See-saw pairs and plethysm

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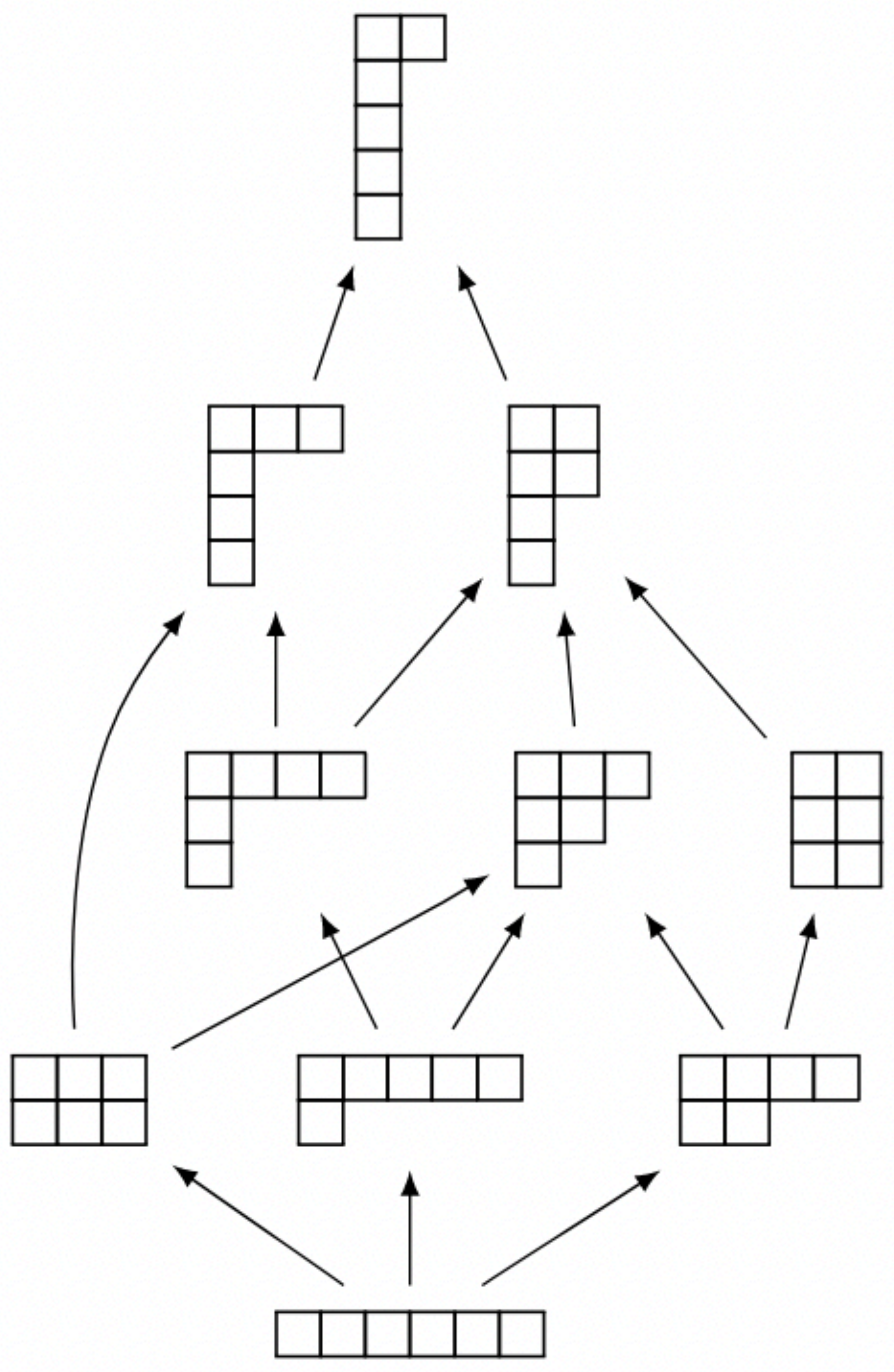
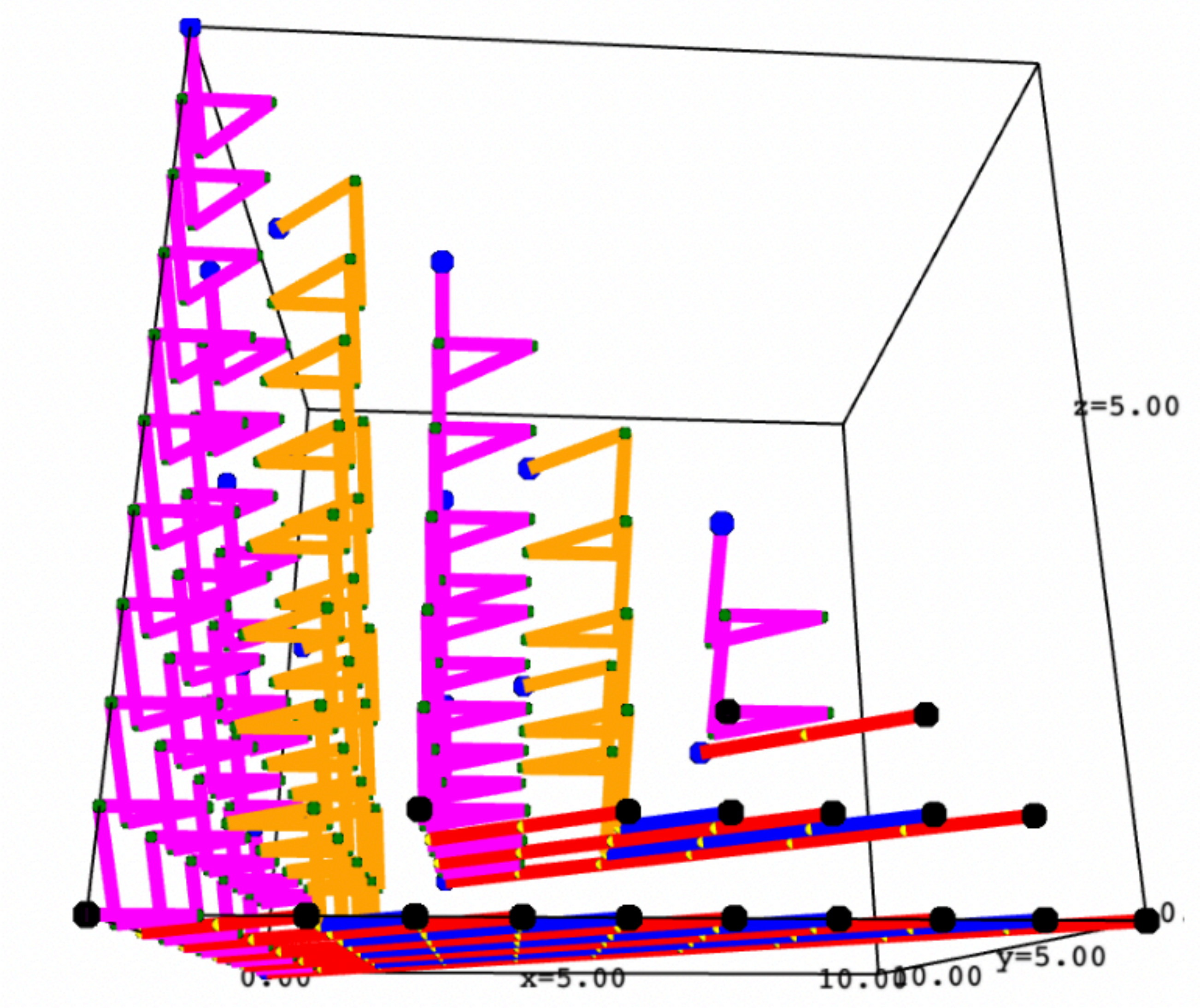
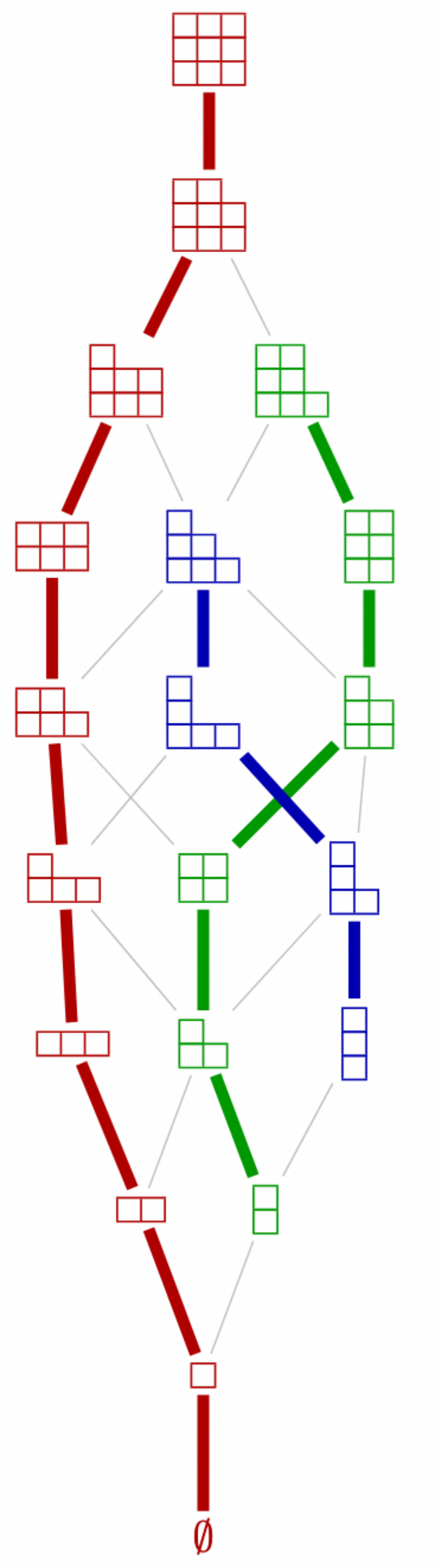






Based on join work with

F. Saliola

A. Schilling

M. Zabrocki

- Plethysm and the algebra of uniform block permutations, 2022.
- The lattice of submonoids of the uniform block permutations containing the symmetric group (arXiv:2405.09710), 2024.
- Special thanks to AIM, ICERM, BIRS at Banff.

The symmetric group

Sagan's book

- The symmetric group S_n.
- A **representation** is a homomorphism: $\rho : S_n \to GL_d$.
- The irreducible representations $\mathbb{S}^\lambda$ are indexed by partitions of n .
- The module, $\mathbb{S}^\lambda$ is spanned by **standard** tableaux of shape λ.

Example: $\mathbb{S}^{(2,2)}$

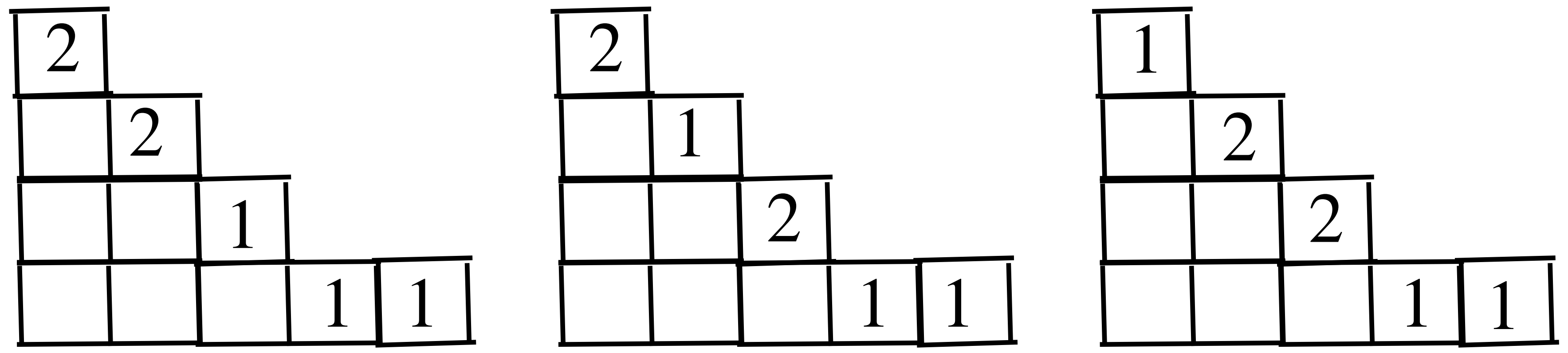
2	4
1	3

3	4
1	2

	(1)(2)(3)	(1,2)(3)	(1,2,3)
$\chi^{(1,1,1)}$	1	-1	1
$\chi^{(2,1)}$	2	0	-1
$\chi^{(3)}$	1	1	1

- The **character** of a representation: $\mathrm{trace}(\rho(\sigma))$, for $\sigma \in S_n$.
- Restriction of representations: $\mathrm{Res}^{S_{k+m}}_{S_k \times S_m} \mathbb{S}^\nu = \bigoplus_{\lambda \vdash k, \mu \vdash m} (\mathbb{S}^\lambda \otimes \mathbb{S}^\mu)^{c^\nu_{\lambda,\mu}}$

Example: $k = 6,\ m = 5.$ The representation $\mathbb{S}^{(3,2,1)} \otimes \mathbb{S}^{(3,2)}$ occurs 3 times in the restriction of $\mathbb{S}^{(5,3,2,1)}$ from S_{11} to $S_6 \times S_5$.



The General Linear Group

- The **general linear group** $GL_n = GL_n(\mathbb{C})$ is the group of invertible $n \times n$ matrices.
- A **representation** is a homomorphism: $\rho : GL_n \to GL_m$.
- For any $g \in GL_n$, $\rho(g)$ is an $m \times m$ matrix.
- Irreducible polynomial representation are indexed by partitions λ with at most n parts: $\mathbb{V}^{\lambda}$

Example: $\mathbb{V}^{(2)}$ $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}$$

- The module, $\mathbb{V}^{\lambda}$ is spanned by semistandard tableaux of shape λ with entries in $\{1,\ldots,n\}$.

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 8 \\ \cline{1-1} 7 & 8 & 9 \\ \cline{1-3} 3 & 5 & 6 & 7 \\ \cline{1-4} 1 & 3 & 3 & 4 & 4 & 5 \\ \hline \end{array}$$

Example: $\mathbb{V}^{(2)}$ in GL_2 has basis $\boxed{1\,|\,1}$ $\boxed{1\,|\,2}$ $\boxed{2\,|\,2}$

$\mathbb{V}^{(2)}$ in GL_3 has basis $\boxed{1\,|\,1}$ $\boxed{1\,|\,2}$ $\boxed{1\,|\,3}$ $\boxed{2\,|\,2}$ $\boxed{2\,|\,3}$ $\boxed{3\,|\,3}$

The General Linear Group

- The **character** of a representation: trace$(\rho(g))$, for $g \in GL_n$.

- The characters of the polynomial irreducible representations of GL_n are evaluations of Schur functions (Schur polynomials).

Schur Functions: $$s_\lambda(x) = \sum_{T \in SSYT(\lambda)} x^T$$ SSYT "semistandard Young tableaux"

Example: $\lambda = (6,4,3,1)$ $T = \begin{array}{|c|c|c|c|c|c|} \hline 8 \\ \hline 7 & 8 & 9 \\ \hline 3 & 5 & 6 & 7 \\ \hline 1 & 3 & 3 & 4 & 4 & 5 \\ \hline \end{array}$ $x^T = x_1 x_3^3 x_4^2 x_5^2 x_6 x_7^2 x_8^2 x_9$

Example: The character of the GL_2 representation $\mathbb{V}^{(2)}$ is $s_{(2)}(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$

$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array}$ $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$ $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array}$

- Computing characters of elements:

If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has eigenvalues θ_1 and θ_2, then trace$(\rho^{(2)}(g)) = s_{(2)}(\theta_1, \theta_2) = \theta_1^2 + \theta_1\theta_2 + \theta_2^2$.

Characters and symmetric functions

Polynomial Representation of GL_n ⟷ Symmetric Functions

"formal (universal) character"

Representation	Character
$\mathbb{V}^\lambda$	$s_\lambda(x_1, \ldots, x_n)$
$\mathrm{Sym}^{\lambda_1} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_\ell} V$	$h_\lambda(x_1, \ldots, x_n)$
$\wedge^{\lambda_1} V \otimes \cdots \otimes \wedge^{\lambda_\ell} V$	$e_\lambda(x_1, \ldots, x_n)$
$\mathbb{V}^\lambda \otimes \mathbb{V}^\mu \cong \bigoplus_\nu \left(\mathbb{V}^\nu\right)^{c^\nu_{\lambda,\mu}}$	$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$

$c^\nu_{\lambda,\mu}$ are the **Littlewood-Richarson coeffs**.

Same as for restriction of S_n -representations.

Plethysm - composing characters

GL_n **representation:** $\rho : GL_n \to GL_m$

GL_m **representation:** $\tau : GL_m \to GL_r$

Then the composition is a representation of GL_n .

$$\tau \circ \rho : GL_n \to GL_r$$

We call the character of the composition: plethysm.

If f **and** g **are symmetric functions, then the plethysm is denoted by** $f[g]$.

Example: $s_{(2)}(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$

[1|1] [1|2] [2|2]

$s_{(2)}(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$

[1|1] [1|2] [1|3] [2|2] [2|3] [3|3]

$$s_{(2)}[s_{(2)}(x_1, x_2)] = s_2(x_1^2, x_1x_2, x_2^2)$$

$$= x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4$$

[[1|1]|[1|1]] [[1|1]|[1|2]] [[1|1]|[2|2]] [[1|2]|[1|2]] [[1|2]|[2|2]] [[2|2]|[2|2]]

$$= s_4(x_1, x_2) + s_{2,2}(x_1, x_2)$$

Plethysm Problem

Problem: Find a **combinatorial interpretation** for the coefficients $a_{\lambda,\mu}^{\nu} \in \mathbb{Z}_{\geq 0}$ in the expansion

$$s_\lambda[s_\mu] = \sum_\nu a_{\lambda,\mu}^{\nu} s_\nu$$

Very few special cases are known:

- Carre and Leclerc: $s_2[s_\mu]$ and $s_{1,1}[s_\mu]$
- COSSZ: $s_\lambda[s_m]$ when $\lambda \vdash 3$.
- Howe: $s_4[s_m]$ complicated expressions for the coefficients.
- Littlewood (see Macdonal page 138): $s_n[s_2]$, $s_n[s_{1,1}]$, $s_1^n[s_2]$ and $s_{1^n}[s_{1,1}]$.
- Bowman, Paget, Wildon - stable coefficients in $s[n](s[m])$.

Remark: A solution for $s_m[s_n]$ would help prove **Foulkes' Conjecture**: For $n > m$

$$s_n[s_m] - s_m[s_n] \text{ is Schur-positive.}$$

Subproblem: We are interested in the plethysm $s_\lambda[s_m]$.

Restriction to the symmetric group

- The symmetric group as the group of permutation matrices is a subgroup of GL_n.

$$S_n \subset GL_n$$

$$GL_n \hookleftarrow S_n$$

The Restriction Problem: Given an irreducible representation of GL_n, decompose it as a representation of S_n.

$$\mathbf{Res}_{S_n}^{GL_n} \mathbb{V}^{\lambda} \cong \bigoplus_{\mu} (\mathbb{S}^{\mu})^{r_{\lambda,\mu}}$$

Example: $\mathbf{Res}_{S_4}^{GL_4} \mathbb{V}^{(2)} \cong \left(\mathbb{S}^{(4)}\right)^{\oplus 2} \oplus \left(\mathbb{S}^{(3,1)}\right)^{\oplus 2} \oplus \mathbb{S}^{(2,2)}$

Littlewood, Buttler and King, Narayanan, Paul, Prasad and Srivastava '22

Restricting Characters

- Every character of GL_n is a character of S_n.

Example: Restrict the representation $\mathbb{V}^{(2)}$ of GL_3 to S_3.

$$s_{(2)}(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

Permutation matrix for each conjugacy class:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Eigenvalues: $1,1,1$ $\quad 1,-1,1$ $\quad 1,\xi,\xi^2$

$$s_2(1,1,1) = 6 \qquad s_2(1,-1,1) = 2 \qquad s_2(1,\xi,\xi^2) = 0$$

The decomposition for $s_{(2)}(x_1, x_2, x_3)$ as an S_3 character is

	$(1)(2)(3)$	$(1,2)(3)$	$(1,2,3)$
$\chi^{(1,1,1)}$	1	-1	1
$\chi^{(2,1)}$	2	0	-1
$\chi^{(3)}$	1	1	1

$$< 6,2,0 > = 2 < 1,1,1 > + 2 < 2,0,-1 >$$

$$= 2\,\chi^{(3)} + 2\,\chi^{(2,1)}$$

Restricting Characters - Symmetric Functions

- Zabrocki and I introduced a new basis of symmetric functions : $\{\tilde{s}_\lambda : \lambda$ a partition $\}$ such that if Ξ_μ are the eigenvalues of permutation matrices, we have

$$\tilde{s}_\alpha(\Xi_\mu) = \chi^{(n-|\alpha|,\alpha)}(\mu)$$

Example:

$$\tilde{s}_{()}(x_1, x_2, x_3) = 1 \qquad \tilde{s}_1(x_1, x_2, x_3) = x_1 + x_2 + x_2 - 1$$

$$\tilde{s}_{1,1}(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3 - x_1 - x_2 - x_3 + 1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Eigenvalues: $1,1,1$ $1,-1,1$ $1,\xi,\xi^2$

$$\tilde{s}_1(1,1,1) = 2 \qquad \tilde{s}_1(1,-1,1) = 0 \qquad \tilde{s}_1(1,\xi,\xi^2) = -1$$

	(1)(2)(3)	(1,2)(3)	(1,2,3)
$\chi^{(1,1,1)}$	1	−1	1
$\chi^{(2,1)}$	2	0	−1
$\chi^{(3)}$	1	1	1

- Using this basis, the restriction problem is reformulated as:

$$s_\lambda = \sum_\mu r_{\lambda,\mu} \tilde{s}_\mu$$

restriction coefficient

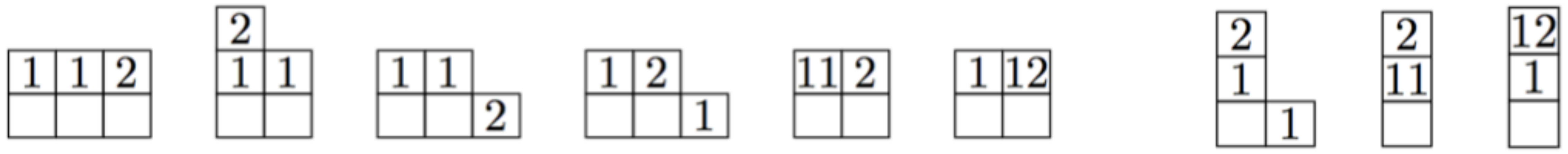
Restricting Characters - Symmetric Functions

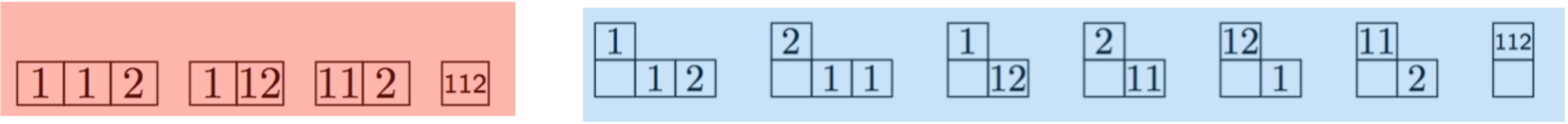
Theorem: [O - Zabrocki]

$$h_\mu = \sum_\lambda M_{\lambda,\mu} \tilde{s}_\lambda$$

where $M_{\lambda,\mu}$ is the number of semi standard multiset filled tableaux of shape $(r, \lambda)/\lambda_1$ and content μ.

Example: If $\mu = (2,1)$ the entries are multisets of 1,1,2.





$$h_{21} = 4\tilde{s}_{.} + 7\tilde{s}_1 + 3\tilde{s}_{11} + 4\tilde{s}_2 + \tilde{s}_{21} + \tilde{s}_3 \ .$$

- Recall: h_μ is the character of $Sym^{\mu_1}V \otimes \cdots \otimes Sym^{\mu_\ell}V$.

Plethysm and Restriction

$$\textbf{Res}^{GL_n}_{S_n}\mathbb{V}^\lambda \cong \bigoplus_\mu (\mathbb{S}^\mu)^{r_{\lambda,\mu}}$$

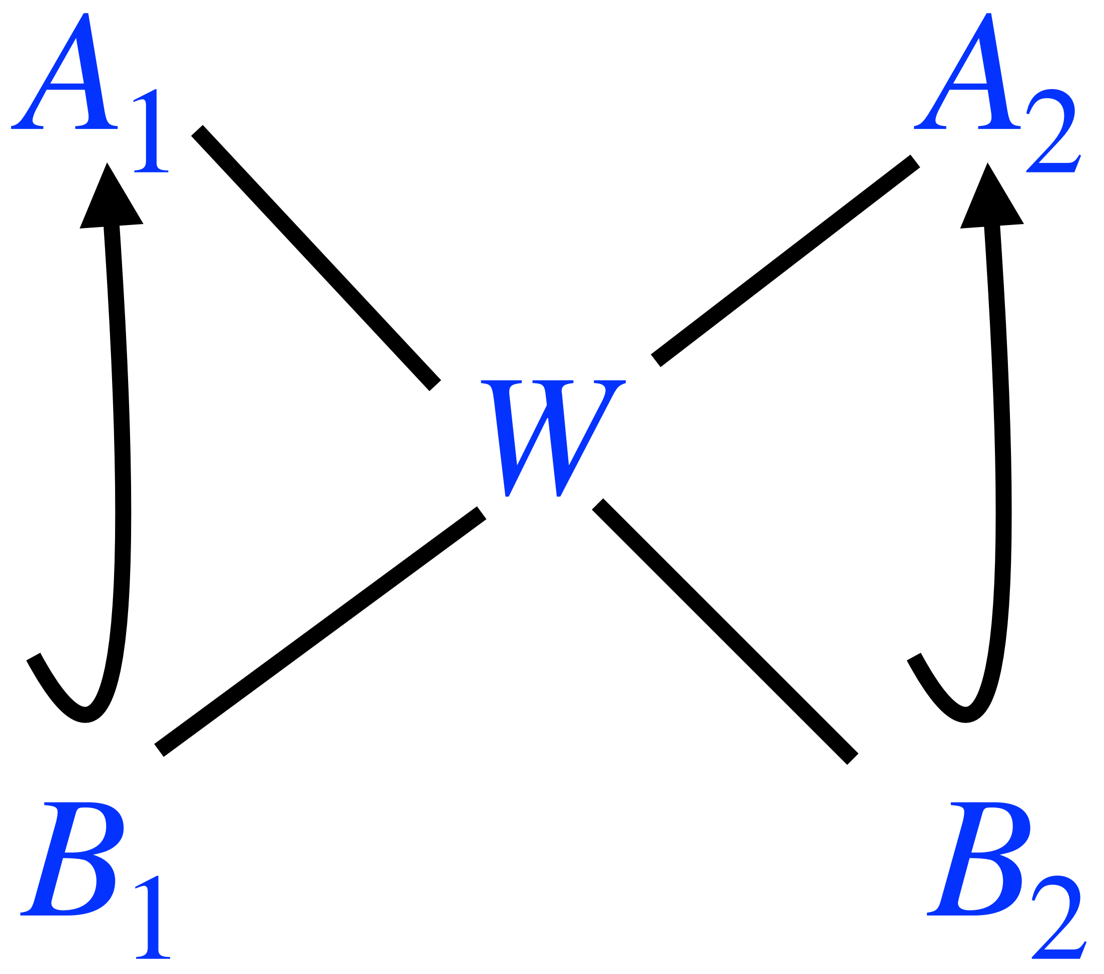
Littlewood '1950s and reformulated by Scharf and Thibon

Theorem: $r_{\lambda,\mu} = \langle s_\lambda, s_\mu[1 + s_1 + s_2 + s_3 + \cdots]\rangle$

Subproblem: We are interested in the plethysm $s_\lambda[s_m]$.

See-saw pairs

Let A_1, A_2, B_1, B_2 be groups/algebras such that $B_1 \subset A_1, B_2 \subset A_2$. All acting on the same vector space W.



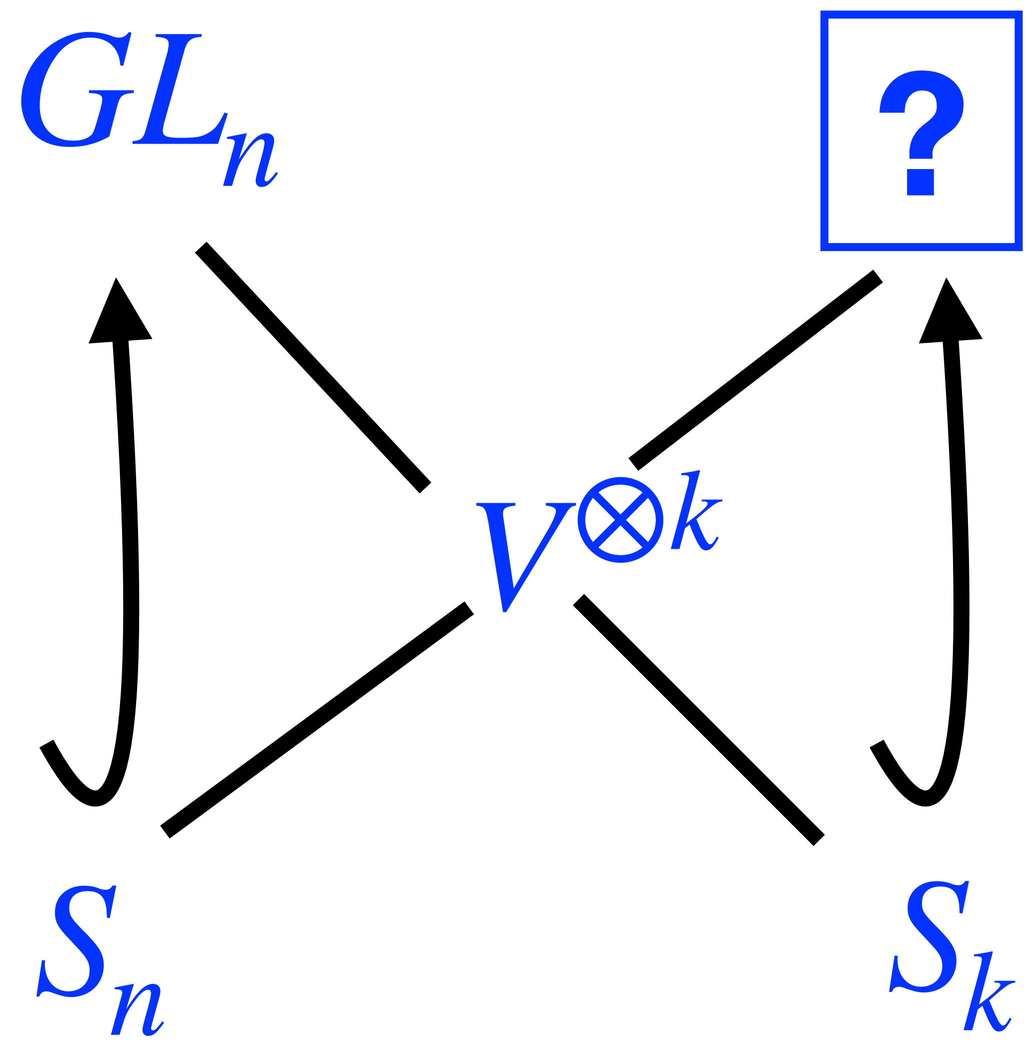
$$A_2 \cong \mathbf{End}_{B_1}(W)$$

$$B_2 \cong \mathbf{End}_{A_1}(W)$$

If (A_2, B_1) and (A_1, B_2) are centralizer pairs then

$$\mathbf{Res}^{A_1}_{B_1} V^{\lambda}_{A_1} \cong \bigoplus_{\mu} (V^{\mu}_{B_1})^{m_{\lambda,\mu}} \quad \text{and} \quad \mathbf{Res}^{A_2}_{B_2} V^{\mu}_{A_2} \cong \bigoplus_{\lambda} (V^{\lambda}_{B_2})^{m_{\lambda,\mu}}$$

Restricting Characters - See-saw approach

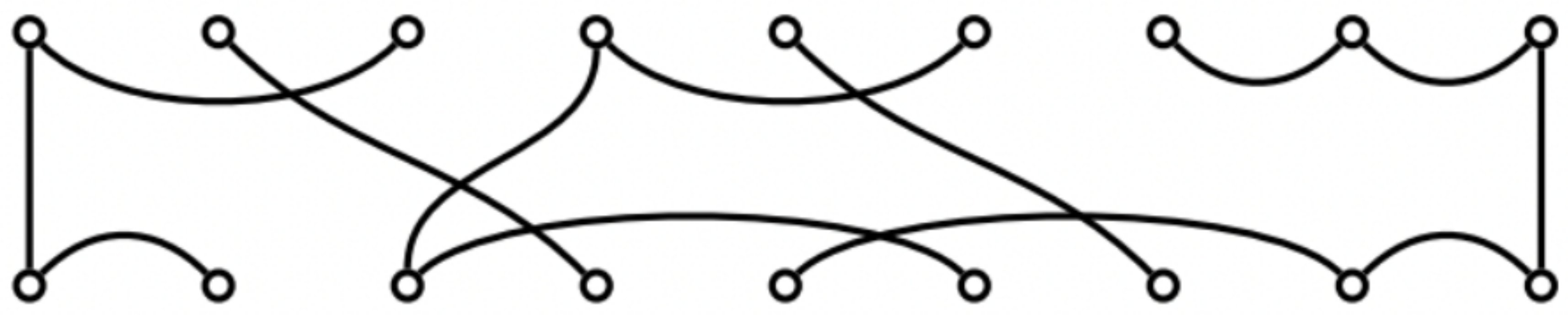


By Schur-Weyl duality (GL_n, S_k) is a centralizer pair

Think of $S_n \subseteq GL_n$ as the subgroup of permutation matrices acting diagonally on $V^{\otimes k}$

$$\sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sigma v_1 \otimes \sigma v_2 \otimes \cdots \otimes \sigma v_k$$

what commutes with this action?



$$\{\{1,3,\bar{1},\bar{2}\},\{2,\bar{4}\},\{4,6,\bar{3},\bar{6}\},\{5,\bar{7}\},\{7,8,9,\bar{5},\bar{8},\bar{9}\}\}$$

1990s: Jones and Martin

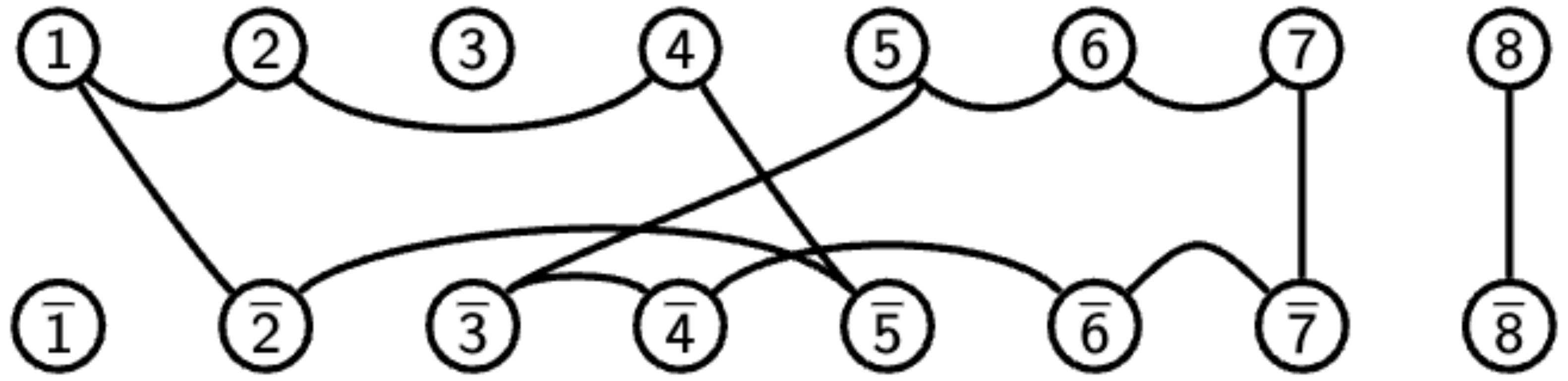
Partition Algebra

- For any positive integer k, let $[k] = \{1,\ldots,k\}$ and $[\bar{k}] = \{\bar{1},\ldots,\bar{k}\}$
- The partition algebra, $P_k(n)$ has

  Basis: set partitions of $[k] \cup [\bar{k}]$

  Example:

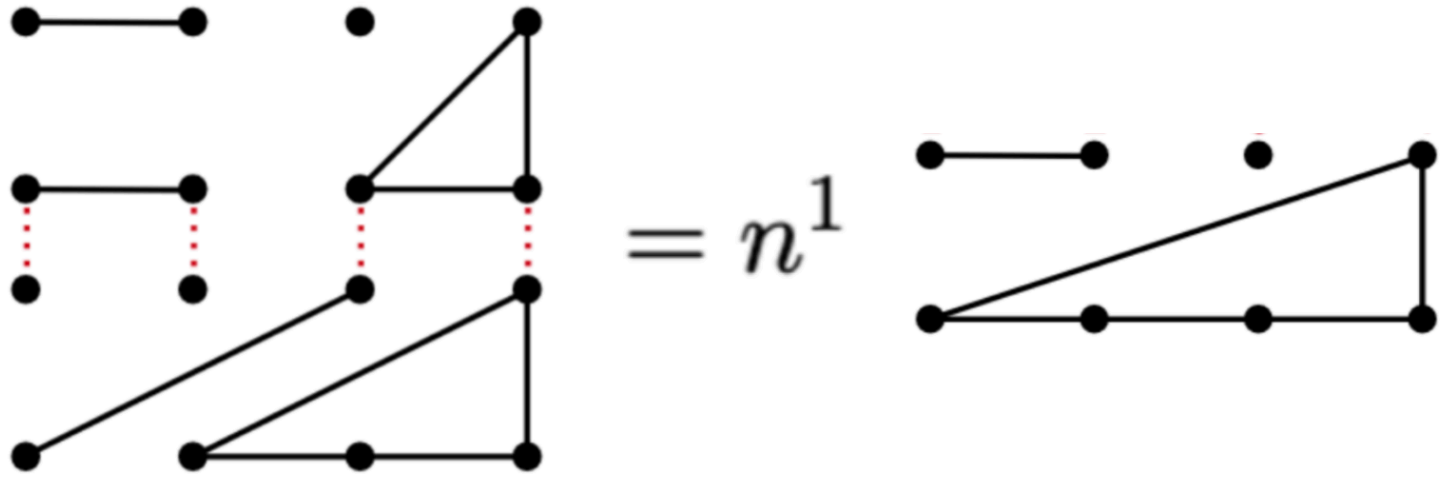
  $k = 9$



$\{\{1,2,4,\bar{2},\bar{5}\},\{3\},\{5,6,7,\bar{3},\bar{4},\bar{6},\bar{7}\},\{8,\bar{8}\},\{\bar{1}\}\}$

  Product:

$dd' =$



- The partition algebra, $P_k(n)$ has an identity $1 = \{\{1,\bar{1}\},\ldots,\{k,\bar{k}\}\}$ and it has dimension equal $B(2k)$, the Bell number.
- Halverson-Ram, Halverson, Jacobson-Halverson, etc.

$P_k(n)$ and S_n form a centralizer pair

The partition algebra is not always semisimple, but in the cases when it is semisimple, we have

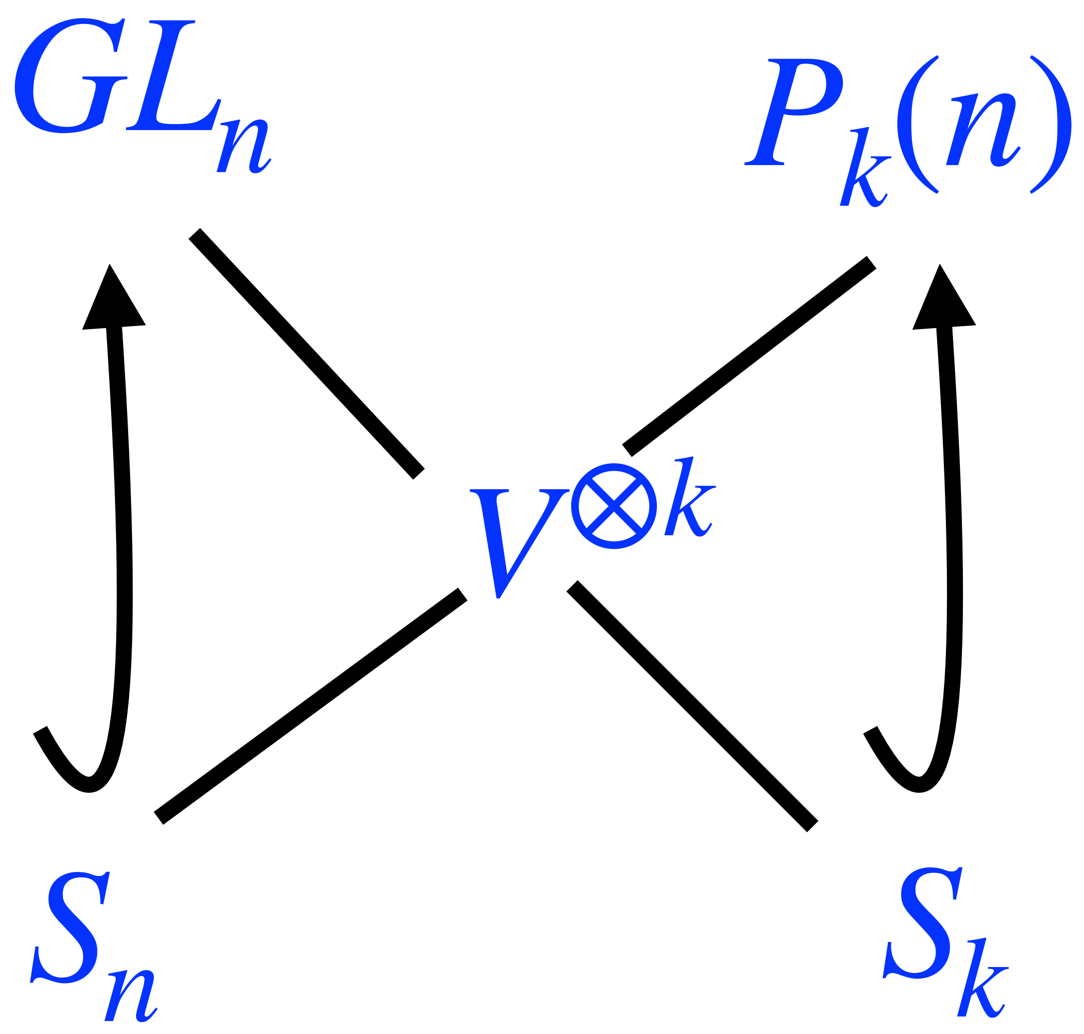
$V_{P_k(n)}^{(n-|\lambda|,\lambda)}$ - **The irreducible representations are indexed by partitions λ such that** $\lambda_1 + \lambda_2 + \cdots \leq k$

Irreducible have bases consisting of standard tableau where entries are sets .

$$V_{P_3(6)}^{(4,2)} = \text{span}\left\{ \begin{array}{|c|c|c|c|}\hline & & & 3 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|}\hline & & & 2 \\ \hline 1 & 3 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|}\hline & & & \\ \hline 12 & 3 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|}\hline & & & \\ \hline 1 & 23 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|}\hline & & & \\ \hline 2 & 13 \\ \cline{1-2}\end{array} \right\}$$

Jones 1994 - $(P_k(n), S_n)$ form a centralizer pair

Our see-saw pair



$$\mathbf{Res}_{S_k}^{P_k(n)} V_{P_k(n)}^{\mu} \cong \bigoplus_{\lambda} (\mathbb{S}^{\lambda})^{\oplus r_{\lambda,\mu}}$$

$$\mathbf{Res}_{S_n}^{GL_n} \mathbb{V}^{\lambda} \cong \bigoplus_{\mu} (\mathbb{S}^{\mu})^{\oplus r_{\lambda,\mu}}$$

Idea: To solve the restriction problem, solve the restriction of $P_k(n)$ to S_k

An approach for restriction

U_k **the uniform block permutation algebra.**

$$\mathsf{S}_k \hookrightarrow \mathsf{U}_k \hookrightarrow \mathsf{P}_k(n)$$

special cases of plethysm **generalized LR coefficients**

Why U_k? It is smaller and has a rich structure.

	0	1	2	3	4	5	6
dim$(\mathsf{P}_k(n))$	1	2	15	877	21,147	678,570	27,644,437
dim(U_k)	1	1	3	16	131	1,496	22,482

Goal: Give a combinatorial construction of representations of U_k using tableaux.

Uniform Block permutations

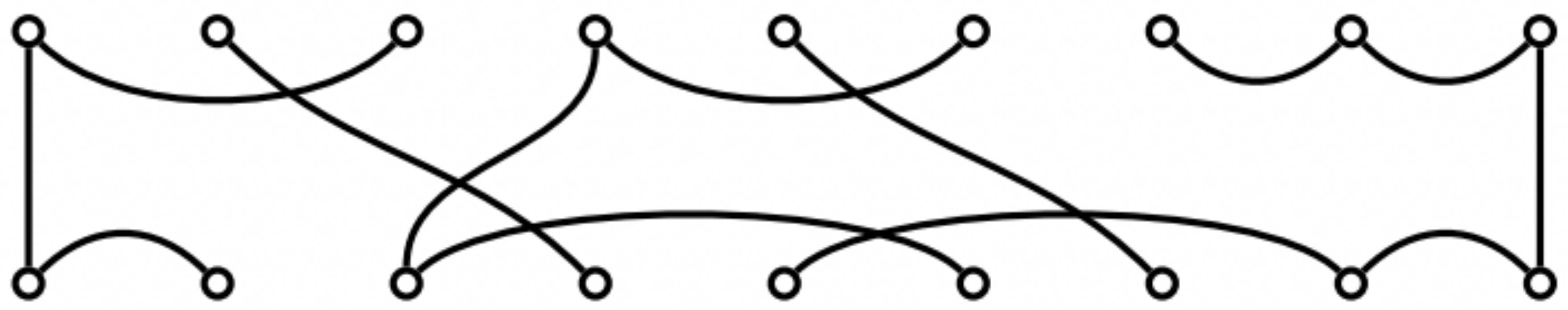
Tanabe and Kosuda: Centralizer algebra for complex reflection groups. "Party Algebra"

Elements: Uniform set partitions

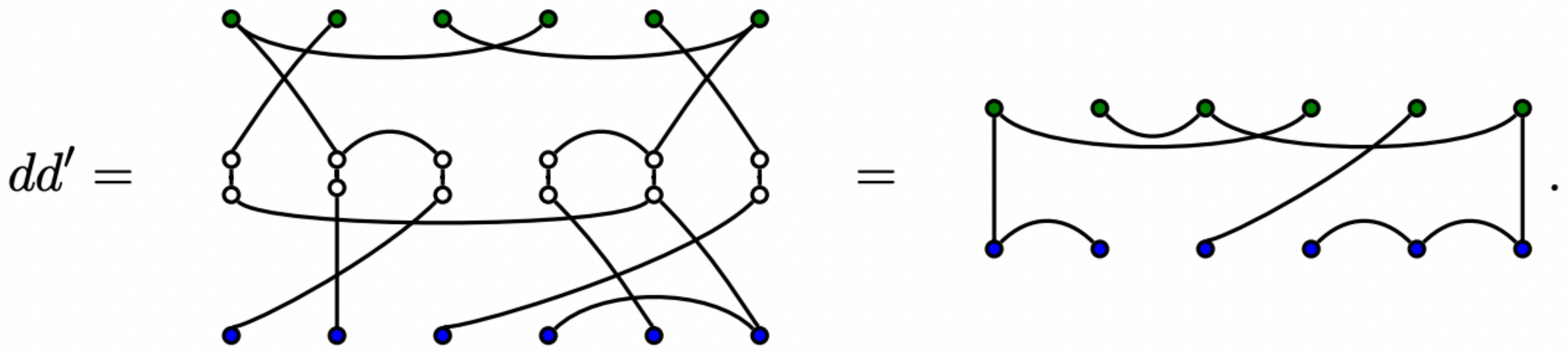
A set partition $d = \{d_1, d_2, \ldots, d_\ell\}$ **of** $[k] \cup [\bar{k}]$ **is uniform if** $|d_i \cap [k]| = |d_i \cap [\bar{k}]|$ **for all** $1 \leq i \leq \ell$.

$$\mathsf{U}_k := \{d \vdash [k] \cup [\bar{k}] \text{ is uniform}\}$$

Example: $\{\{1,3,\bar{1},\bar{2}\},\{2,\bar{4}\},\{4,6,\bar{3},\bar{6}\},\{5,\bar{7}\},\{7,8,9,\bar{5},\bar{8},\bar{9}\}\}$



Product:



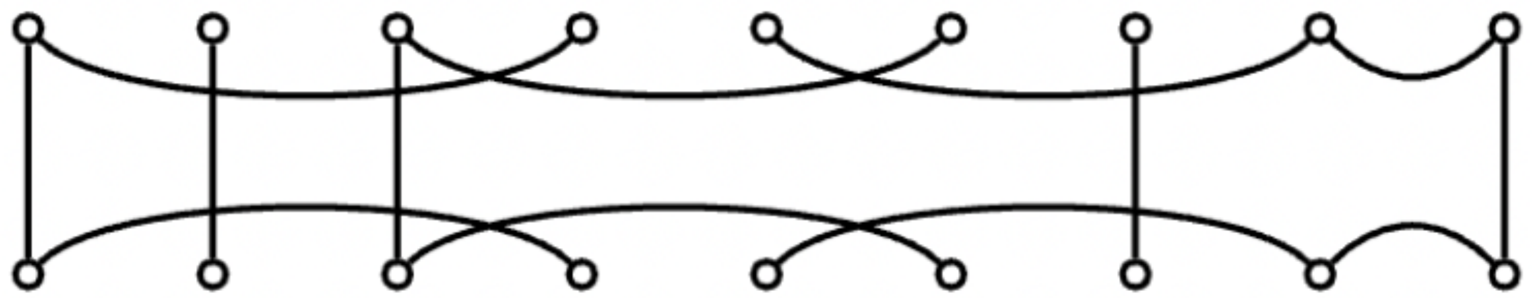
No parameter!

Note: U_k **is a monoid algebra.**

Idempotents and $\mathcal{J}$-classes

Idempotents: For each $\pi \vdash [k]$, we define an idempotent

$$e_\pi = \{A \cup \overline{A} : A \in \pi\}$$

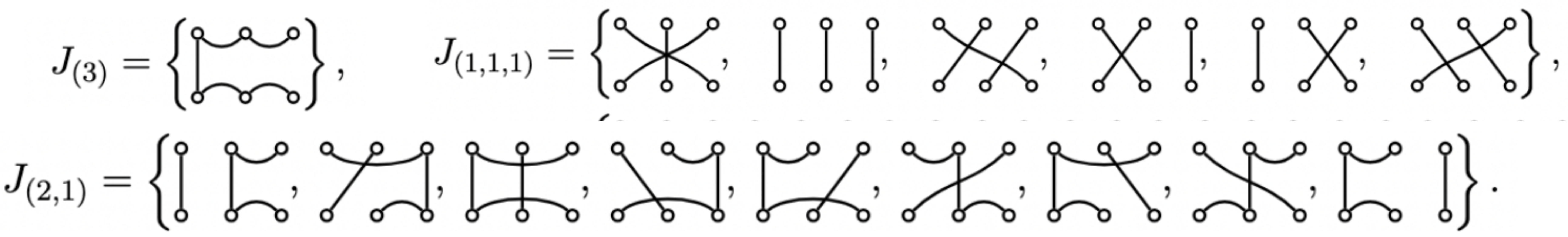
Example: $e_{\{\{2\},\{7\},\{1,4\},\{3,6\},\{5,8,9\}\}} =$ 

This idempotent has type $\lambda = (3,2,2,1,1)$

The set $E(\mathrm{U}_k) = \{e_\pi : \pi \vdash [k]\}$ is a complete set of idempotents.

$\mathcal{J}$ − classes: For each $\lambda \vdash k$ and $\pi \vdash [k]$ of type λ the set : $J_\lambda = \{\sigma e_\pi \tau : \sigma, \tau \in S_n\}$

Example:



Note: U_k is the union of $\mathcal{J}$ − classes.

Uniform block permutations

U_k is semisimple and its irreducible representations are indexed by

$$I_k = \left\{ \left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}\right) : \lambda^{(i)} \text{ are partitions such that } \sum_{i=1}^{k} i|\lambda^{(i)}| = k \right\}$$

Example: $I_3 = \{((3),\emptyset,\emptyset),((2,1),\emptyset,\emptyset),((1,1,1),\emptyset,\emptyset),((1),(1),\emptyset),(\emptyset,\emptyset,(1))\}$

A uniform tableau $T = \left(T^{(1)}, \ldots, T^{(k)}\right)$ of shape $\vec{\lambda} \in I_k$ is a tableau where each $T^{(i)}$ is filled with blocks of size i and the blocks in T form a set partition of $[k]$.

The irreducible representations of U_k :

$$V^{\vec{\lambda}}_{\mathsf{U}_k} := \text{span}\left\{ T \text{ is a uniform tableau of shape } \vec{\lambda} \right\}$$

Example: $$V^{((1),(1),\cdot)}_{U_3} = \text{span}\left\{ \left(\boxed{1}, \boxed{23}\right), \left(\boxed{2}, \boxed{13}\right), \left(\boxed{3}, \boxed{12}\right) \right\}$$

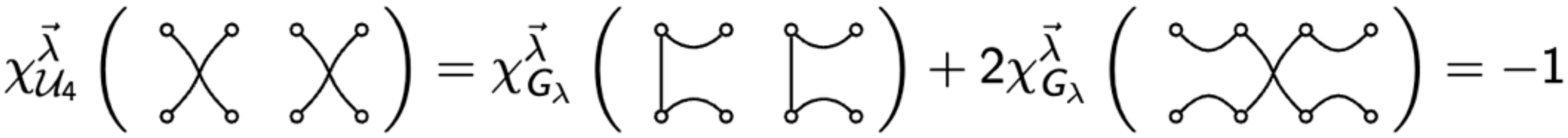
Characters for UBP

Explicit formulas for the characters!

Theorem: [OSSZ] For $\vec{\lambda}, \vec{\mu} \in I_k, a_i = |\lambda^{(i)}|, \lambda = (1^{a_1}2^{a_2}\ldots k^{a_k})$ **and** $G_\lambda \cong S_{a_1} \times \cdots \times S_{a_k}$

$$\chi^{\vec{\lambda}}_{\mathsf{U}_k}(d_{\vec{\mu}}) = \sum_{\vec{\nu}\in I_k : |\nu^{(i)}| = a_i} b^{\vec{\nu}}_{\vec{\mu}} \chi^{\vec{\lambda}}_{G_\lambda}(d_{\vec{\nu}})$$

Example: **Let** $\mu = (\cdot, (1,1), \cdot, \cdot)$, **so that** $\lambda = (2,2)$

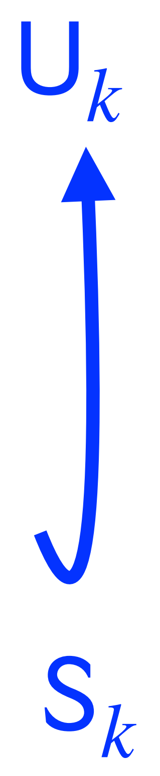


Note: Coefficients are always integers. We found and explicit formula for $b^{\vec{\nu}}_{\vec{\mu}}$ **.**

Connection to plethysm

Our problem: **We want to compute:**

$$\mathbf{Res}^{\mathsf{U}_k}_{\mathsf{S}_k} V^{\vec{\lambda}}_{U_k}$$



Defined a Frobenius map and connected to symmetric functions.

Theorem: **Multiplicity of** $\mathbb{S}^{\mu}$ **in** $\mathbf{Res}^{\mathsf{U}_k}_{S_k} V^{\vec{\lambda}}_{U_k}$ **is** $\langle s_{\lambda^{(1)}}[s_1] s_{\lambda^{(2)}}[s_2] \cdots s_{\lambda^{(k)}}[s_k], s_\mu \rangle = a_{\vec{\lambda},\mu}$

$$\mathbf{Res}^{\mathsf{U}_k}_{S_k} V^{\vec{\lambda}}_{U_k} \cong \bigoplus_{\mu \vdash k} (\mathbb{S})^{\oplus a_{\vec{\lambda},\mu}}$$

If $\vec{\lambda} = (\,\cdot\,, \ldots, \cdot\,, \lambda, \cdots)$ **where** λ **is in** m^{th} **position:**

$$\langle s_\lambda[s_m], s_\mu > = a^{\mu}_{\lambda,(m)}$$

Question: **How do we do the restriction/induction so that we get new information?**

Submonoids of UBP

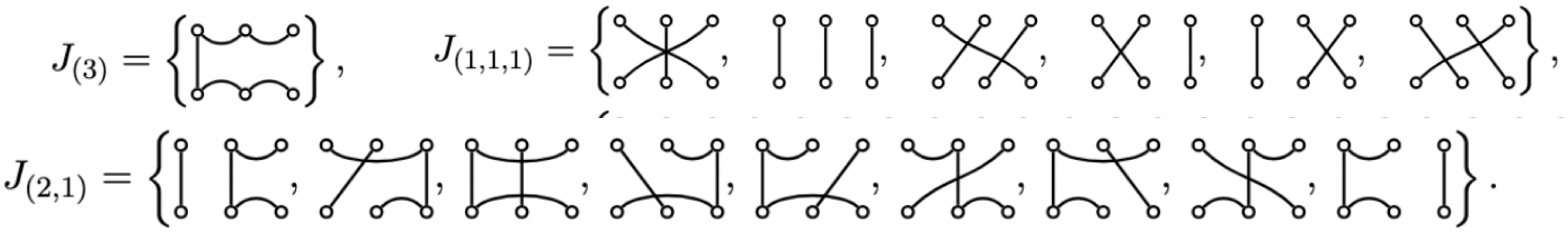
We are interested in the submonoids of U_k that contain S_k and how they are related.

Proposition: **Every submonoid of U_k containing S_k is the union of $\mathscr{J}-$ classes.**

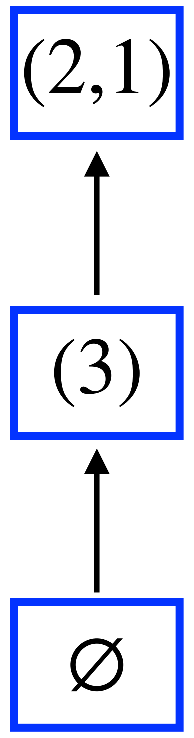
Theorem: **The set $\{M$ monoid such that $S_k \subseteq M \subseteq \mathsf{U}_k\}$ with order $\subseteq$ is a distributive lattice.**

Example:

$k = 3$



Monoids : $S_3 = J_{(1,1,1)} \subset \left(S_3 \cup J_{(3)}\right) \subset \left(S_3 \cup J_{(3)} \cup J_{(2,1)}\right) = \mathsf{U}_k$

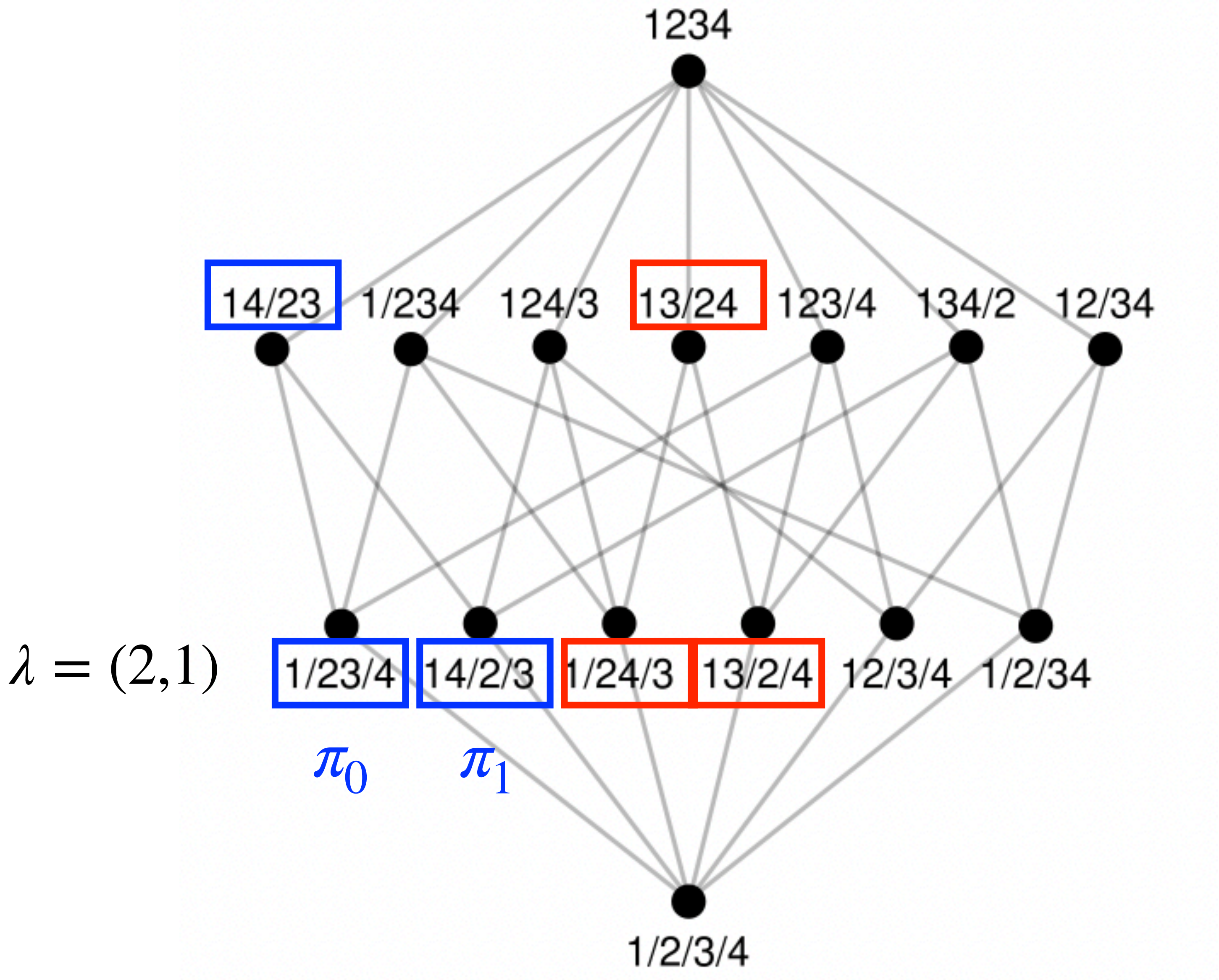


Number of submonoids:

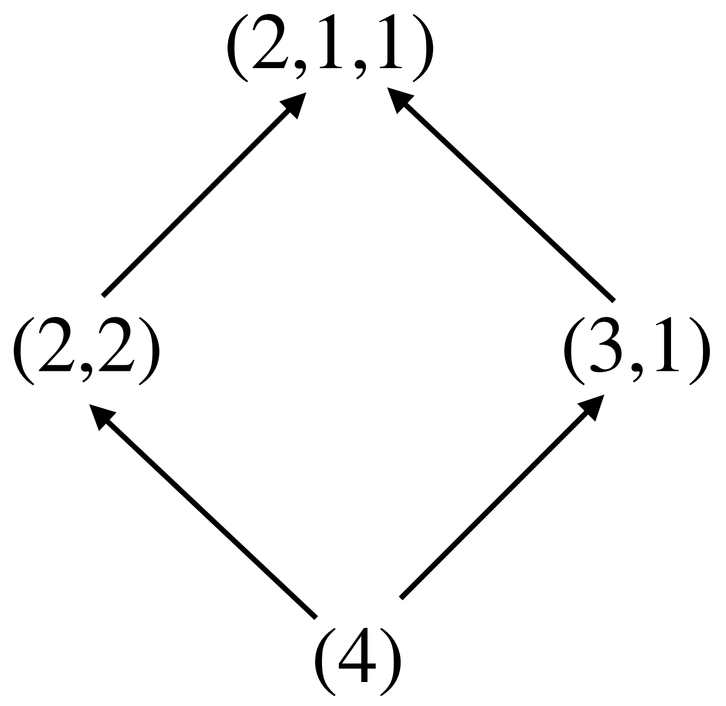
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
n_k	1	2	3	6	10	31	63	287	1099	8640	62658	1546891	29789119	2525655957

A new order on partitions

Definition: $\lambda, \mu \vdash k$, then $\mu \preceq \lambda$ if there are set partitions $\pi_0, \pi_1, \ldots, \pi_\ell \vdash [k]$ of type λ with join $\pi_o \vee \ldots \vee \pi_\ell$ of type μ .



Example:



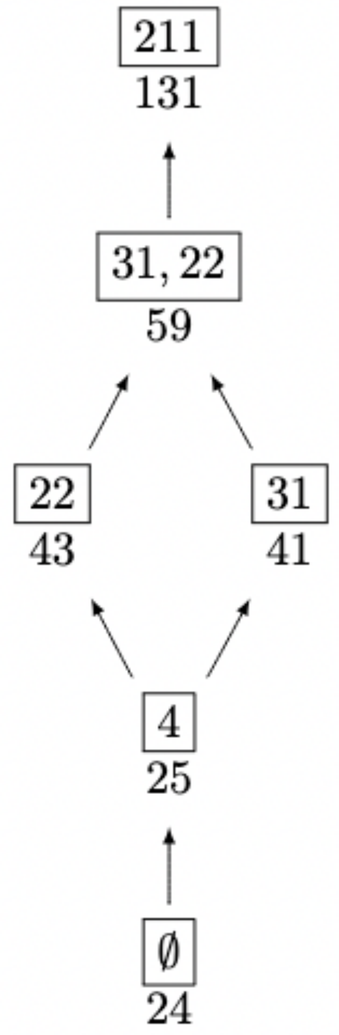
Poset: $\left(\text{Par}_k \backslash \{(1^k)\}, \preceq\right)$

Theorem: Every monoid $M \subseteq \mathsf{U}_k$ containing S_k is of the form:

$$M = S_k \cup \bigcup_{\mu \in I} J_\mu$$

where I is an order ideal of $\left(\text{Par}_k \backslash \{(1^k)\}, \preceq\right)$

Example:

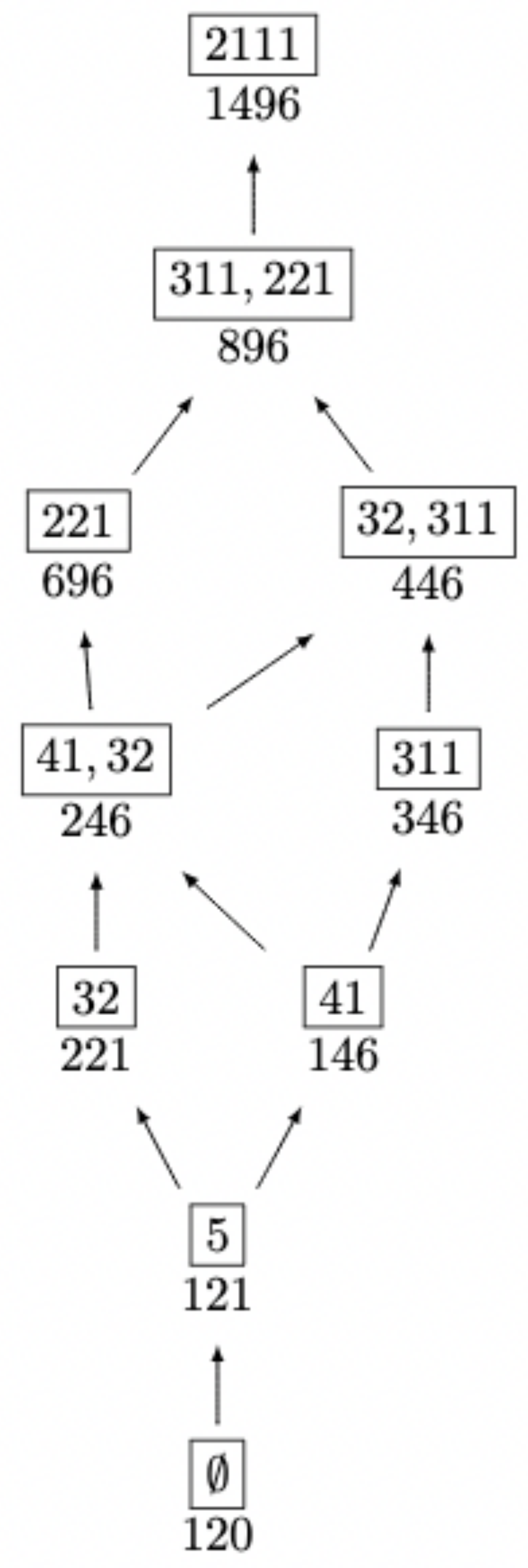


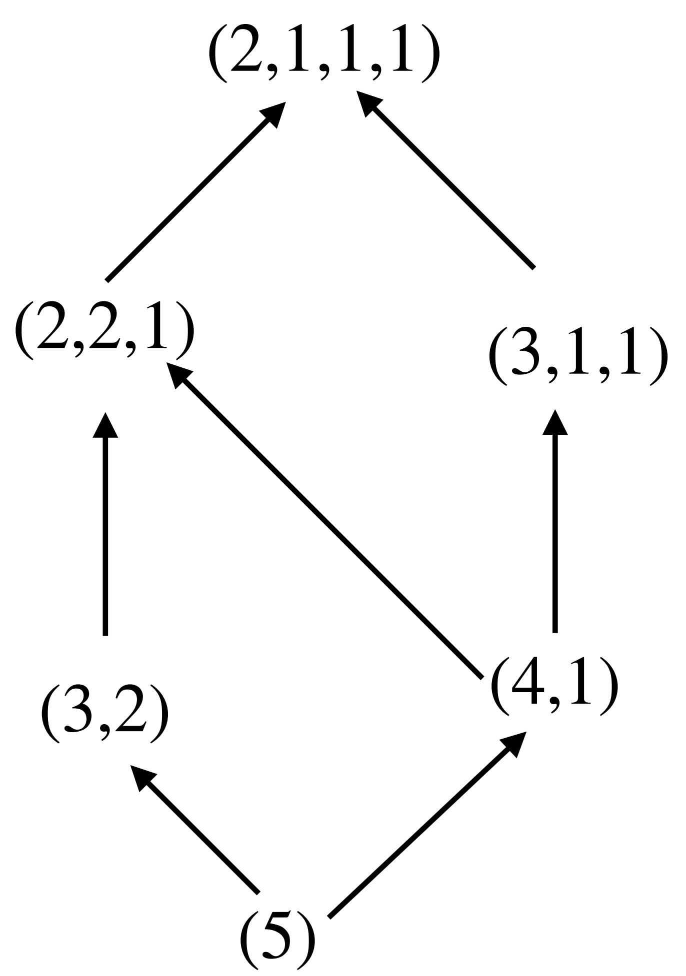
Theorem: $\lambda, \mu \in \text{Part}_k \backslash \{(1^k)\}$. Then

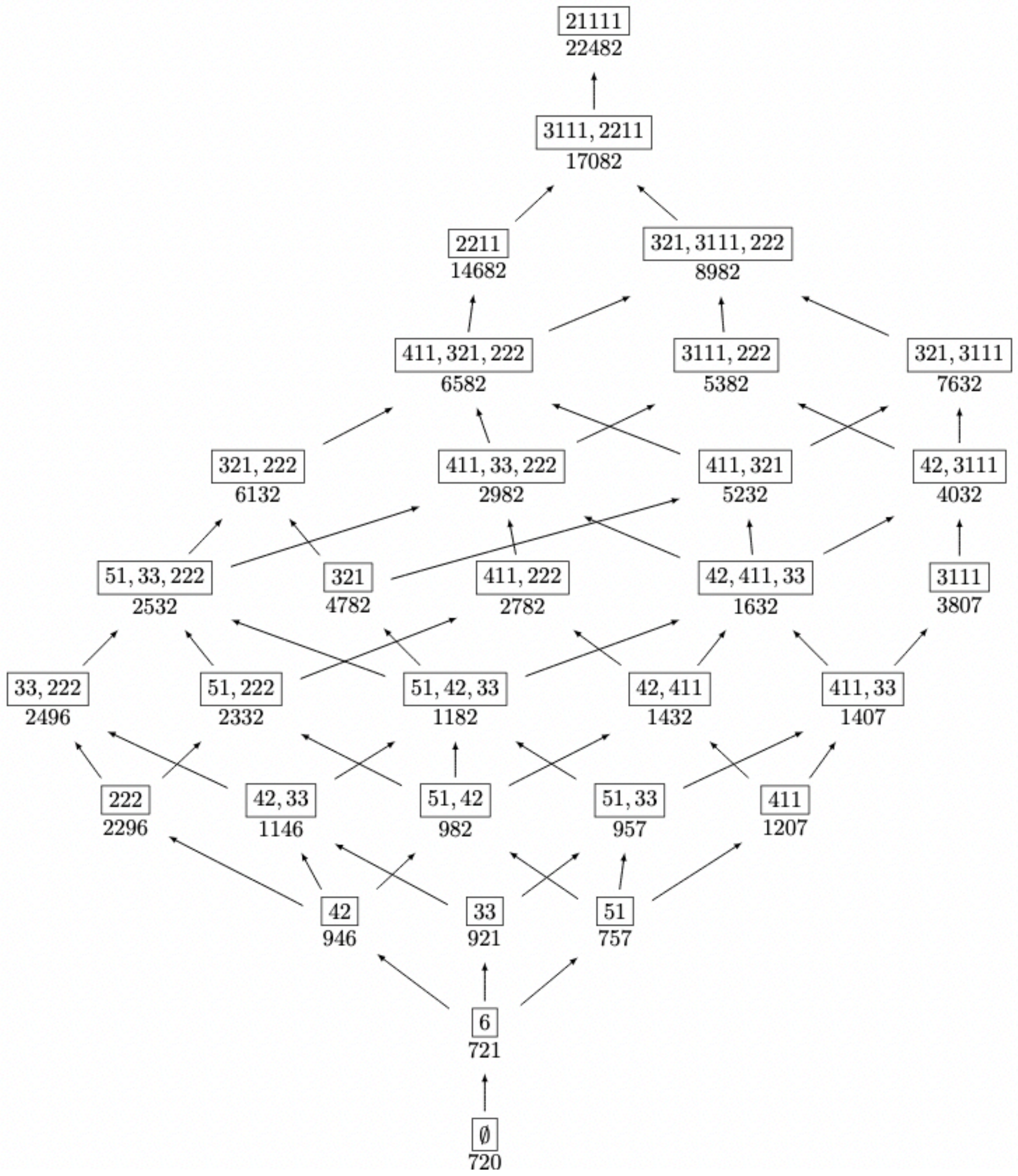
$\mu \preceq \lambda$ iff μ is coarser than λ and $SP_{>1}(\mu) \geq SP_{>1}(\lambda)$.

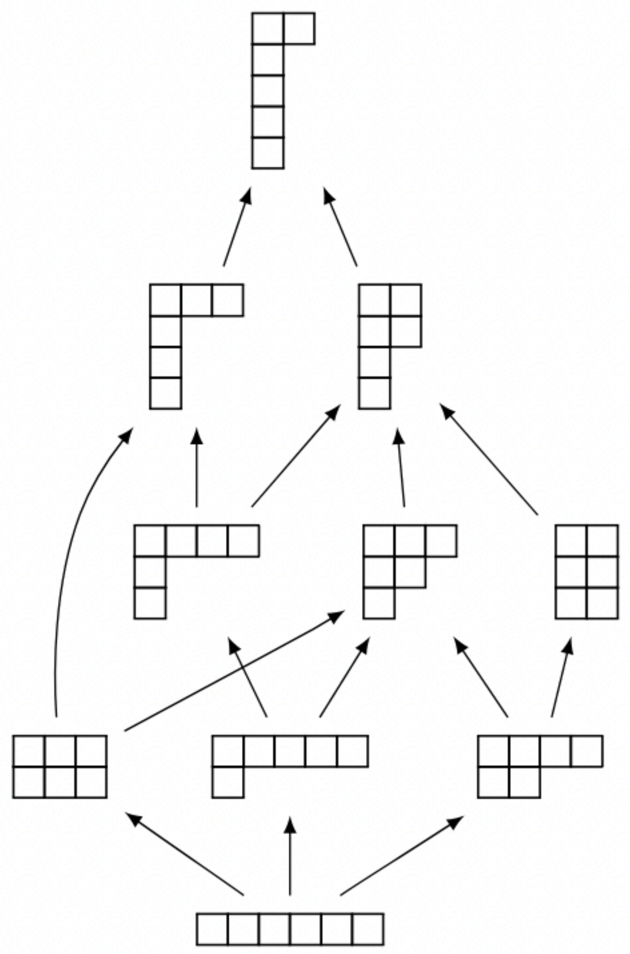
Final Remarks

New combinatorics:





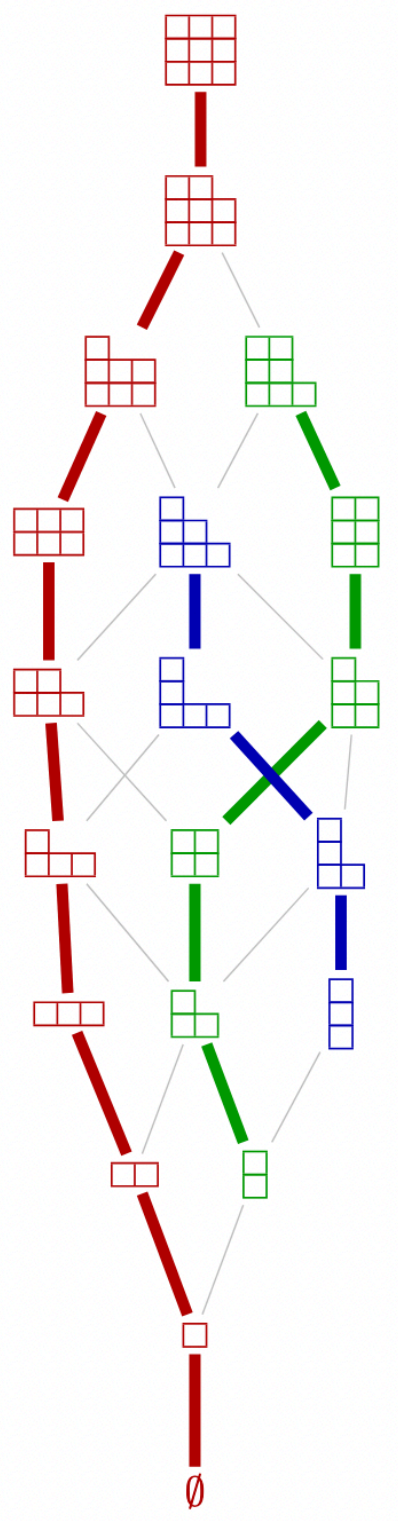


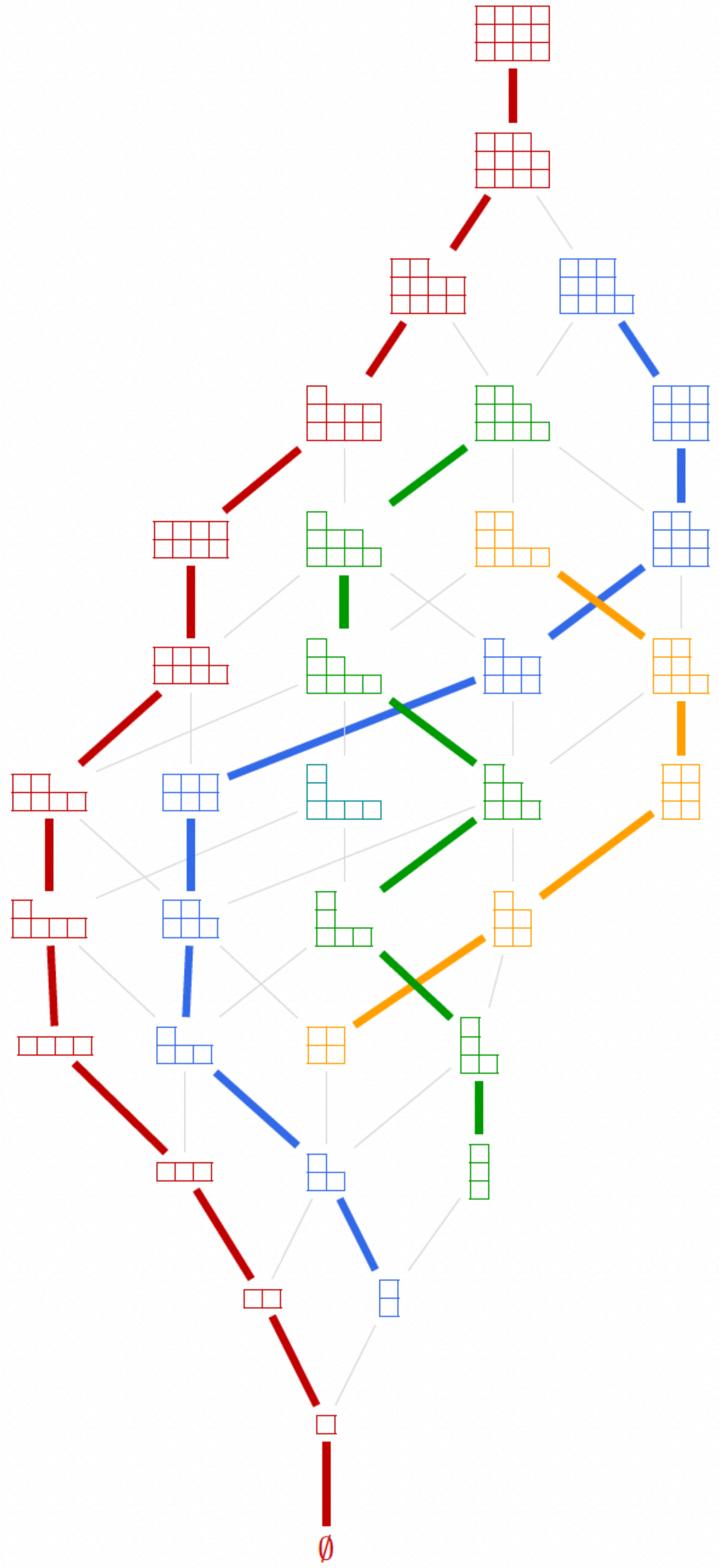


Final Remarks

Interesting connections: **Partitions in box $w \times h$, symmetric chain decompositions correspond to the plethysm:** $s_w[s_h(x,y)] = \sum_{\nu} a^{\nu}_{(w),(h)} s_{\nu}$

For ν partitions with at most two parts.





Faulstich, Sturmfels, and Sverrisdottir - connections of UBP to algebraic varieties.

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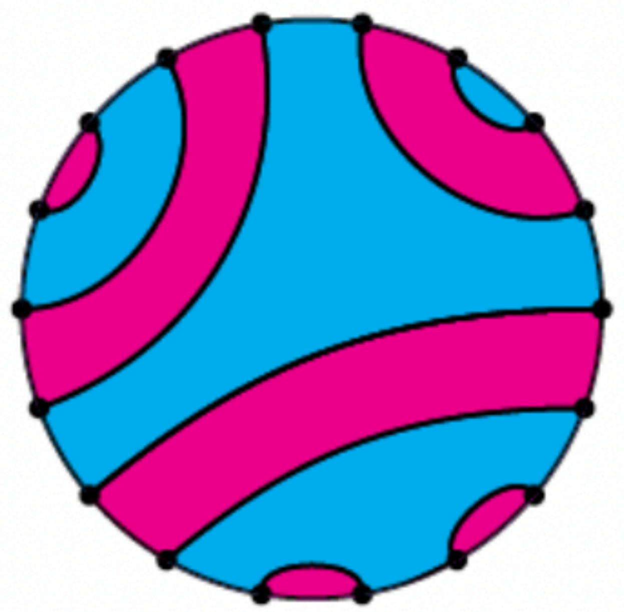
Special Semester @ ICERM

Categorification and Computation in Algebraic Combinatorics



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