |\alpha|+\alpha_1,$$

$$\overline{g}(\alpha,\beta,\gamma) = c^\alpha_{\beta\gamma} \quad \text{for} \quad |\alpha| = |\beta|+|\gamma|,$$

The reduced Kronecker coefficients

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}} \qquad \mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_\nu \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

$$\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g(\alpha[n],\beta[n],\gamma[n]), \qquad \alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots),\ n \geq |\alpha|+\alpha_1,$$

$$\overline{g}(\alpha,\beta,\gamma) = c^\alpha_{\beta\gamma} \quad \text{for} \quad |\alpha| = |\beta| + |\gamma|,$$

Kirillov: $\overline{g}$ are the "extended Littlewood-Richardson coefficients".

The reduced Kronecker coefficients

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}} \qquad \mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_\nu \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

$$\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g\left(\alpha[n],\beta[n],\gamma[n]\right), \qquad \alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots),\ n \geq |\alpha|+\alpha_1,$$

$$\overline{g}(\alpha,\beta,\gamma) = c^\alpha_{\beta\gamma} \quad \text{for} \quad |\alpha| = |\beta| + |\gamma|,$$

Kirillov: $\overline{g}$ are the "extended Littlewood-Richardson coefficients".

Problem [Kirillov, 2004]: Find a combinatorial interpretation for the reduced Kronecker coefficients.

The reduced Kronecker coefficients

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}} \qquad \mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_\nu \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

$$\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g(\alpha[n],\beta[n],\gamma[n]), \qquad \alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots), \; n \geq |\alpha| + \alpha_1,$$

$$\overline{g}(\alpha,\beta,\gamma) = c^\alpha_{\beta\gamma} \quad \text{for} \quad |\alpha| = |\beta| + |\gamma|,$$

Kirillov: $\overline{g}$ are the "extended Littlewood-Richardson coefficients".

Problem [Kirillov, 2004]: Find a combinatorial interpretation for the reduced Kronecker coefficients.

Some combinatorial interpretations: [Ballantine–Orellana], [Colmenarejo–Rosas]; Properties and computation: [Briand–Orellana–Rosas], [Murnaghan],[Kirilov], [Klyachko]; Partition algebra: [Bowman–De Vischer–Orellana]; As structure constants: [Orellana–Zabrocki].

The reduced Kronecker coefficients

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}} \qquad \mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_\nu \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

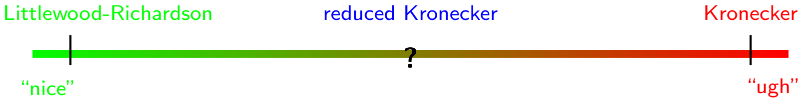
$$\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g(\alpha[n],\beta[n],\gamma[n]), \qquad \alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots),\ n \geq |\alpha|+\alpha_1,$$

$$\overline{g}(\alpha,\beta,\gamma) = c^\alpha_{\beta\gamma} \quad \text{for} \quad |\alpha| = |\beta|+|\gamma|,$$

Kirillov: $\overline{g}$ are the "extended Littlewood-Richardson coefficients".

Problem [Kirillov, 2004]: Find a combinatorial interpretation for the reduced Kronecker coefficients.

Some combinatorial interpretations: [Ballantine–Orellana], [Colmenarejo–Rosas]; Properties and computation: [Briand–Orellana–Rosas], [Murnaghan],[Kirilov], [Klyachko]; Partition algebra: [Bowman–De Vischer–Orellana]; As structure constants: [Orellana–Zabrocki].

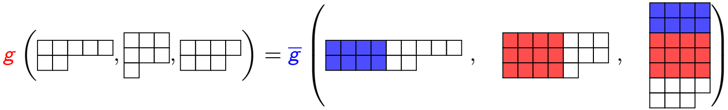


Kronecker g = reduced Kronecker $\overline{g}$

Theorem (Ikenmeyer-Panova, 2023)

For every $\lambda, \mu, \nu \vdash n$ *we have*

$$g(\lambda,\mu,\nu) = \overline{g}(\nu_1^{\ell(\lambda)} + \lambda, \nu_1^{\ell(\mu)} + \mu, \nu_1^{\ell(\lambda)+\ell(\mu)} \cup \nu)$$

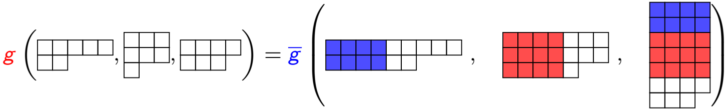


Kronecker g = reduced Kronecker $\overline{g}$

Theorem (Ikenmeyer-Panova, 2023)

For every $\lambda, \mu, \nu \vdash n$ *we have*

$$g(\lambda, \mu, \nu) = \overline{g}(\nu_1^{\ell(\lambda)} + \lambda, \nu_1^{\ell(\mu)} + \mu, \nu_1^{\ell(\lambda)+\ell(\mu)} \cup \nu)$$



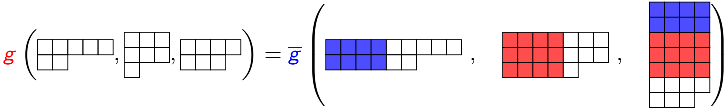
Corollaries:

- Deciding positivity of reduced Kroneckers is NP-hard.
- Computing the reduced Kroneckers is #P-hard. [Pak-Panova'2020]
- Saturation fails [Pak-Panova'2020]

Theorem (Ikenmeyer-Panova, 2023)

For every $\lambda, \mu, \nu \vdash n$ we have

$$g(\lambda, \mu, \nu) = \overline{g}(\nu_1^{\ell(\lambda)} + \lambda, \nu_1^{\ell(\mu)} + \mu, \nu_1^{\ell(\lambda)+\ell(\mu)} \cup \nu)$$



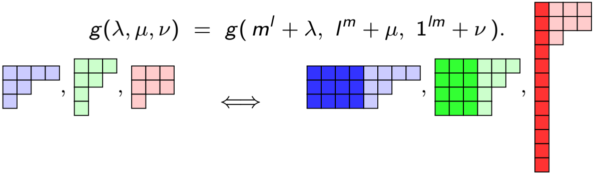
Corollaries:

- Deciding positivity of reduced Kroneckers is NP-hard.
- Computing the reduced Kroneckers is #P-hard. [Pak-Panova'2020]
- Saturation fails [Pak-Panova'2020]

Lemma

Let λ, μ, ν be partitions with $\ell(\lambda) \le l$, $\ell(\mu) \le m$. Then

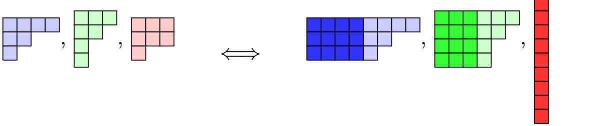
$$g(\lambda, \mu, \nu) \ = \ g(\, m^l + \lambda,\ l^m + \mu,\ 1^{lm} + \nu\,).$$



Lemma

Let λ, μ, ν be partitions with $\ell(\lambda) \leq l$, $\ell(\mu) \leq m$. Then

$$g(\lambda, \mu, \nu) \; = \; g(\, m^l + \lambda,\; l^m + \mu,\; 1^{lm} + \nu\,).$$



Let $\hat{\nu} = 1^{lm} + \nu$. Variables $x_1, \ldots, x_\ell$ and $y_1, \ldots, y_m$:

$$s_{\hat{\nu}}[x \cdot y] = \sum_{\theta,\tau} g(\hat{\nu}, \theta, \tau) s_\theta(x) s_\tau(y).$$

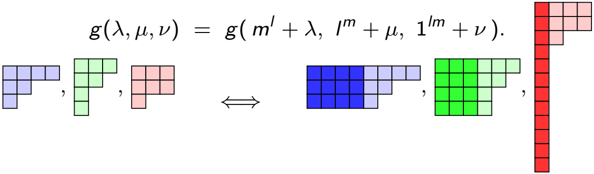
$$s_{\hat{\nu}}[x \cdot y] = s_\nu[x \cdot y] \prod_{i,j} x_i y_j = (x_1 \ldots x_l)^m (y_1, \ldots, y_m)^l \sum_{\rho,\eta} g(\nu, \rho, \eta) s_\rho(x) s_\eta(y)$$

$$s_{l^m+\mu}(y_1, \ldots, y_m) = (y_1 \ldots y_m)^l s_\mu(y),\; s_{m^l+\lambda}(x_1, \ldots, x_l) = (x_1 \ldots x_l)^m s_\lambda(x).$$

Lemma

Let λ, μ, ν be partitions with $\ell(\lambda) \le l$, $\ell(\mu) \le m$. Then

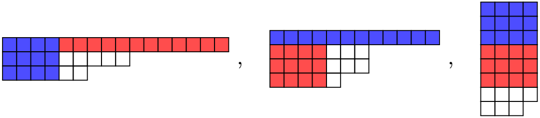
$$g(\lambda,\mu,\nu) \;=\; g(\, m^l + \lambda,\; l^m + \mu,\; 1^{lm} + \nu\,).$$



Lemma

Let λ, μ, ν be partitions of the same size, and let $l \ge \ell(\lambda)$, $m \ge \ell(\mu)$ and $c \ge \nu_1$. Let $d = (m+1)c$, $e = (l+1)c$. Then

$$g(\lambda,\mu,\nu) \;=\; g\big(\,(d)\cup(c^l+\lambda),\; (e)\cup(c^m+\mu),\; c^{l+m+1}\cup\nu\,\big).$$



Lemma

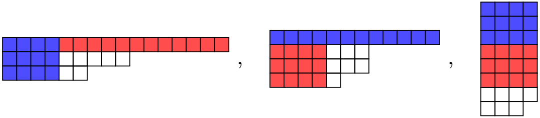
Let λ, μ, ν be partitions with $\ell(\lambda) \leq l$, $\ell(\mu) \leq m$. Then

$$g(\lambda, \mu, \nu) \; = \; g(\, m^l + \lambda,\; l^m + \mu,\; 1^{lm} + \nu\,).$$

Lemma

Let λ, μ, ν be partitions of the same size, and let $l \geq \ell(\lambda)$, $m \geq \ell(\mu)$ and $c \geq \nu_1$. Let $d = (m+1)c$, $e = (l+1)c$. Then

$$g(\lambda, \mu, \nu) \; = \; g\big(\,(d) \cup (c^l + \lambda),\; (e) \cup (c^m + \mu),\; c^{l+m+1} \cup \nu\,\big).$$



Theorem (Ikenmeyer-P)

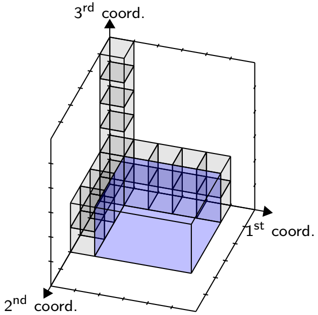
Let λ, μ, ν be partitions of the same size, such that $\lambda_1 \geq \ell(\mu) \cdot \nu_1$ and $\mu_1 \geq \ell(\lambda) \cdot \nu_1$. Then for every $h \geq 0$ we have

$$g(\lambda, \mu, \nu) = g(\,\lambda + h,\; \mu + h,\; \nu + h\,).$$

Point configurations in 3d aka 3d binary contingency arrays

$Q \subseteq \mathbb{N}^3$,
2d marginals:
$Q_{i**} := \sum_{j,k} Q_{i,j,k}$ $Q_{*i*} := \sum_{j,k} Q_{j,i,k}$ $Q_{**i} := \sum_{j,k} Q_{j,k,i}$

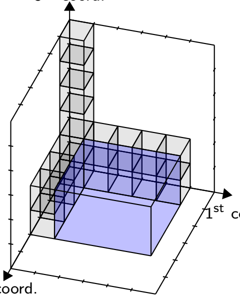
$$\mathcal{C}(\alpha,\beta,\gamma) := \{Q \subseteq \mathbb{N}^3 \mid Q_{i**} = \alpha_i,\ Q_{*i*} = \beta_i,\ Q_{**i} = \gamma_i \text{ for every } i\}.$$



Point configurations in 3d aka 3d binary contingency arrays

$Q \subseteq \mathbb{N}^3$,
2d marginals:
$Q_{i**} := \sum_{j,k} Q_{i,j,k}$ $Q_{*i*} := \sum_{j,k} Q_{j,i,k}$ $Q_{**i} := \sum_{j,k} Q_{j,k,i}$

$$\mathcal{C}(\alpha,\beta,\gamma) := \{Q \subseteq \mathbb{N}^3 \mid Q_{i**} = \alpha_i,\ Q_{*i*} = \beta_i,\ Q_{**i} = \gamma_i \text{ for every } i\}.$$



Lemma:
α, β, γ – compositions with $|\alpha| = |\beta| = |\gamma|$.
$a := \ell(\alpha)$, $b := \ell(\beta)$, $c + h \geq \ell(\gamma)$ and $\sum_{i>c} \gamma_i \leq h$, $\alpha_1 \geq bc + h$, $\beta_1 \geq ac + h$.
Then, for every $Q \in \mathcal{C}(\alpha,\beta,\gamma)$ we have

$$\{1\} \times [b] \times [c] \subseteq Q, [a] \times \{1\} \times [c] \subseteq Q,$$

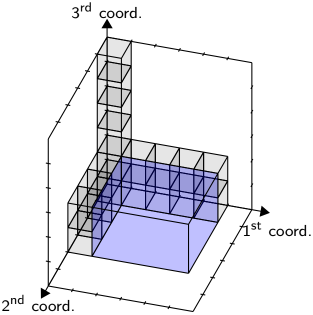
$$\{1\} \times \{1\} \times [c+h] \subseteq Q,$$

$$Q \cap (\mathbb{N} \times \mathbb{N} \times [c+1, c+h]) = \{1\} \times \{1\} \times [c+1, c+h].$$

Point configurations in 3d aka 3d binary contingency arrays

$Q \subseteq \mathbb{N}^3$,
2d marginals:
$Q_{i**} := \sum_{j,k} Q_{i,j,k}$ $Q_{*i*} := \sum_{j,k} Q_{j,i,k}$ $Q_{**i} := \sum_{j,k} Q_{j,k,i}$

$$\mathcal{C}(\alpha,\beta,\gamma) := \{Q \subseteq \mathbb{N}^3 \mid Q_{i**} = \alpha_i,\ Q_{*i*} = \beta_i,\ Q_{**i} = \gamma_i \text{ for every } i\}.$$



Lemma:
α,β,γ – compositions with $|\alpha| = |\beta| = |\gamma|$.
$a := \ell(\alpha)$, $b := \ell(\beta)$, $c + h \geq \ell(\gamma)$ and $\sum_{i>c} \gamma_i \leq h$, $\alpha_1 \geq bc + h$, $\beta_1 \geq ac + h$.
Then, for every $Q \in \mathcal{C}(\alpha,\beta,\gamma)$ we have

$$\{1\} \times [b] \times [c] \subseteq Q, [a] \times \{1\} \times [c] \subseteq Q,$$

$$\{1\} \times \{1\} \times [c+h] \subseteq Q,$$

$$Q \cap (\mathbb{N} \times \mathbb{N} \times [c+1, c+h]) = \{1\} \times \{1\} \times [c+1, c+h].$$

$\mathcal{C}(\alpha,\beta,\gamma) \neq \emptyset \implies \gamma_i = 1$ for all $c+1 \leq i \leq c+h$, and $\alpha_1 = bc + h$, $\beta_1 = ac + h$, $\alpha_2 \leq bc$, and $\beta_2 \leq ac$.

Kronecker coefficients via 3d binary contingency arrays

$$\sum_{\alpha,\beta,\gamma} g(\alpha,\beta,\gamma) s_\alpha(x) s_\beta(y) s_{\gamma'}(z) = \prod_{i,j,k} (1 + x_i y_j z_k),$$

$$g(\alpha,\beta,\gamma) = \sum_{\sigma \in S_a, \pi \in S_b, \rho \in S_{\gamma_1}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi)\,\mathrm{sgn}(\rho)\, C(\alpha+\sigma-\mathrm{id}, \beta+\pi-\mathrm{id}, \gamma'+\rho-\mathrm{id}).$$

Kronecker coefficients via 3d binary contingency arrays

$$\sum_{\alpha,\beta,\gamma} g(\alpha,\beta,\gamma) s_\alpha(x) s_\beta(y) s_{\gamma'}(z) = \prod_{i,j,k}(1 + x_i y_j z_k),$$

$$g(\alpha,\beta,\gamma) = \sum_{\sigma\in S_a,\pi\in S_b,\rho\in S_{\gamma_1}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi)\,\mathrm{sgn}(\rho)\, C(\alpha+\sigma-\mathrm{id}, \beta+\pi-\mathrm{id}, \gamma'+\rho-\mathrm{id}).$$

If $c := \nu_1$, $\lambda_1 \geq bc$ and $\mu_1 \geq ac \Longrightarrow g(\lambda,\mu,\nu) = g(\,\lambda+h,\ \mu+h,\ \nu+h\,)$?
$(\gamma = \nu + h)$

Kronecker coefficients via 3d binary contingency arrays

$$\sum_{\alpha,\beta,\gamma} g(\alpha,\beta,\gamma) s_\alpha(x) s_\beta(y) s_{\gamma'}(z) = \prod_{i,j,k} (1 + x_i y_j z_k),$$

$$g(\alpha,\beta,\gamma) = \sum_{\sigma \in S_a, \pi \in S_b, \rho \in S_{\gamma_1}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi)\,\mathrm{sgn}(\rho) C(\alpha+\sigma-\mathrm{id}, \beta+\pi-\mathrm{id}, \gamma'+\rho-\mathrm{id}).$$

If $c := \nu_1$, $\lambda_1 \geq bc$ and $\mu_1 \geq ac \implies g(\lambda,\mu,\nu) = g(\,\lambda + h,\; \mu + h,\; \nu + h\,)$?
$(\gamma = \nu + h)$

$$\begin{aligned} Q_{1**} &:= \sum_{j,k} Q_{1,j,k} = \lambda_1 + \sigma_1 - 1 \geq bc + h, \\ Q_{*1*} &:= \sum_{i,k} Q_{i,1,k} = \mu_1 + \pi_1 - 1 \geq ac + h, \\ Q_{**k} &:= \sum_{i,j} Q_{i,j,k} = 1 + \rho_k - k, \text{ for } k = c+1, \ldots, c+h. \end{aligned}$$

Kronecker coefficients via 3d binary contingency arrays

$$\sum_{\alpha,\beta,\gamma} g(\alpha,\beta,\gamma) s_\alpha(x) s_\beta(y) s_{\gamma'}(z) = \prod_{i,j,k}(1 + x_i y_j z_k),$$

$$g(\alpha,\beta,\gamma) = \sum_{\sigma\in S_a, \pi\in S_b, \rho\in S_{\gamma_1}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi)\,\mathrm{sgn}(\rho) C(\alpha+\sigma-\mathrm{id}, \beta+\pi-\mathrm{id}, \gamma'+\rho-\mathrm{id}).$$

If $c := \nu_1$, $\lambda_1 \geq bc$ and $\mu_1 \geq ac \implies g(\lambda,\mu,\nu) = g(\,\lambda+h,\ \mu+h,\ \nu+h\,)$?
$(\gamma = \nu + h)$

$$\begin{aligned} Q_{1**} &:= \sum_{j,k} Q_{1,j,k} = \lambda_1 + \sigma_1 - 1 \geq bc + h, \\ Q_{*1*} &:= \sum_{i,k} Q_{i,1,k} = \mu_1 + \pi_1 - 1 \geq ac + h, \\ Q_{**k} &:= \sum_{i,j} Q_{i,j,k} = 1 + \rho_k - k, \text{ for } k = c+1, \ldots, c+h. \\ \sum_{k=c+1}^{c+h} Q_{**k} &= h + \sum_{k=c+1}^{c+h} \rho_k - \sum_{k=c+1}^{c+h} k \leq h, \end{aligned}$$

Kronecker coefficients via 3d binary contingency arrays

$$\sum_{\alpha,\beta,\gamma} g(\alpha,\beta,\gamma) s_\alpha(x) s_\beta(y) s_{\gamma'}(z) = \prod_{i,j,k} (1 + x_i y_j z_k),$$

$$g(\alpha,\beta,\gamma) = \sum_{\sigma \in S_a, \pi \in S_b, \rho \in S_{\gamma_1}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)\operatorname{sgn}(\rho) C(\alpha+\sigma-\mathrm{id}, \beta+\pi-\mathrm{id}, \gamma'+\rho-\mathrm{id}).$$

If $c := \nu_1$, $\lambda_1 \geq bc$ and $\mu_1 \geq ac \implies g(\lambda,\mu,\nu) = g(\lambda+h,\ \mu+h,\ \nu+h)$?
$(\gamma = \nu + h)$

$$\begin{aligned} Q_{1**} &:= \sum_{j,k} Q_{1,j,k} = \lambda_1 + \sigma_1 - 1 \geq bc + h, \\ Q_{*1*} &:= \sum_{i,k} Q_{i,1,k} = \mu_1 + \pi_1 - 1 \geq ac + h, \\ Q_{**k} &:= \sum_{i,j} Q_{i,j,k} = 1 + \rho_k - k, \text{ for } k = c+1, \ldots, c+h. \\ \sum_{k=c+1}^{c+h} Q_{**k} &= h + \sum_{k=c+1}^{c+h} \rho_k - \sum_{k=c+1}^{c+h} k \leq h, \end{aligned}$$

Lemma $\implies Q_{1**} = bc + h$, $Q_{*1*} = ac + h$, $\rho = \bar{\rho}, (c+1), \ldots, (c+h)$ for $\bar{\rho} \in S_c$

Kronecker coefficients via 3d binary contingency arrays

$$\sum_{\alpha,\beta,\gamma} g(\alpha,\beta,\gamma) s_\alpha(x) s_\beta(y) s_{\gamma'}(z) = \prod_{i,j,k}(1 + x_i y_j z_k),$$

$$g(\alpha,\beta,\gamma) = \sum_{\sigma\in S_a, \pi\in S_b, \rho\in S_{\gamma_1}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi)\,\mathrm{sgn}(\rho) C(\alpha+\sigma-\mathrm{id}, \beta+\pi-\mathrm{id}, \gamma'+\rho-\mathrm{id}).$$

If $c := \nu_1$, $\lambda_1 \geq bc$ and $\mu_1 \geq ac \implies g(\lambda,\mu,\nu) = g(\,\lambda+h,\ \mu+h,\ \nu+h\,)$?
$(\gamma = \nu + h)$

$$\begin{aligned} Q_{1**} &:= \sum_{j,k} Q_{1,j,k} = \lambda_1 + \sigma_1 - 1 \geq bc + h, \\ Q_{*1*} &:= \sum_{i,k} Q_{i,1,k} = \mu_1 + \pi_1 - 1 \geq ac + h, \\ Q_{**k} &:= \sum_{i,j} Q_{i,j,k} = 1 + \rho_k - k, \text{ for } k = c+1, \ldots, c+h. \\ \sum_{k=c+1}^{c+h} Q_{**k} &= h + \sum_{k=c+1}^{c+h} \rho_k - \sum_{k=c+1}^{c+h} k \leq h, \end{aligned}$$

Lemma $\implies Q_{1**} = bc + h$, $Q_{*1*} = ac + h$, $\rho = \bar{\rho}, (c+1), \ldots, (c+h)$ for $\bar{\rho} \in S_c$

$$g(\underbrace{\lambda+h}_{\alpha}, \underbrace{\mu+h}_{\beta}, \underbrace{\nu+h}_{\gamma'}) = \textstyle\sum_{\sigma\in S_a,\pi\in S_b,\rho\in S_{c+h}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi)\,\mathrm{sgn}(\rho)\, C(\alpha+\sigma-id, \beta+\pi-id, \gamma+\rho-id)$$

$$= \textstyle\sum_{\sigma\in S_a,\pi\in S_b,\eta\in S_c} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi)\,\mathrm{sgn}(\eta)\, C(\lambda+\sigma-id, \mu+\pi-id, \nu'+\eta-id) = g(\lambda,\mu,\nu),$$

Via the General Linear group

$GL_a \times GL_b \times GL_c$'s irreducible representations are $V_\alpha \otimes V_\beta \otimes V_\gamma$

$$g(\alpha, \beta, \gamma') = \dim \left(\mathrm{HWV}_{\alpha,\beta,\gamma} \bigwedge^D (\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) \right),$$

Via the General Linear group

$\mathrm{GL}_a \times \mathrm{GL}_b \times \mathrm{GL}_c$'s irreducible representations are $V_\alpha \otimes V_\beta \otimes V_\gamma$

$$g(\alpha,\beta,\gamma') = \dim\left(\mathrm{HWV}_{\alpha,\beta,\gamma} \bigwedge^D (\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)\right),$$

Raising operators $\mathcal{E} = \{(E_{i-1,i},0,0),(0,E_{i-1,i},0),(0,0,E_{i-1,i}) | i=2,\ldots\}$, e.g. $(E_{i,j},0,0)e_{r,1,1} = e_{i,1,1}$ iff $r=j$ and 0 otherwise, where $e_{i,j,k} := e_i \otimes e_j \otimes e_k$.

A **highest weight vector (HWV)** of weight (α,β,γ) is

$$u = \sum_{P \in \mathcal{C}(\alpha,\beta,\gamma)} c_P e_{P_1} \wedge e_{P_2} \wedge \cdots \quad s.t.\ Eu = 0\ \forall E \in \mathcal{E}$$

Via the General Linear group

$GL_a \times GL_b \times GL_c$'s irreducible representations are $V_\alpha \otimes V_\beta \otimes V_\gamma$

$$g(\alpha,\beta,\gamma') = \dim\left(\mathrm{HWV}_{\alpha,\beta,\gamma}\bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)\right),$$

Raising operators $\mathcal{E} = \{(E_{i-1,i},0,0),(0,E_{i-1,i},0),(0,0,E_{i-1,i}) | i = 2,\ldots\}$, e.g. $(E_{i,j},0,0)e_{r,1,1} = e_{i,1,1}$ iff $r = j$ and 0 otherwise, where $e_{i,j,k} := e_i \otimes e_j \otimes e_k$.

A **highest weight vector (HWV)** of weight (α,β,γ) is

$$u = \sum_{P \in \mathcal{C}(\alpha,\beta,\gamma)} c_P e_{P_1} \wedge e_{P_2} \wedge \cdots \quad s.t.\ Eu = 0\ \forall E \in \mathcal{E}$$

$$t := e_{1,1,1} \wedge e_{2,1,1} \wedge e_{1,2,2} + e_{1,1,1} \wedge e_{1,2,1} \wedge e_{2,1,2} + e_{1,1,1} \wedge e_{1,1,2} \wedge e_{2,2,1}$$

is a HWV of weight $((2,1),(2,1),(2,1))$ in $\bigwedge^3(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$:

$$\begin{aligned}(E_{1,2},0,0)t &= e_{1,1,1} \wedge e_{1,2,1} \wedge e_{1,1,2} + e_{1,1,1} \wedge e_{1,1,2} \wedge e_{1,2,1} = 0,\\ (0,E_{1,2},0)t &= e_{1,1,1} \wedge e_{2,1,1} \wedge e_{1,1,2} + e_{1,1,1} \wedge e_{1,1,2} \wedge e_{2,1,1} = 0,\\ (0,0,E_{1,2})t &= e_{1,1,1} \wedge e_{2,1,1} \wedge e_{1,2,1} + e_{1,1,1} \wedge e_{1,2,1} \wedge e_{2,1,1} = 0.\end{aligned}$$

Via the General Linear group

$\mathrm{GL}_a \times \mathrm{GL}_b \times \mathrm{GL}_c$'s irreducible representations are $V_\alpha \otimes V_\beta \otimes V_\gamma$

$$g(\alpha,\beta,\gamma') = \dim\left(\mathrm{HWV}_{\alpha,\beta,\gamma} \bigwedge^D (\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)\right),$$

Raising operators $\mathcal{E} = \{(E_{i-1,i},0,0),(0,E_{i-1,i},0),(0,0,E_{i-1,i}) | i = 2,\ldots\}$, e.g. $(E_{i,j},0,0)e_{r,1,1} = e_{i,1,1}$ iff $r = j$ and 0 otherwise, where $e_{i,j,k} := e_i \otimes e_j \otimes e_k$.

A **highest weight vector (HWV)** of weight (α,β,γ) is

$$u = \sum_{P \in \mathcal{C}(\alpha,\beta,\gamma)} c_P e_{P_1} \wedge e_{P_2} \wedge \cdots \quad s.t.\ Eu = 0\ \forall E \in \mathcal{E}$$

$$\begin{array}{rccc} \varphi: & \bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) & \to & \bigwedge^{D+h}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^{c+h}) \\ & v & \mapsto & v \wedge e_{1,1,c+1} \wedge e_{1,1,c+2} \wedge \cdots \wedge e_{1,1,c+h} \end{array}$$

Claim: φ is an isomorphism $\mathrm{HWV}_{\lambda,\mu,\gamma} \leftrightarrow \mathrm{HWV}_{\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma}}$.

Via the General Linear group

$\mathrm{GL}_a \times \mathrm{GL}_b \times \mathrm{GL}_c$'s irreducible representations are $V_\alpha \otimes V_\beta \otimes V_\gamma$

$$g(\alpha,\beta,\gamma') = \dim\left(\mathrm{HWV}_{\alpha,\beta,\gamma}\bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)\right),$$

Raising operators $\mathcal{E} = \{(E_{i-1,i},0,0),(0,E_{i-1,i},0),(0,0,E_{i-1,i}) | i = 2,\ldots\}$, e.g. $(E_{i,j},0,0)e_{r,1,1} = e_{i,1,1}$ iff $r = j$ and 0 otherwise, where $e_{i,j,k} := e_i \otimes e_j \otimes e_k$.

A **highest weight vector (HWV)** of weight (α,β,γ) is

$$u = \sum_{P \in \mathcal{C}(\alpha,\beta,\gamma)} c_P e_{P_1} \wedge e_{P_2} \wedge \cdots \quad s.t.\ Eu = 0\ \forall E \in \mathcal{E}$$

$$\begin{array}{rccc} \varphi: & \bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) & \to & \bigwedge^{D+h}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^{c+h}) \\ & v & \mapsto & v \wedge e_{1,1,c+1} \wedge e_{1,1,c+2} \wedge \cdots \wedge e_{1,1,c+h} \end{array}$$

Claim: φ is an isomorphism $\mathrm{HWV}_{\lambda,\mu,\gamma} \leftrightarrow \mathrm{HWV}_{\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma}}$.

If $w = \sum_Q a_Q e_{Q_1} \wedge e_{Q_2} \cdots \in \mathrm{HWV}_{\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma}}$, where $Q \in \mathcal{C}(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$.
3d binary CTs Lemma: $\{1\} \times \{1\} \times [c+1, c+h] \subset Q$ and $Q \cap (\mathbb{N} \times \mathbb{N} \times \{i\}) = \{(1,1,i)\}$ for all $c+1 \le i \le c+h$, so $w = u \wedge e_{1,1,c+1} \wedge \cdots \wedge e_{1,1,c+h}$ for $u \in \mathrm{HWV}_{\lambda,\mu,\gamma}$.

Using multi-Littlewood-Richardson coefficients

Set $\hat{\mu} = \mu + h$, $\hat{\lambda} = \lambda' \cup (1^h) = (\lambda + h)'$ and $\hat{\nu} = \nu' \cup (1^h) = (\nu + h)'$

$$g(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots}$$

Using multi-Littlewood-Richardson coefficients

Set $\hat{\mu} = \mu + h$, $\hat{\lambda} = \lambda' \cup (1^h) = (\lambda + h)'$ and $\hat{\nu} = \nu' \cup (1^h) = (\nu + h)'$

$$g(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \text{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots}$$

$c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} := \langle s_{\hat{\lambda}}, s_{\alpha^1} s_{\alpha^2} \cdots \rangle = \#$ certain SSYTs of type $(\alpha^1 \cup \alpha^2 \cup \ldots \cup \alpha^c \cup \ldots)$, shape $\hat{\lambda}$:

1	1	1	1	4	4	6
2	2	2	4	5	7	
3	5	5	6	6		

and

1	1	1	1	4	4	6
2	2	2	4	6	6	
3	5	5	5	7		

multi-LR tableaux of shape $\lambda = (7, 6, 5)$ and types $\alpha^1 = (4, 3, 1)$, $\alpha^2 = (3, 3)$, $\alpha^3 = (3, 1)$.

Using multi-Littlewood-Richardson coefficients

Set $\hat{\mu} = \mu + h$, $\hat{\lambda} = \lambda' \cup (1^h) = (\lambda + h)'$ and $\hat{\nu} = \nu' \cup (1^h) = (\nu + h)'$

$$g(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \dots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \dots}$$

$c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \dots} := \langle s_{\hat{\lambda}}, s_{\alpha^1} s_{\alpha^2} \cdots \rangle = \#$ certain SSYTs of type $(\alpha^1 \cup \alpha^2 \cup \ldots \cup \alpha^c \cup \ldots)$, shape $\hat{\lambda}$:

1	1	1	1	4	4	6
2	2	2	4	5	7	
3	5	5	6	6		

and

1	1	1	1	4	4	6
2	2	2	4	6	6	
3	5	5	5	7		

multi-LR tableaux of shape $\lambda = (7, 6, 5)$ and types $\alpha^1 = (4, 3, 1)$, $\alpha^2 = (3, 3)$, $\alpha^3 = (3, 1)$.

$\Longrightarrow \alpha^i \subset \hat{\mu}$, $\alpha^i \subset \hat{\lambda}$, so $\ell(\alpha^i) \leq \ell(\mu) = b$, $\alpha^i_1 \leq \hat{\lambda}_1 = a$.

Using multi-Littlewood-Richardson coefficients

Set $\hat{\mu} = \mu + h$, $\hat{\lambda} = \lambda' \cup (1^h) = (\lambda + h)'$ and $\hat{\nu} = \nu' \cup (1^h) = (\nu + h)'$

$$g(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \operatorname{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots}$$

$c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} := \langle s_{\hat{\lambda}}, s_{\alpha^1} s_{\alpha^2} \cdots \rangle = \#$ certain SSYTs of type $(\alpha^1 \cup \alpha^2 \cup \ldots \cup \alpha^c \cup \ldots)$, shape $\hat{\lambda}$:

1	1	1	1	4	4	6
2	2	2	4	5	7	
3	5	5	6	6		

and

1	1	1	1	4	4	6
2	2	2	4	6	6	
3	5	5	5	7		

multi-LR tableaux of shape $\lambda = (7, 6, 5)$ and types $\alpha^1 = (4, 3, 1)$, $\alpha^2 = (3, 3)$, $\alpha^3 = (3, 1)$.

$\Longrightarrow \alpha^i \subset \hat{\mu}$, $\alpha^i \subset \hat{\lambda}$, so $\ell(\alpha^i) \leq \ell(\mu) = b$, $\alpha^i_1 \leq \hat{\lambda}_1 = a$.

$$h \leq \ell(\alpha^{c+1}) + \cdots + \ell(\alpha^{c+h}) \leq |\alpha^{c+1}| + \cdots + |\alpha^{c+h}| = \sum_{i=c+1}^{c+h} 1 - i + \sigma_i \leq h,$$

Using multi-Littlewood-Richardson coefficients

Set $\hat{\mu} = \mu + h$, $\hat{\lambda} = \lambda' \cup (1^h) = (\lambda + h)'$ and $\hat{\nu} = \nu' \cup (1^h) = (\nu + h)'$

$$g(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \operatorname{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots}$$

$c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} := \langle s_{\hat{\lambda}}, s_{\alpha^1} s_{\alpha^2} \cdots \rangle = \#$ certain SSYTs of type $(\alpha^1 \cup \alpha^2 \cup \ldots \cup \alpha^c \cup \ldots)$, shape $\hat{\lambda}$:

1	1	1	1	4	4	6
2	2	2	4	5	7	
3	5	5	6	6		

and

1	1	1	1	4	4	6
2	2	2	4	6	6	
3	5	5	5	7		

multi-LR tableaux of shape $\lambda = (7, 6, 5)$ and types $\alpha^1 = (4, 3, 1)$, $\alpha^2 = (3, 3)$, $\alpha^3 = (3, 1)$.

$\Longrightarrow \alpha^i \subset \hat{\mu}$, $\alpha^i \subset \hat{\lambda}$, so $\ell(\alpha^i) \leq \ell(\mu) = b$, $\alpha^i_1 \leq \hat{\lambda}_1 = a$.

$$h \leq \ell(\alpha^{c+1}) + \cdots + \ell(\alpha^{c+h}) \leq |\alpha^{c+1}| + \cdots + |\alpha^{c+h}| = \sum_{i=c+1}^{c+h} 1 - i + \sigma_i \leq h,$$

$$|\alpha^{c+1}| + \cdots + |\alpha^{c+h}| = h, \ell(\alpha^i) = |\alpha^i|.$$

Using multi-Littlewood-Richardson coefficients

Set $\hat\mu = \mu + h$, $\hat\lambda = \lambda' \cup (1^h) = (\lambda+h)'$ and $\hat\nu = \nu' \cup (1^h) = (\nu+h)'$

$$g(\lambda+h, \mu+h, \nu+h) = \sum_{\sigma \in S_{c+h}} \operatorname{sgn}(\sigma) \sum_{\alpha^i \vdash \hat\nu_i - i + \sigma_i} c^{\hat\lambda}_{\alpha^1\alpha^2\dots} c^{\hat\mu}_{\alpha^1\alpha^2\dots}$$

$$|\alpha^{c+1}| + \cdots + |\alpha^{c+h}| = h, \ell(\alpha^i) = |\alpha^i|.$$

$\alpha^i \subset (\lambda+h)'$, so $\alpha^i_i \leq a$. $\alpha^i \subset \hat\mu$. Multi-LR of type $(\alpha^1 \cup \alpha^2 \cdots)$ shape $\hat\mu$, so

$$ac + h = \hat\mu_1 \leq \sum_i \alpha^i_1 \leq \sum_{i=1}^{c} a + \sum_{i=c+1}^{c+h} \alpha^i_1.$$

$\implies \alpha_1^{c+1} + \cdots + \alpha_1^{c+h} \geq h$. $\implies \alpha^i = (1)$ for all $i > c$ and $\sigma_i = i$ for $i = c+1, \dots, c+h$.

Using multi-Littlewood-Richardson coefficients

Set $\hat{\mu} = \mu + h$, $\hat{\lambda} = \lambda' \cup (1^h) = (\lambda + h)'$ and $\hat{\nu} = \nu' \cup (1^h) = (\nu + h)'$

$$g(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots}$$

$$|\alpha^{c+1}| + \cdots + |\alpha^{c+h}| = h, \ell(\alpha^i) = |\alpha^i|.$$

$\alpha^i \subset (\lambda + h)'$, so $\alpha^i_i \leq a$. $\alpha^i \subset \hat{\mu}$. Multi-LR of type $(\alpha^1 \cup \alpha^2 \cdots)$ shape $\hat{\mu}$, so

$$ac + h = \hat{\mu}_1 \leq \sum_i \alpha^i_1 \leq \sum_{i=1}^{c} a + \sum_{i=c+1}^{c+h} \alpha^i_1.$$

$\implies \alpha_1^{c+1} + \cdots + \alpha_1^{c+h} \geq h$. $\implies \alpha^i = (1)$ for all $i > c$ and $\sigma_i = i$ for $i = c+1, \ldots, c+h$.

$$c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots \alpha^{c+h}} = c^{\lambda'}_{\alpha^1 \ldots \alpha^c} \quad \text{and} \quad c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots \alpha^{c+h}} = c^{\mu}_{\alpha^1 \ldots \alpha^c}.$$

$$g(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots}$$

$$= \sum_{\sigma \in S_c} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \nu'_i - i + \sigma_i} c^{\lambda'}_{\alpha^1 \alpha^2 \ldots} c^{\mu}_{\alpha^1 \alpha^2 \ldots} = g(\nu', \lambda', \mu) = g(\lambda, \mu, \nu),$$

Vielen Dank für Ihre Aufmerksamkeit!