Smirnov words and the Delta conjectures

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FPSAC 2024, Bochum (Tief im Westen)

What this talk is about

We are interested a symmetric function in $(x_i)_{i\geq 1}$, with coeffs in $\mathbb{Q}(q)$:

$$\mathrm{OT}_q(n,k,l) := \Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}(x_1,x_2,\dots)\big|_{t=0}$$

where Θ_f, ∇ are certain operators with parameters q,t.

(Motivation: it is conjecturally the "graded Frobenius characteristic of the (k,l)-component of the $\mathbb{S}_n$-coinvariants $R_{1,2}(n)$" as we should see in $\simeq$ 20 minutes)

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We first prove a combinatorial expansion: let $\mathrm{SMIR}(n,k,l)$ be the set of *segmented Smirnov words* with k descents and l rises.
We will determine a statistic sminv such that

Theorem ([Iraci, N., Vanden Wyngaerd '24]).

$$\mathrm{OT}_q(n,k,l) = \sum_{w\in\mathrm{SMIR}(n,k,l)} q^{\mathrm{sminv}(w)} x_{w_1}x_{w_2}\cdots x_{w_n}.$$

We will then explain the connection to the Delta conjectures.

(Segmented) Smirnov words

A Smirnov word of length n is a word $w = w_1 w_2 \cdots w_n$ with $w_i \in \mathbb{Z}_{>0}$ such that $w_i \neq w_{i+1}$ for all $i < n$.

Example :$w = 1474273435$ has length $n = 10$.

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A segmented Smirnov word w of shape $\alpha = (\alpha_1, \ldots, \alpha_s) \vDash n$ has the form $w = w^1 w^2 \cdots w^s$ with w^i a Smirnov word of length α_i.

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Definition. Let $\mathrm{SMIR}(n, k, l)$ be the set of segmented Smirnov words of length n with k descents and l ascents.

Note: Such words have exactly $n - k - l$ blocks.

The statistic sminv

Definition. For a segmented Smirnov word w, we say that $i < j$ is a sminversion if $w_i > w_j$ and one of the following holds:

1. w_j is the first letter of its block ("initial");
2. $w_{j-1} > w_i$;
3. $i \neq j-1$, $w_{j-1} = w_i$, and w_{j-1} is initial;
4. $i \neq j-1$ and $w_{j-2} > w_{j-1} = w_i$.

We denote by $\mathrm{sminv}(w)$ the number of sminversions of w.

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Let $\mu = (\mu_1, \mu_2, \ldots, \mu_m, 0, 0, \ldots)$ with $\mu_i \geq 0$.
Define $w \in \mathrm{SMIR}(\mu, k, l)$ if w has μ_i occurrences of i for any i, and

$$\mathrm{SMIR}_q(\mu, k, l) := \sum_{w \in \mathrm{SMIR}(\mu,k,l)} q^{\mathrm{sminv}(w)}.$$

Key recurrence

If $\mu = (\mu_1, \dots, \mu_m, 0, \dots)$, define $\mu^- = (\mu_1, \dots, \mu_{m-1}, 0, 0, \dots)$.

Proposition[Iraci, N., Vanden Wyngaerd '24] Denote $B = \sum_i \mu_i - k - l$.

$$\mathrm{SMIR}_q(\mu, k, l) = \sum_{i=0}^{\mu_m} \sum_{r=i}^{\mu_m} \sum_{a=i}^{\mu_m} q^{\binom{r-i}{2}} \begin{bmatrix} B - (\mu_m - r - a + i) \\ r - i \end{bmatrix}_q q^{\binom{a-i}{2}} \begin{bmatrix} B - (\mu_m - r - a + i) \\ a - i \end{bmatrix}_q \times \begin{bmatrix} B \\ \mu_m - r - a + i \end{bmatrix}_q \begin{bmatrix} B - \mu_m + r + a - 1 \\ i \end{bmatrix}_q \mathrm{SMIR}_q(\mu^-, k - r, l - a).$$

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Note that this is a recurrence with positive integer coefficients, it thus requires a combinatorial proof that we provide.

We sketch it in the standard case $\mu = (1, 1, \dots, 1, 0, \dots) = (1^n)$, that is when we consider segmented permutations.

Proof sketch in the standard case

The recurrence simplifies greatly:

$$\begin{aligned}\mathrm{SMIR}_q(1^n, k, l) = [n-k-l]_q \big(& \mathrm{SMIR}_q(1^{n-1}, k, l) + \mathrm{SMIR}_q(1^{n-1}, k-1, l) \\ & + \mathrm{SMIR}_q(1^{n-1}, k, l-1) + \mathrm{SMIR}_q(1^{n-1}, k-1, l-1) \big)\end{aligned}$$

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Proof. Insert the value n in a segmented permutation of length $n-1$.

1. replacing a block separator.
2. at the beginning of a block, or
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4 distinct ways

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In each case, there are B ways to do this, where $B = n-k-l$ is the number of blocks in the resulting segmented permutation.

Moreover, the statistic sminv can augment by $0, 1, \ldots, B-1$, leading to the factor $[n-k-l]_q$. □

Remark. This is essentially the "Laguerre history" for permutations, linked to orhtogonal polynomials and continued fractions.

Recurrence for the symmetric function

Let $\mathrm{OT}_q(\mu, k, l)$ be the coefficient of $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots$ in $\mathrm{OT}_q(n, k, l)$.

Proposition ([Iraci, N., Vanden Wyngaerd '24]). Denote $B = \sum_i \mu_i - k - l$.

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Finally our main theorem

$$\mathrm{OT}_q(n, k, l) = \sum_{w \in \mathrm{SMIR}(n,k,l)} q^{\mathrm{sminv}(w)} x_{w_1} x_{w_2} \cdots x_{w_n}$$

follows since coefficients on both sides satisfy the same recurrences.

From $\mathbb{S}_n$-representations to combinatorics

Finite dimensional $\mathbb{S}_n$-representation V over $\mathbb{C}$.

↓ Frobenius characteristic Frob

Symmetric function F_V

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If the $\mathbb{S}_n$ representation preserves certain $\mathbb{N}$-gradings
$\Rightarrow$ use *graded* Frobenius characteristic $\text{grFrob}_{q_1,q_2,\ldots}$ with indeterminates $q_1, q_2, \ldots$ to record these gradings in F_V.

Coinvariant spaces

$\mathbb{S}_n$ acts on $P(n) := \mathbb{C}[t_1, \ldots, t_n]$ via $\sigma \cdot f(t_1, \ldots, t_n) := f(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$.

$\mathbb{S}_n$ acts on $\Lambda(n) := Ext(\xi_1, \ldots, \xi_n)$ via $\sigma \cdot f(\xi_1, \ldots, \xi_n) := f(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)})$.

(Here we have anticommuting variables $\xi_i \xi_j + \xi_j \xi_i = 0$.)

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Let us combine them.

$T_{a,b}(n) = P(n)^{\otimes a} \otimes \Lambda(n)^{\otimes b}$ with $\mathbb{S}_n$ acting diagonally.

$\mathbb{S}_n$-invariants in $T_{a,b}(n)$ with zero constant term generate an ideal $T^+_{a,b}(n)$.

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The action preserves all $a + b$ gradings $\Rightarrow$ $\mathrm{grFrob}_{q_1,\ldots,q_a;u_1,\ldots,u_b} R_{a,b}(n)$.

Let us now see small values of a, b.

Delta and Theta

Conjecture.[Zabrocki '18] $\mathrm{grFrob}_{q,t;z}(R_{2,1}(n))$ is equal to $\sum_k z^k \Delta'_{e_{n-k-1}} e_n$.

where $\Delta'_{e_{n-k-1}}$ is a certain operator with parameters q, t.

Conjecture.[D'Adderio, Iraci and Vanden Wyngaerd '21]
$\mathrm{grFrob}_{q,t;u,v}(R_{2,2}(n))$ is equal to $\sum_{k,l} u^k v^l \Theta_{e_k} \Theta_{e_l} \nabla e_{n-k-l}$.

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$\Rightarrow$ Thus $OT_q(n,k,l)$ is conjecturally the graded Frobenius of $R_{1,2}(n)$.

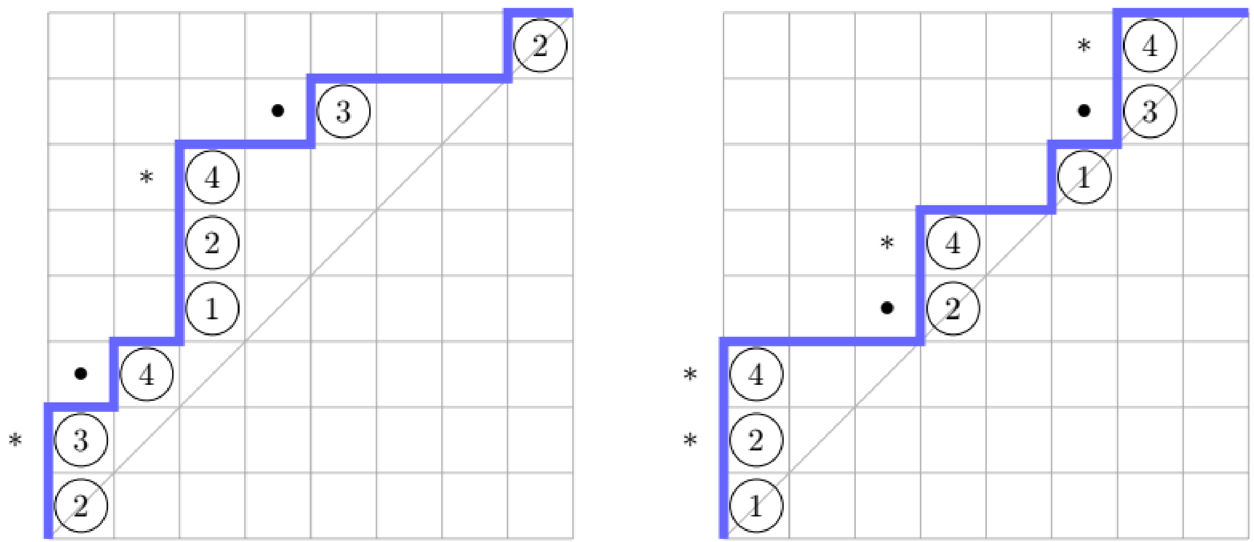
Combinatorics: The Delta conjectures (Haglund and al '18) claim that

$$\Delta'_{e_{n-k-1}} e_n = \sum_{D \in \mathrm{LD}(n)^{*k}} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)} x^D = \sum_{D \in \mathrm{LD}(n)^{\bullet k}} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)} x^D,$$

rise version — valley version

The rise version is now a theorem due to D'Adderio and Mellit '22.

Labeled Dyck paths



$LD(n)$ is the set of labeled Dyck paths of length n, where labels stricly increase on consecutive North steps.

A rise is a North step preceded by a North step. A contractible valley is a valley such that "removing the east step gives a good labeling".

Let $\mathrm{LD}(n)^{*k,\bullet l}$ be the subset of $LD(n)$ with k decorated rises and l decorated contractible valleys, and $\mathrm{LD}(\mu)^{*k,\bullet l}$ those with content μ.

The area of a path is the number of cells under the path that are *not* east of a decorated valley.

Labeled Dyck paths of area zero

D'Adderio, Iraci and Vanden Wyngaerd conjecture in addition

$$\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}\big|_{q=1} = \sum_{D\in \mathrm{LD}(n)^{*k,\bullet l}} t^{\mathrm{area}(D)} x^D$$

A potential unified Delta conjecture consists in finding a q-statistic on paths that restricts to the two Delta conjectures when $k = 0$ or $l = 0$.

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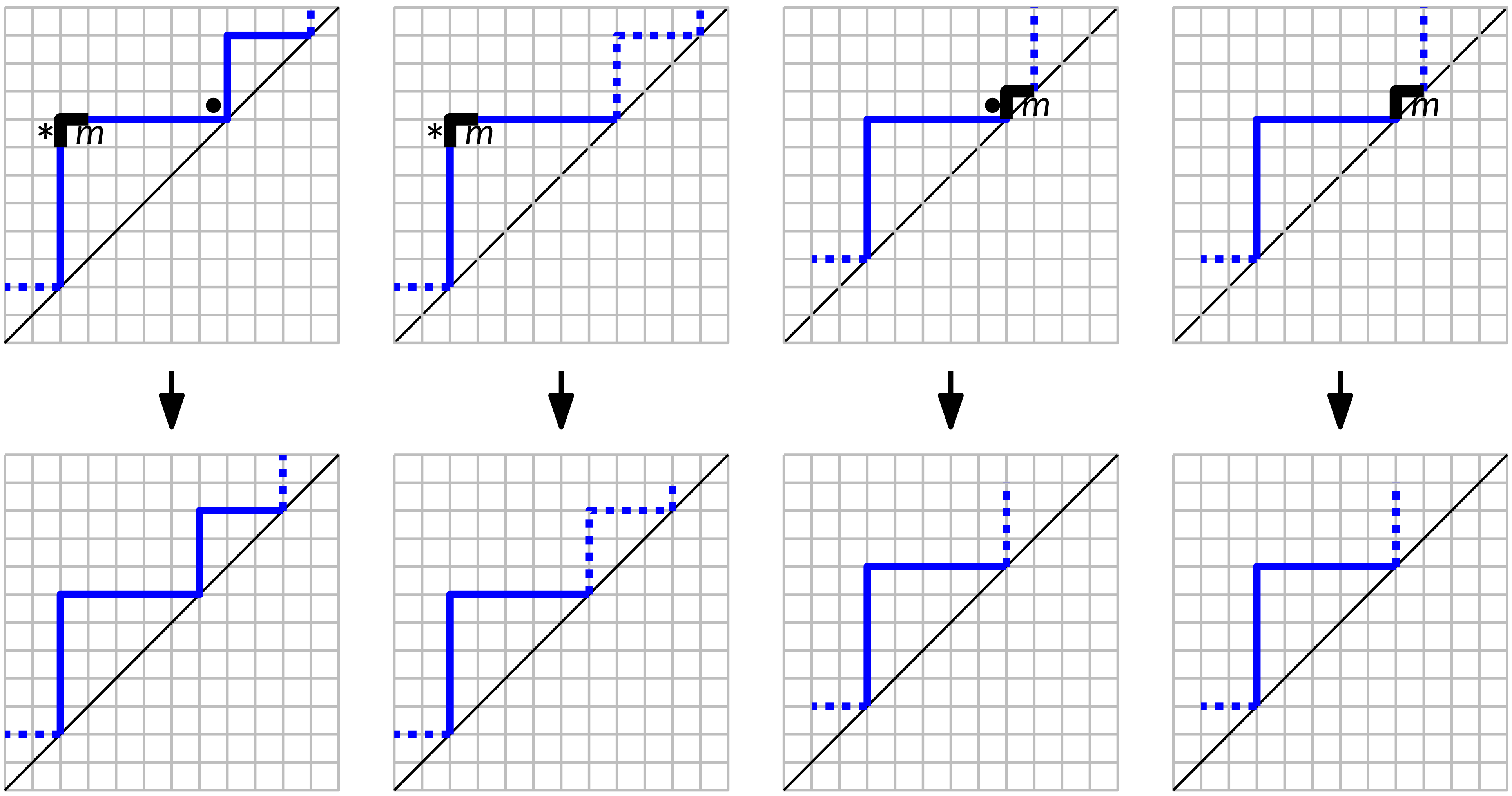
When $t=0$, we get back our series $\mathrm{OT}_q(n,k,l)$ at $q=1$. Thus a special case of the conjecture is the following result:

Theorem. [Iraci, N., Vanden Wyngaerd '24] The subset $\mathrm{LD}_0(n)^{*k,\bullet l}$ of paths of area zero is in bijection with $\mathrm{SMIR}(n,k,l)$.

Recursion for area zero paths.



Unified Delta theorem at $t = 0$

We can moreover define a statistic $\mathrm{sdinv} \neq \mathrm{sminv}$ that coincides with dinv in the Delta conjectures at $t = 0$ (via the previous bijection).

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