PERMUTATION ACTION ON CHOW RINGS OF MATROIDS

Anastasia Nathanson

joint work with Robbie Angarone and Vic Reiner

University of Minnesota

FPSAC 2024
BOCHUM, GERMANY

OUTLINE

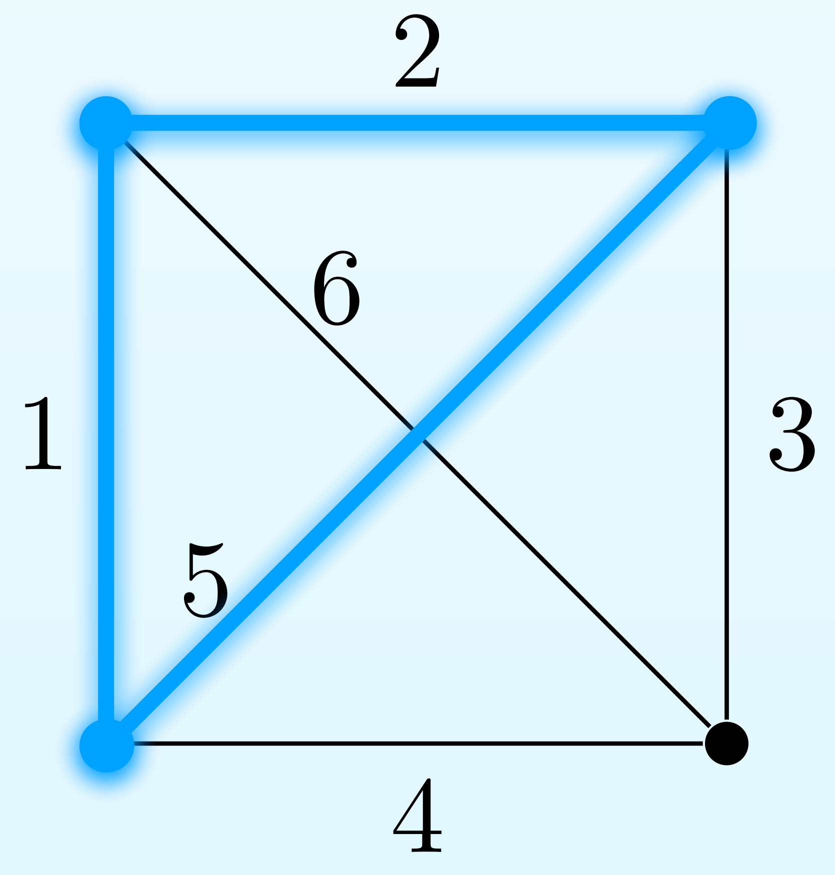
1. *Background*
2. *Our work*
3. *Beyond unimodality*

PERMUTATION ACTION ON CHOW RINGS OF MATROIDS

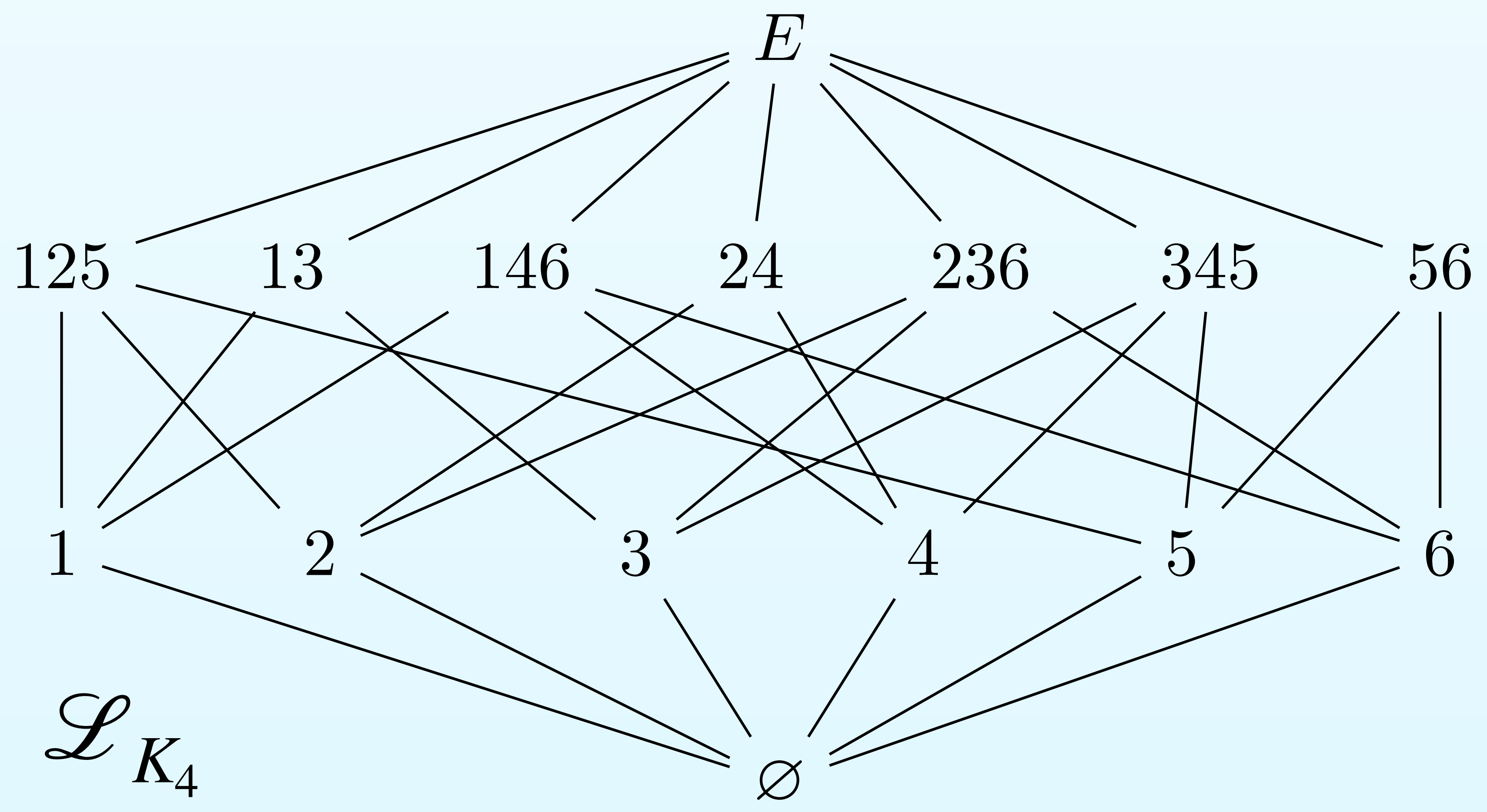
Matroids and Flats

We think of a **matroid** $M = (E, \mathscr{L}_M)$, on **ground set** E by its lattice of **flats** $\mathscr{L}_M$.

If E is the edge set of a graph, the flats are transitively closed subgraphs.



The flats of a matroid form a geometric lattice under inclusion, $\mathscr{L}_M$.



$$\mathrm{rk}(M) = \mathrm{rk}_{\mathscr{L}_M}(E) = r + 1$$

Affirming Conjectures

This and related conjectures were **affirmed** by **Adiprasito, Huh,** and **Katz** in **2015** by proving that the Chow ring of a matroid satisfies the Kähler package.

Conjecture **(Read '68, Hoggar '74, Rota '71, Heron '72, Welsh '76):** The sequence of Whitney numbers of the first kind $(w_0, w_1, \ldots, w_r)$ is log-concave for any matroid:

$$w_i^2 \geq w_{i-1} w_{i+1}$$

the log-concavity of $(A^0, A^1, \ldots, A^r)$
is only conjectured

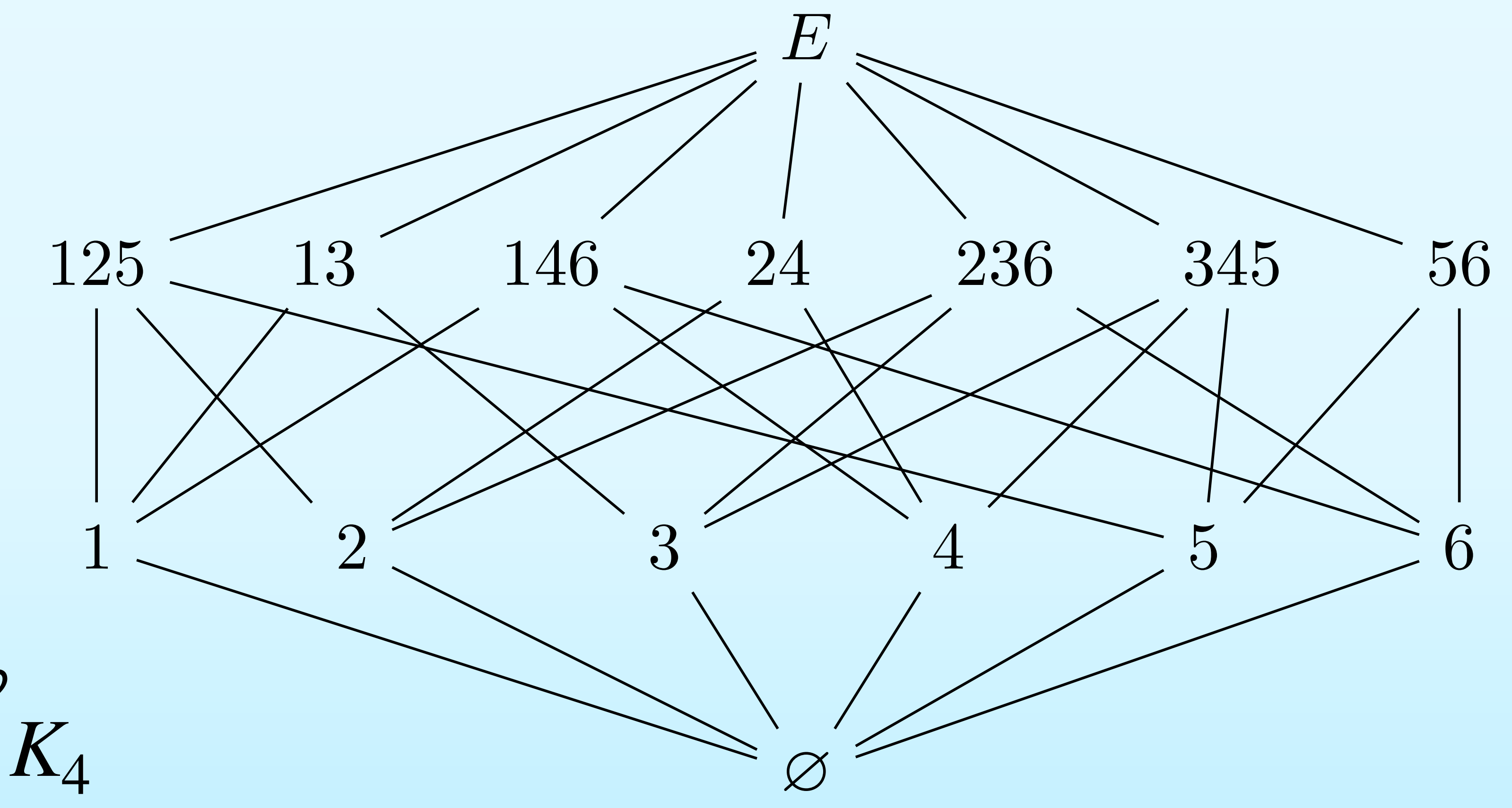
The **Chow ring of a matroid** $M = (E, \mathscr{L}_M)$ of rank $r + 1$ is

$$A(M) := S_M/(I_M + J_M) = \bigoplus_{k=0}^{r} A^k$$

$$S_M = \mathbb{Z}[x_F\colon \varnothing \neq F \in \mathscr{L}_M]$$

$$I_M = \big(x_F x_G\colon F, G \text{ incomparable in } \mathscr{L}_M\big)$$

$$J_M = \Big(\sum_{F \ni a} x_F\colon a \text{ an atom in } \mathscr{L}_M\Big)$$



$\mathscr{L}_{K_4}$

Chow Ring of a Matroid

The **Chow ring of a matroid** $M = (E, \mathscr{L}_M)$ of rank $r+1$ is

$$A(M) := S_M/(I_M + J_M) = \bigoplus_{k=0}^{r} A^k$$

$$S_M = \mathbb{Z}[x_F \colon \varnothing \neq F \in \mathscr{L}_M]$$

$$I_M = \left(x_F x_G \colon F, G \text{ incomparable in } \mathscr{L}_M\right)$$

$$J_M = \left(\sum_{F \ni a} x_F \colon a \text{ an atom in } \mathscr{L}_M\right)$$

Theorem (Feichtner-Yuzvinsky '03):
A basis for the Chow ring is given by monomials of the following form:

$$\prod_{i=1}^{t} x_{F_i}^{m_i} \colon \varnothing = F_0 \subset F_1 \subset \cdots \subset F_t$$
$$m_i < \operatorname{rk}(F_i) - \operatorname{rk}(F_{i-1})$$

We call these **Feichtner-Yuzvinsky** (or FY) **monomials.**

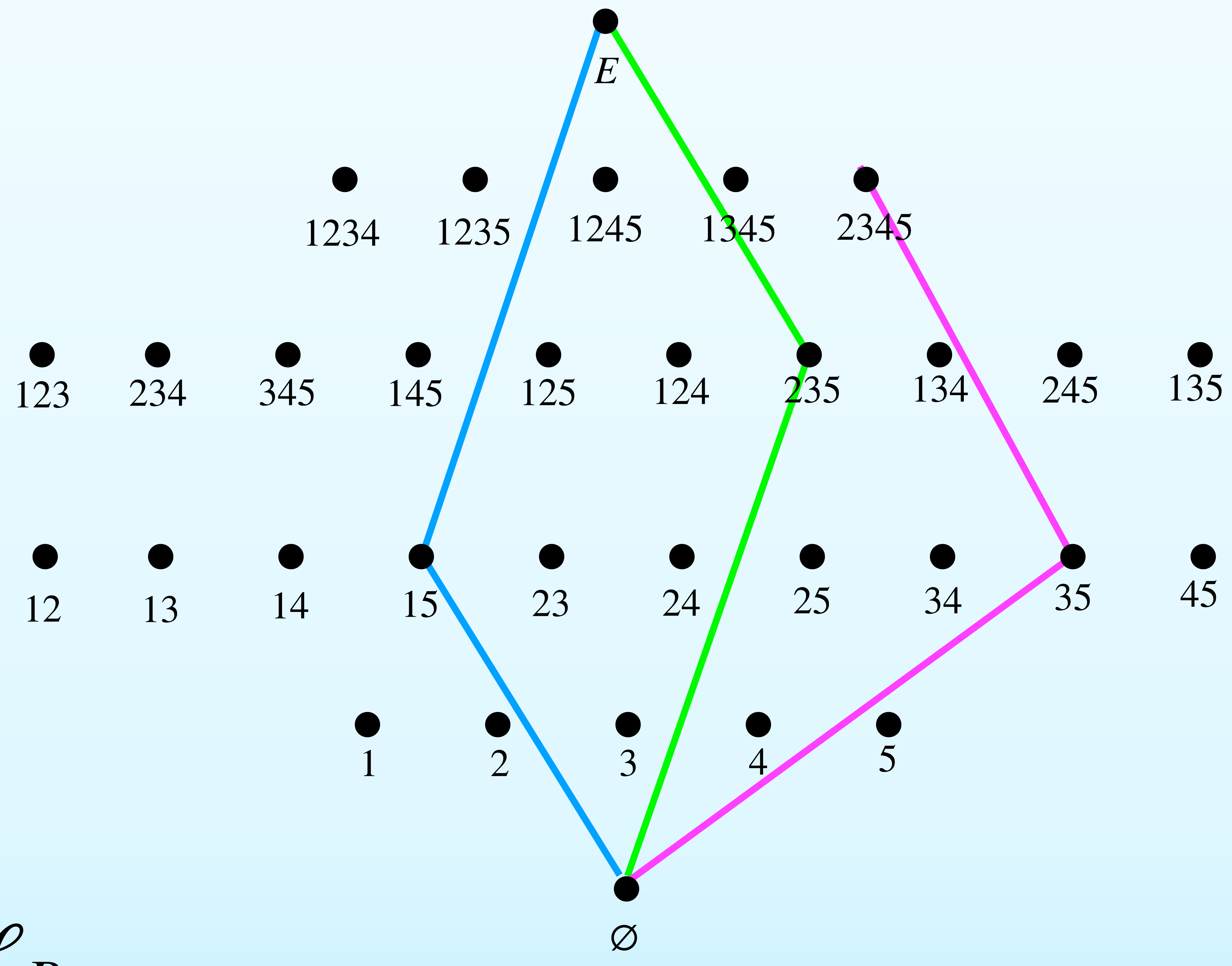
Degree k FY monomials, FY^k, form a basis for A^k.

Chow ring of B_5

FY-monomials:

$$\prod_{i=1}^{t} x_{F_i}^{m_i} : \varnothing = F_0 \subset F_1 \subset \ldots \subset F_t$$

$$m_i < \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1})$$



$\mathscr{L}_{B_5}$

$A^4(B_5)$	x_E^4 1	1
$A^3(B_5)$	$x_{ij}x_E^2$, 10 $x_{ijk}^2 x_E$, 10 x_{ijkl}^3, 5 x_E^3 1	26
$A^2(B_5)$	x_{ijk}^2, 10 x_{ijkl}^2, 5 x_E^2 1 $x_{ij}x_E$, 10 $x_{ijk}x_E$, 10 $x_{ij}x_{ijkl}$ 30	66
$A^1(B_5)$	x_{ij}, 10 x_{ijk}, 10 x_{ijkl}, 5 x_E 1	26
$A^0(B_5)$	1 1	1

Categorifications of unimodality and symmetry

Angarone – N. – Reiner

$$\mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$$

(as G-sets)

$$\Downarrow$$

$$A^k \hookrightarrow A^{k+1}$$

$$\Downarrow$$

$$\dim_{\mathbb{C}} A^k \leq \dim_{\mathbb{C}} A^{k+1}$$

unimodality

$$\mathrm{FY}^k \cong \mathrm{FY}^{r-k}$$

(as G-sets)

$$\Downarrow$$

$$A^k \cong A^{r-k}$$

$$\Downarrow$$

$$\dim_{\mathbb{C}} A^k = \dim_{\mathbb{C}} A^{r-k}$$

symmetry

Group Actions on Matroids

Let $\mathrm{Aut}(M) := \{\text{maps } E \to E \text{ preserving } M\} \cong \{\text{poset automorphisms of } \mathscr{L}_M\} \subseteq \mathfrak{S}_E$

Let $G \subseteq \mathrm{Aut}(M)$ be any subgroup of $\mathrm{Aut}(M)$.

The Chow ring is a **representation** of G:

- $g \in G$ acts on $\mathbb{C}[x_F : F \in \mathscr{L}_M]$ by extending $g \curvearrowright \mathscr{L}_M$ to send $g \cdot x_F \to x_{g \cdot F}$
- $g \in G$ preserves $I + J$, making the quotient $A(M)$ a representation of G.

Even further, the Chow ring is a **permutation representation** of G.

Any automorphism will preserve inclusion of flats and degrees of monomials,

so it permutes the **Feichtner-Yuzvinsky basis.**

$$M = B_5 \qquad G = \mathfrak{S}_5$$

$$(12)(345) \cdot \left(x_{23}x_{1235}\right) = x_{14}x_{1234}$$

$$\mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$$

$$\mathrm{FY}^k \cong \mathrm{FY}^{r-k}$$

OUR WORK

Extended Support of a Monomial

Define **extended support** of a monomial $a = x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell}$

$$\textbf{supp}_+(a) := \{F_1, \ldots, F_\ell\} \cup \{E\} = \begin{cases} \{F_1, \ldots, F_\ell\} \cup \{E\} & \text{if } F_\ell \subsetneq E, \\ \{F_1, \ldots, F_\ell\} & \text{if } F_\ell = E\,. \end{cases}$$

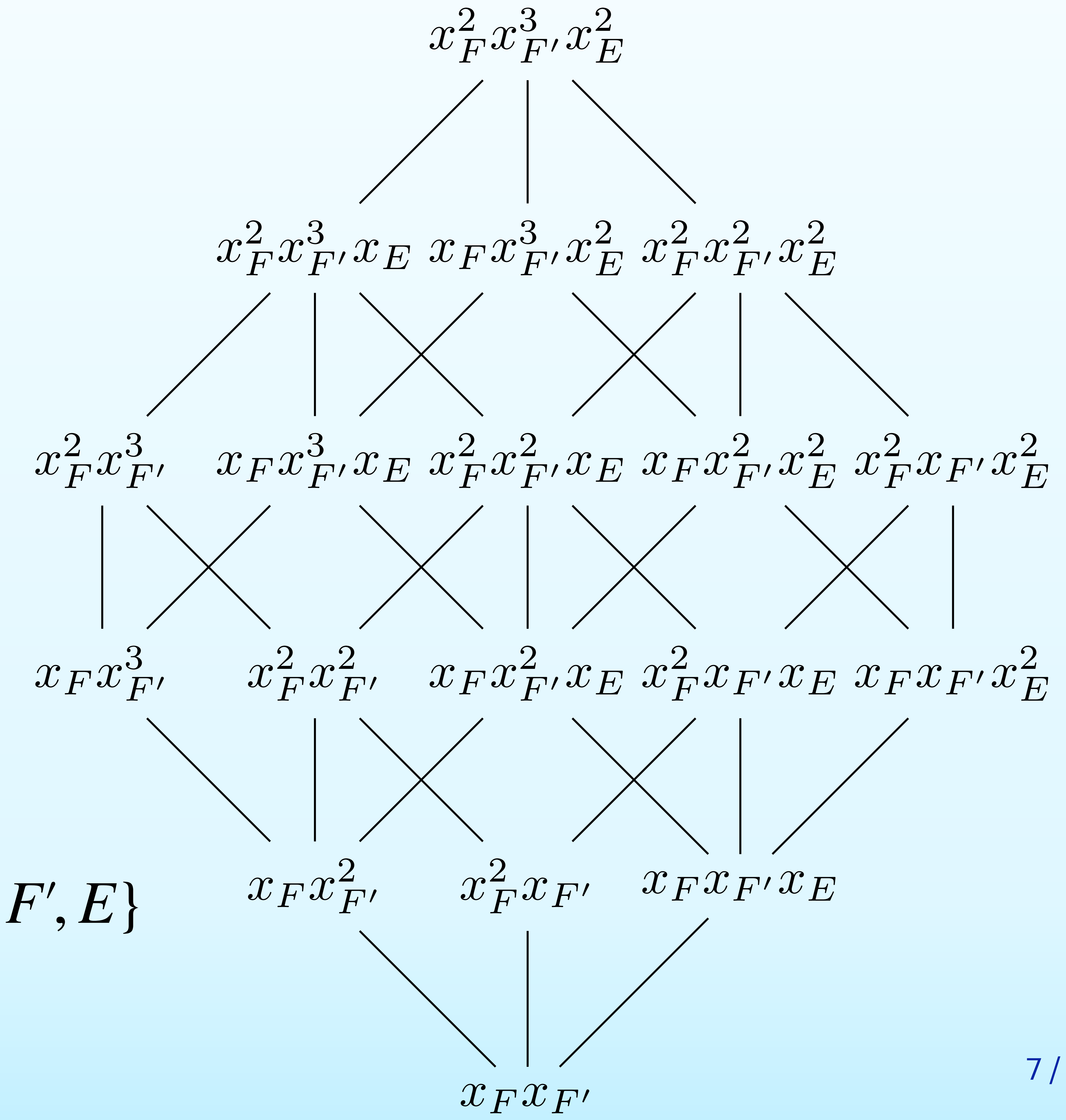
Define an order on the FY-monomials

in the fiber of $\mathrm{supp}_+^{-1}(F_1, \ldots F_\ell, E)$

$$a <_+ b \qquad \text{if } a \text{ divides } b$$

$$[1,2] \times [1,3] \times [0,2] \simeq \ \mathrm{supp}_+(x_F x_{F'}) = \{F, F', E\}$$

Assume $\mathrm{rk}(E) = 10$ and we have a pair of nested flats $F \subset F'$ with $\mathrm{rk}(F) = 3$ and $\mathrm{rk}(F') = 7$



Symmetric Chain Decomposition

Every product of chains has a symmetric chain decomposition.

Fix one ... SC $\{C\}$

Theorem (Angarone — N. — Reiner, '24+)

For every matroid M of rank $r+1$, there exist

- G-eqivariant bijections $\mathrm{FY}^k \leftrightarrow \mathrm{FY}^{r-k}$, and
- G-equivariant injections $\mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$

for $k < r/2$.

Any m ... SCD

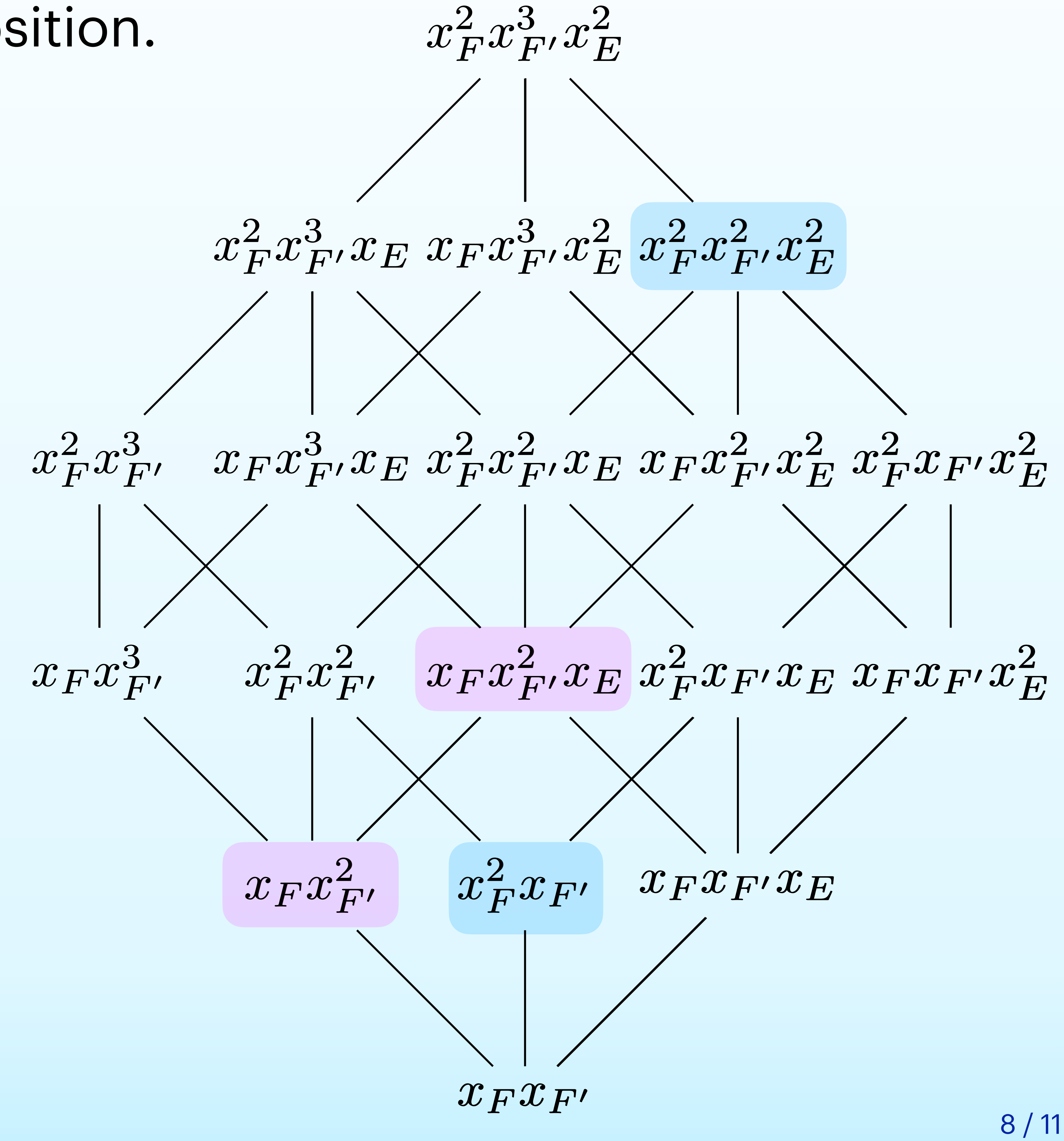
of the ...

We map monomial $a_k \in \mathrm{FY}^k$ where $k \leq r/2$, to another monomial in C

$$a_k \overset{\pi}{\mapsto} a_{r-k} \qquad a_k \overset{\lambda}{\mapsto} a_{k+1}$$

bijection $\mathrm{FY}^k \leftrightarrow \mathrm{FY}^{r-k}$

injection $\mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$



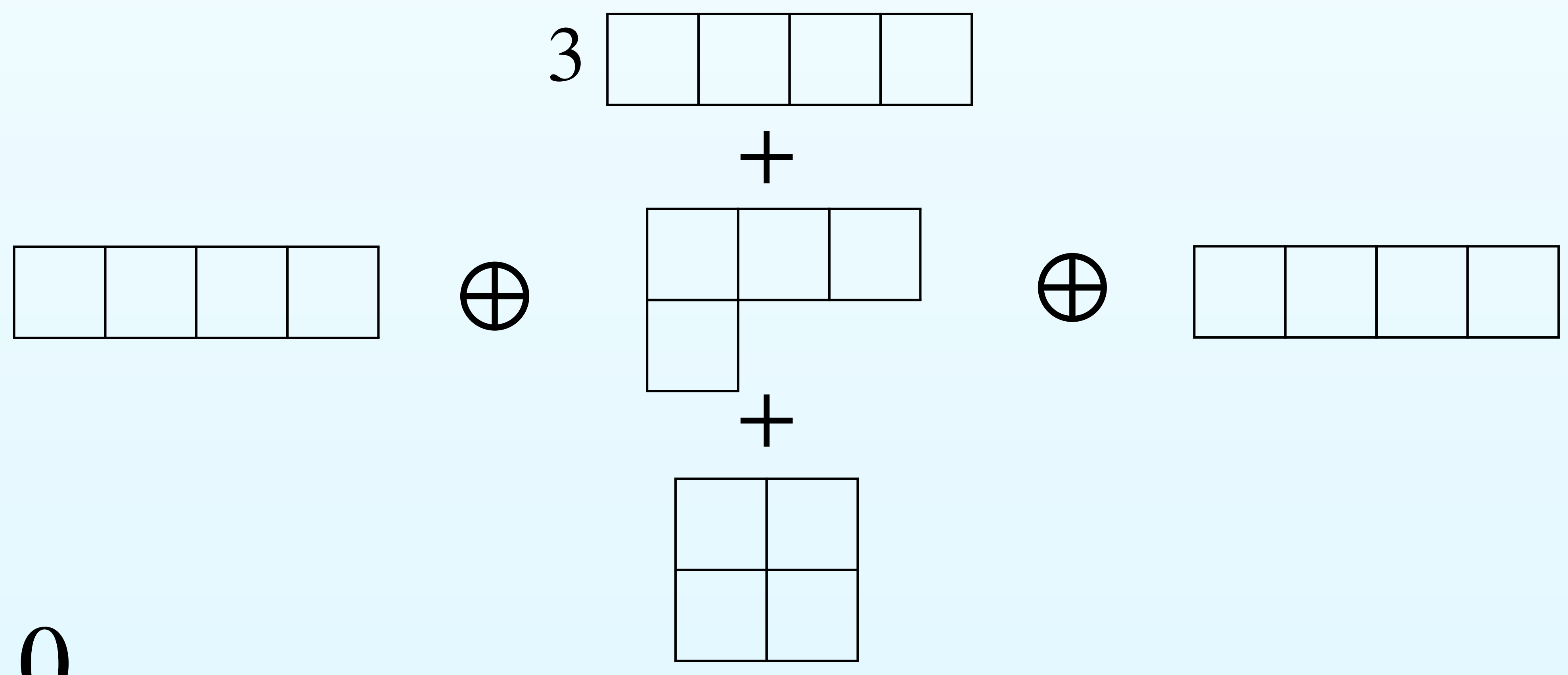
BEYOND UNIMODALITY

Equivariant and Burnside Nonnegativity

The virtual character ring $R(G)$ is a free $\mathbb{Z}$ module with a $\mathbb{Z}$ basis of irreducible characters.

Equivariantly nonnegative $X \geq_{R(G)} 0$ $\longleftrightarrow$ X is a positive linear combination of **irreducible representations** of G

$$A(K_4) = A^0 \oplus A^1 \oplus A^2 =$$



$$A^1 - A^0 \geq_{R(G)} 0$$

Burnside nonnegative $X \geq_{B(G)} 0$ $\longleftrightarrow$ X is a positive linear combination of **permutation representations** of G

We've shown Burnside nonnegativity of $\mathrm{FY}^{k+1} - \mathrm{FY}^k \geq_{B(G)} 0$, or **Burnside unimodality.**

Current Conjectures

real rootedness
$\Downarrow$
log concavity
$\Downarrow$
unimodality

Of $(A^0, A^1, \ldots, A^r)$, we conjecture

- Burnside log-concavity
- Equivariant real-rootedness

$$\mathsf{FY}^j \cdot \mathsf{FY}^k \geq_{B(G)} \mathsf{FY}^i \cdot \mathsf{FY}^\ell$$

for $i \leq j \leq k \leq \ell$ with $i + \ell = j + k$

$$\begin{vmatrix} \mathsf{FY}^j & \mathsf{FY}^i \\ \mathsf{FY}^\ell & \mathsf{FY}^k \end{vmatrix} \geq_{B(G)} 0$$

Non-equivariant real-rootedness and log-concavity are only conjectured.

We've show that $\mathsf{FY}^{j+k} \hookrightarrow \mathsf{FY}^j \times \mathsf{FY}^k$:

For $a = x_{F_1}^{m_1} \cdots x_{F_\ell}^{m_\ell} \in \mathsf{FY}^{j+k}$, factor

$$a = \underbrace{x_{F_1}^{m_1} x_{F_2}^{m_{p-1}} \cdots x_{F_{p-1}}^{m_{p-1}}}_{b:=} \cdot \underbrace{x_{F_p}^{m_p} x_{F_{p+1}}^{m_{p+1}} \cdots x_{F_\ell}^{m_\ell}}_{c:=}$$

where $b \in \mathsf{FY}^j$ and $c \in \mathsf{FY}^k$.

Corollary

It follows that for any matroid of $\mathrm{rk}(M) \leq 6$, the sequence $(\mathsf{FY}^0, \ldots, \mathsf{FY}^5)$ is Burnside log-concave.

Current Conjectures

real rootedness
$\Downarrow$
log concavity
$\Downarrow$
unimodality

Of $(A^0, A^1, \ldots, A^r)$, we conjecture

- Burnside log-concavity
- Equivariant real-rootedness

The polynomial with positive coefficients

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_r t^r$$

is **real-rooted** if and only if every minor of this matrix is nonnegative:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\begin{pmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & \cdots \\ 0 & A_0 & A_1 & A_2 & A_3 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & 0 & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\begin{vmatrix} \mathrm{FY}^1 & \mathrm{FY}^2 & \mathrm{FY}^3 \\ \mathrm{FY}^0 & \mathrm{FY}^1 & \mathrm{FY}^2 \\ 0 & \mathrm{FY}^0 & \mathrm{FY}^1 \end{vmatrix} \geq_{B(G)} 0$$

$$(\mathrm{FY}^2 \times \mathrm{FY}^1) \sqcup (\mathrm{FY}^1 \times \mathrm{FY}^2)$$
$$\hookleftarrow$$
$$(\mathrm{FY}^1 \times \mathrm{FY}^1 \times \mathrm{FY}^1) \sqcup \mathrm{FY}^3$$

$$a_1 \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} - a_0 \begin{vmatrix} a_2 & a_3 \\ a_0 & a_1 \end{vmatrix} + 0 \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}$$

$$A_1 \otimes \begin{vmatrix} A_1 & A_2 \\ A_0 & A_1 \end{vmatrix} - A_0 \otimes \begin{vmatrix} A_2 & A_3 \\ A_0 & A_1 \end{vmatrix} + 0 \otimes \begin{vmatrix} A_2 & A_3 \\ A_1 & A_2 \end{vmatrix}$$

$$a_1^3 - 2a_0a_1a_2 + a_0^2 a_3 \geq 0$$

$$A_1^3 - 2A_0A_1A_2 + A_0^2A_3 \geq_{R(G)} 0$$

Theorem (McCullough-Maestroni '22):
Chow rings are Koszul.

CIAO!