Stanley-Reisner rings and triangulated manifolds

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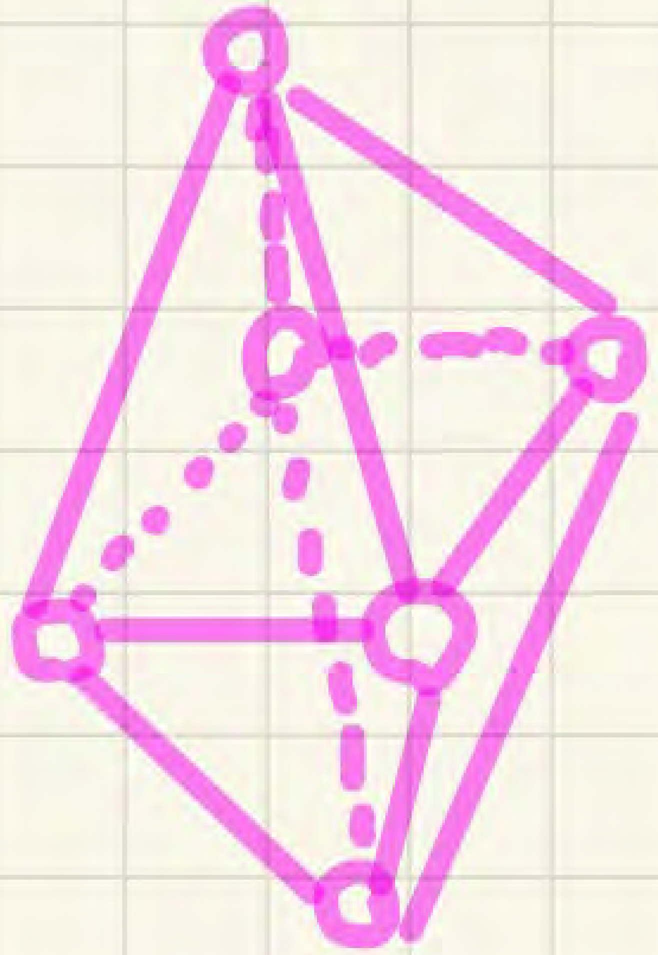
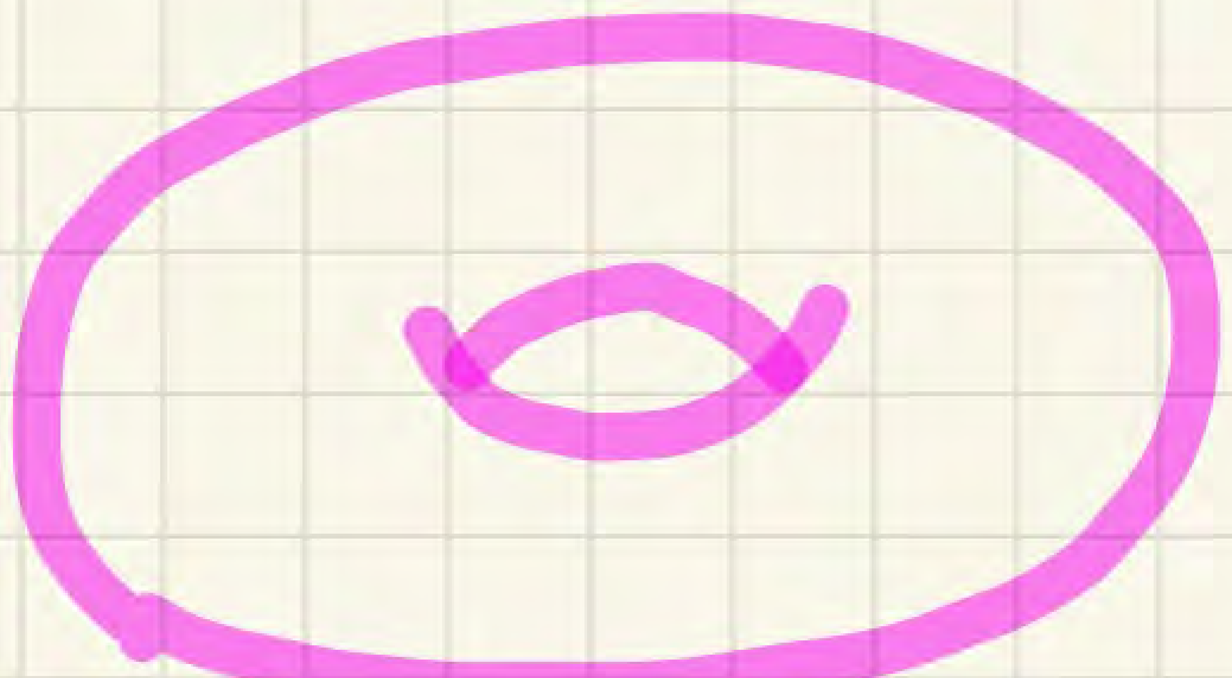




Outline of today's talk

① A combinatorial question
on triangulated manifolds

② Quick intro on Stanley-Reisner rings

③ Stanley-Reisner rings of
triangulated manifolds

Quick intro to vertex minimal triangulations

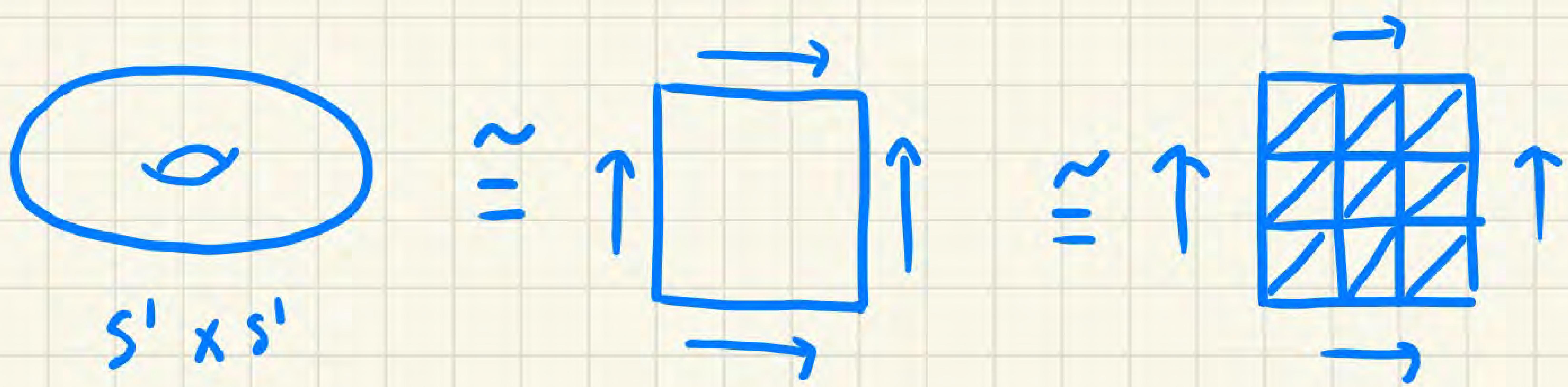
- Δ : abstract simplicial complex on a finite set V

  (Δ is a collection of subsets of V closed under inclusion)

- M : connected topological manifold (mfd)

- Δ is a triangulation of M

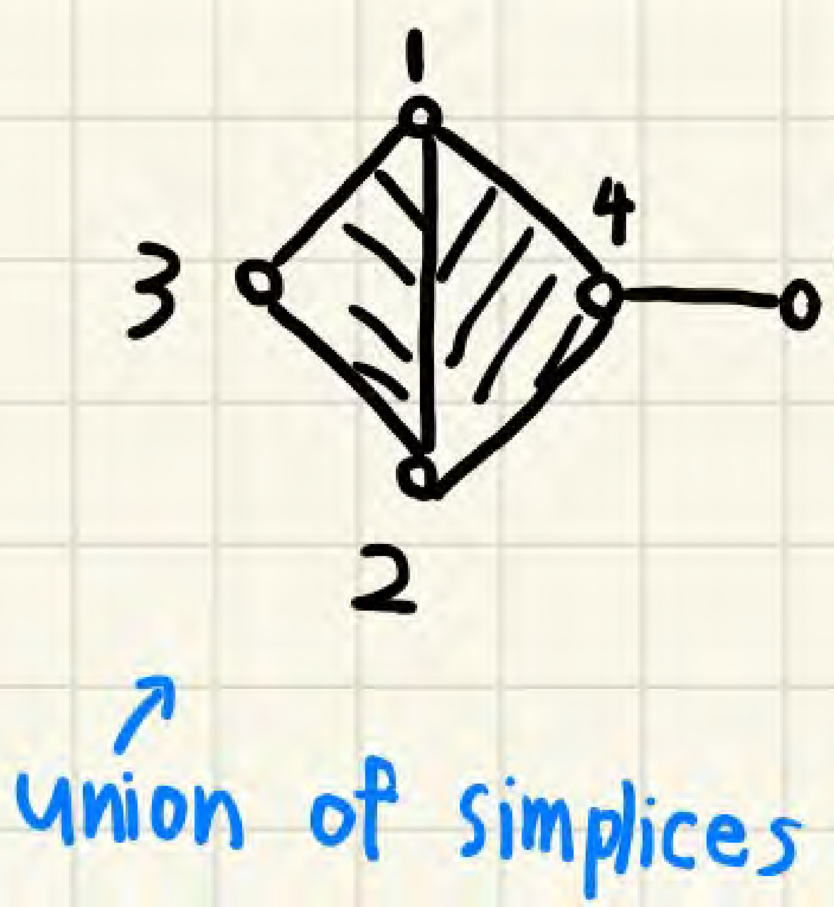
  $\overset{\text{def}}{\iff} |\Delta| \cong_{\text{homeo}} M$



Δ has a geometric realization $|\Delta|$

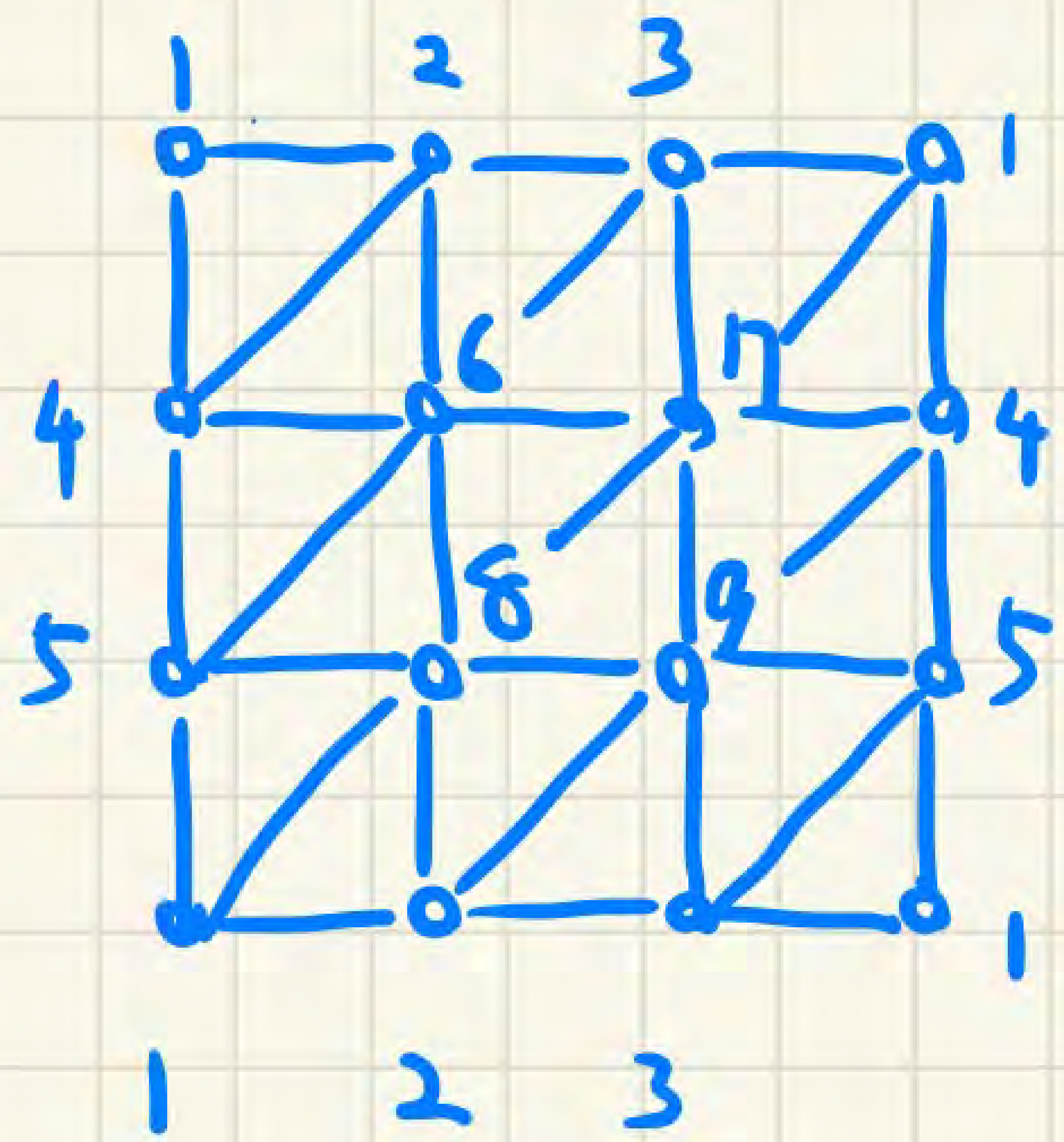
$$\Delta = \left\{ \begin{matrix} 123, 124, \\ 12, 13, 23, 14, 24, 45 \\ 1, 2, 3, 4, 5 \end{matrix} \right\}$$

$\Updownarrow$

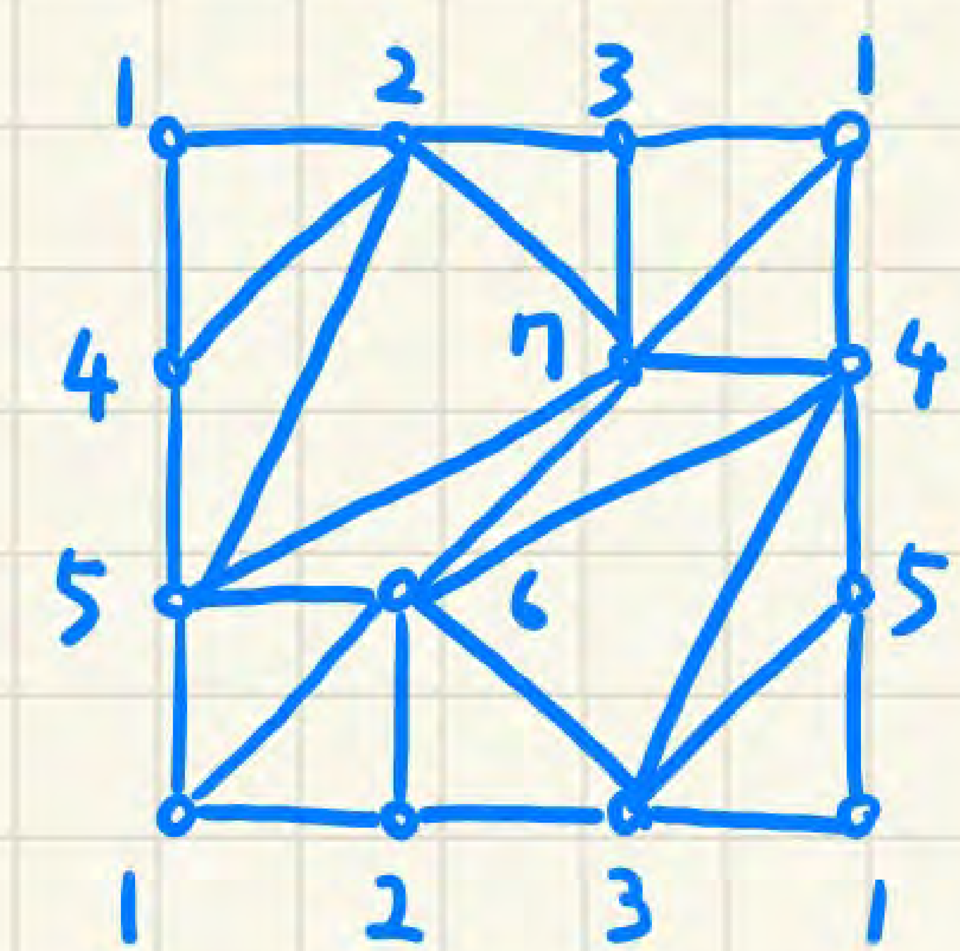


Quick intro to vertex minimal triangulations

Q How many vertices do we need to triangulate a given mfd M?



9 vertex triangulation of $S^1 \times S^1$



7 vert. triangulation of $S^1 \times S^1$

Quick intro to vertex minimal triangulations

> **Q** How many vertices do we need to triangulate a given mfd M? $=$ Compute $f_0^{\min}(M)$

Set

$$f_i(\Delta) = \text{num. of } i\text{-dim faces of } \Delta$$

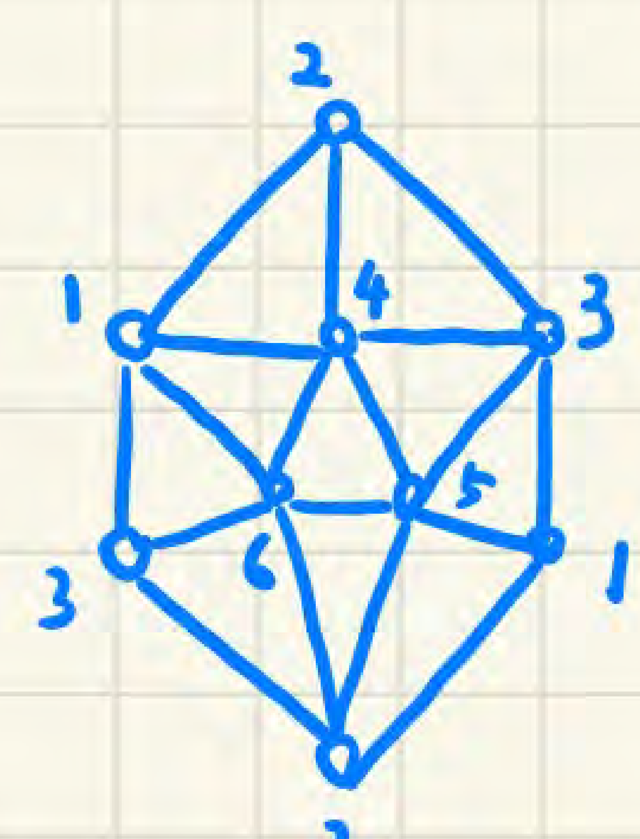
$$f_0^{\min}(M) = \min\{ f_0(\Delta) \mid \Delta \text{ is a triangulation of } M \}$$

ex

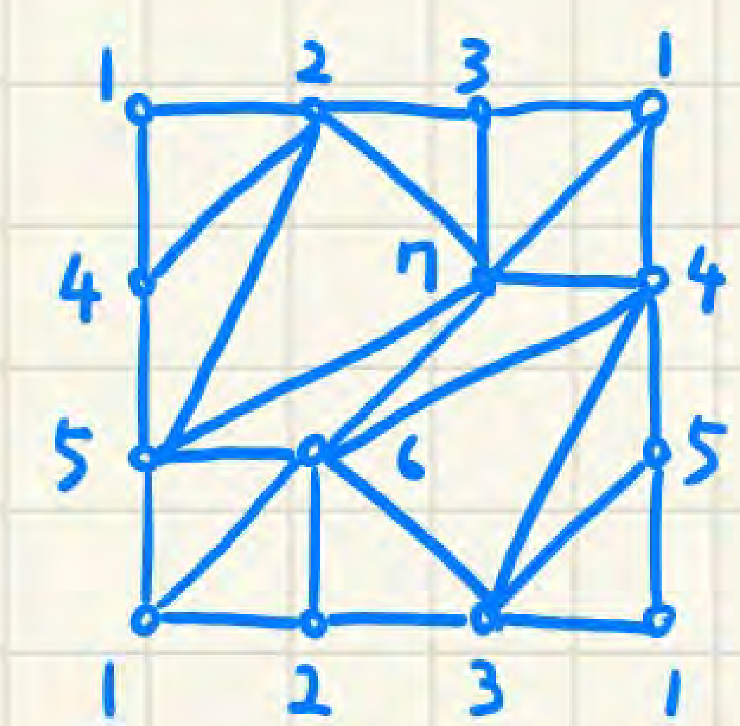
① $f_0^{\min}($  $) = 4$

② $f_0^{\min}(\mathbb{R}P^2) = 6$

③ $f_0^{\min}($  $) = 7$



6 vert. triang. of $\mathbb{R}P^2$



7 vert. triangulation of $S^1 \times S^1$

Classical result : surface case

- $S_g \overset{\text{def}}{=} (S^1 \times S^1)^{\# g} = \underbrace{(S^1\times S^1) \# (S^1 \times S^1) \# \cdots \# (S^1 \times S^1)}_{g \text{ copies}}$

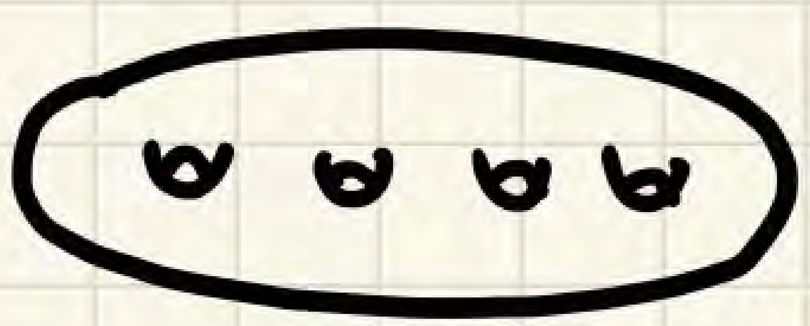
- $N_g \overset{\text{def}}{=} (\mathbb{RP}^2)^{\# g}$

Theorem (Ringel 1955, Jungerman-Ringel 1980))

If M is a closed surface with $M \neq S_2, N_2, N_3$, then

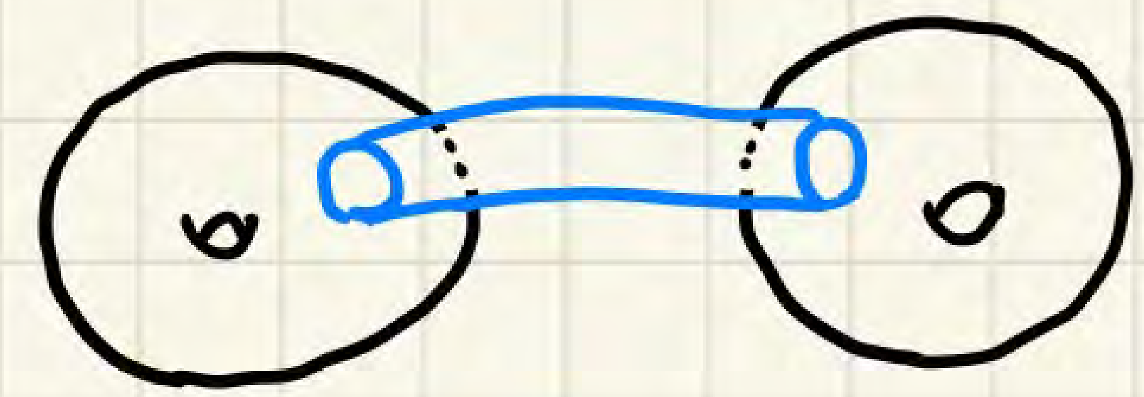
$$f_0^{\min}(M) = \min\left\{ f_0 \mid \binom{f_0 - 3}{2} \geq 3(2 - \chi(M)) \right\}$$

ex

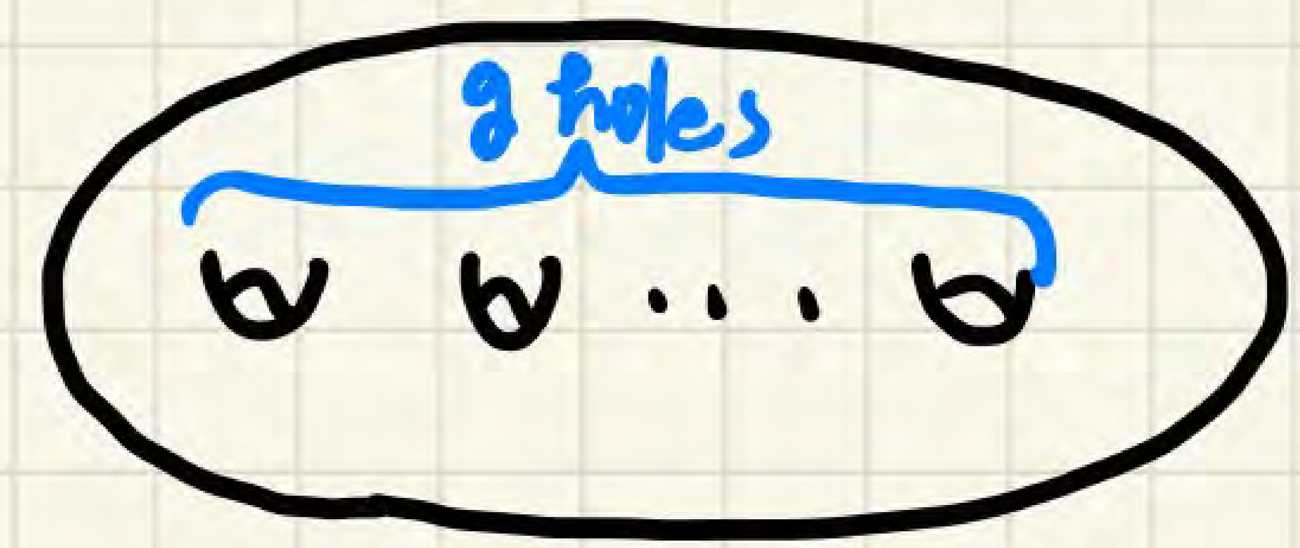
$f_0^{\min}\Big($  $\Big) = 11$

$\chi(M) = -6$

$\#$ means connected sum



$(S^1 \times S^1) \# (S^1 \times S^1)$



S_g

Classical result: surface case

Theorem (Ringel 1955, Jungerman-Ringel 1980))

If M is a closed surface with $M \neq S_2, N_2, N_3$, then

$$f_0^{\min}(M) = \min\left\{ f_0 \mid \underbrace{\binom{f_0 - 3}{2} \geq 3(2 - \chi(M))}_{\text{Heawood inequality}} \right\}$$

Proving similar result in higher dim is hard even if M looks very simple.

Two difficulties

① Lower bound of $f_0^{\min}(M)$? Generalization of Heawood inequality?

② Constructing an actual triangulation is hard.

Known

$f_0^{\min}(S^1 \times S^d) = 2d+5$ or $2d+6$

$f_0^{\min}(S^2 \times S^2) = 11$

$f_0^{\min}(S^2 \times S^3) = 12$

$f_0^{\min}(S^3 \times S^3) = 13$

$f_0^{\min}(S^2 \times S^4) = ?$

$f_0^{\min}(S^1 \times S^1 \times S^1) = ?$

$f_0^{\min}(\mathbb{C}P^2) = 9$

$f_0^{\min}(\mathbb{R}P^3) = 11$

$f_0^{\min}(\mathbb{R}P^4) = 16$

$f_0^{\min}(\mathbb{R}P^5) = ?$

Recent progress: Generalization of Heawood inequality

$r=2,\frac{d}{2}$ 2009 general

Theorem (Novik-Swartz, Adiprasito(Conjectured by Kühnel))

If Δ is a triangulation of a closed $(d-1)$-mfd then

$$\binom{f_0(\Delta)-d-2+r}{r} \geq \binom{d+1}{r}\beta_{r-1}(\Delta) \quad (r < \frac{d-1}{2})$$

$$\binom{f_0(\Delta)-d-1+r}{r+1} \geq \binom{d}{r}\beta_{r-1}(\Delta) \quad (r = \frac{d-1}{2})$$

old

$$f_0^{min}((S^1 \times S^1)\# \cdots \# (S^1 \times S^1)) \Rightarrow$$

new

$$f_0^{min}((S^1 \times S^d)\# \cdots \# (S^1 \times S^d))$$

$\beta_i(-)$: ith Betti number

Applications

$$f_0^{min}((S^3 \times S^1)^{\#143}) = 71$$

$$f_0^{min}((S^3 \times S^1)^{\#342}) = 101$$

$$f_0^{min}((S^2 \underline{\times} S^1)^{\#99}) = 49$$

$$f_0^{min}((S^2 \underline{\times} S^1)^{\#209}) = 69$$

$$f_0^{min}((S^2 \underline{\times} S^1)^{\#546}) = 109$$

(by Burton-Datta-Signh-Spreer)

Quick intro on Stanley-Reisner ring

Stanley-Reisner rings of polytopes & spheres

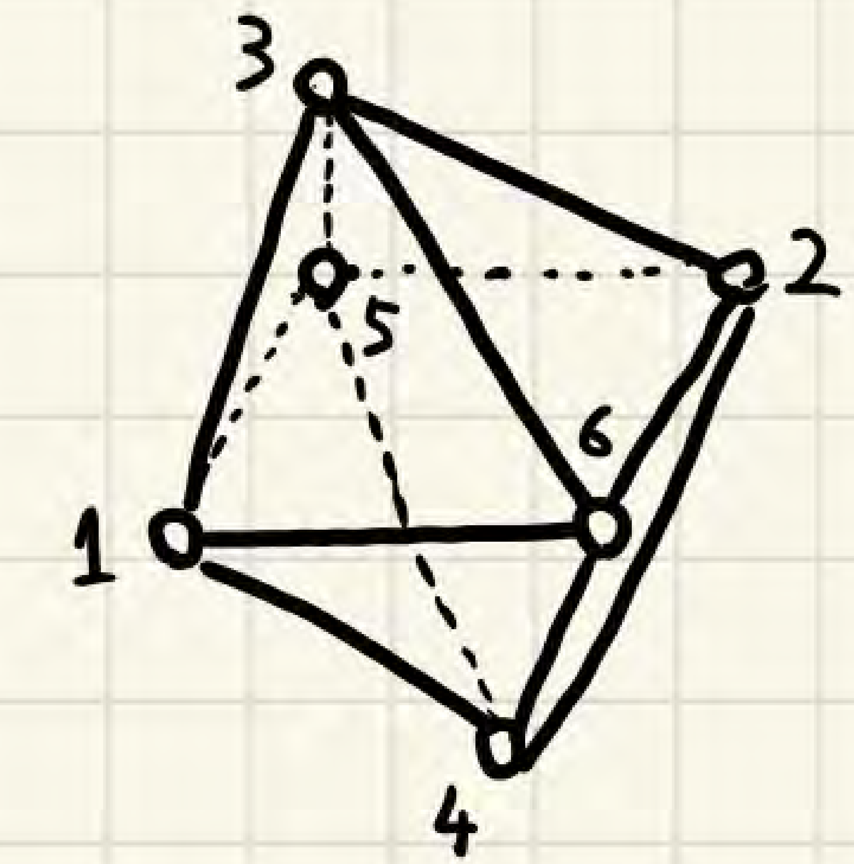
Stanley-Reisner ring

- Δ : simplicial complex on $[n]=\{1,2,\dots,n\}$
- $S=\mathbb{R}[x_1,\dots,x_n]$ $(\deg x_i = 1)$
- $\mathbb{R}[\Delta]=S/I_\Delta$: Stanley-Reisner ring (SR ring)

$$\left(I_\Delta=\left(x^F=\prod_{i\in F} x_i \;\middle|\; F \text{ is a (minimal) non-face of } \Delta\right)\right)$$

I will mainly disscuss

- (boundary complex of) simplicial d-polytope
- Simplicial $(d-1)$-sphere = triangulation of a $(d-1)$-sphere



minimal non-face are

$\{1,2\}$, $\{3,4\}$, $\{5,6\}$

so

$I_\Delta=(x_1x_2,\ x_3x_4,\ x_5x_6)$

Why Stanley-Reisner rings are useful?

Nice comb. property of Δ $\leftarrow$ Nice alg. property of $\mathbb{R}[\Delta]$

Combinatorial side | **Algebraic side**

① Dehn-Sommerville equation $\longleftarrow$ Poincaré duality of (Artinian reduction) of $\mathbb{R}[\Delta]$

② Lower bound theorem $\longleftarrow$ Hard Lefschetz Property of $\mathbb{R}[\Delta]$

ex① Dehn-Sommerville equation

① P : simplicial d-polytope

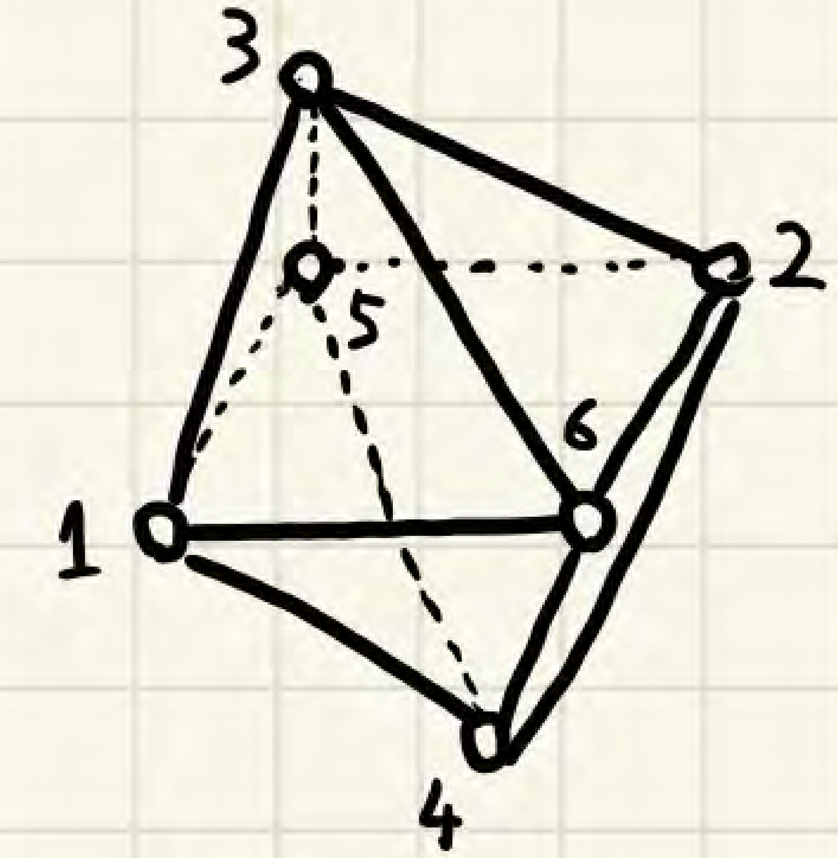
Theorem (Dehn–Sommerville equation)

$$\sum_{j=k}^{d}(-1)^j\binom{j}{k}f_{j-1}(P) = (-1)^d f_{k-1}(P) \quad (k = 0, 1, \ldots, d)$$

$\boxed{f_{-1} = 1}$

Fact: Dehn-Sommerville equation implies

$f_0(P), f_1(P), \ldots, f_{\lfloor \frac{d}{2} \rfloor - 1}(P)$ determine all $f_i(P)$



$f_0 = 6$

$f_0 - f_1 + f_2 = 2$

$3f_2 = 2f_1$

$f_1 = 3f_0 - 6$

$f_2 = 2f_0 - 4$

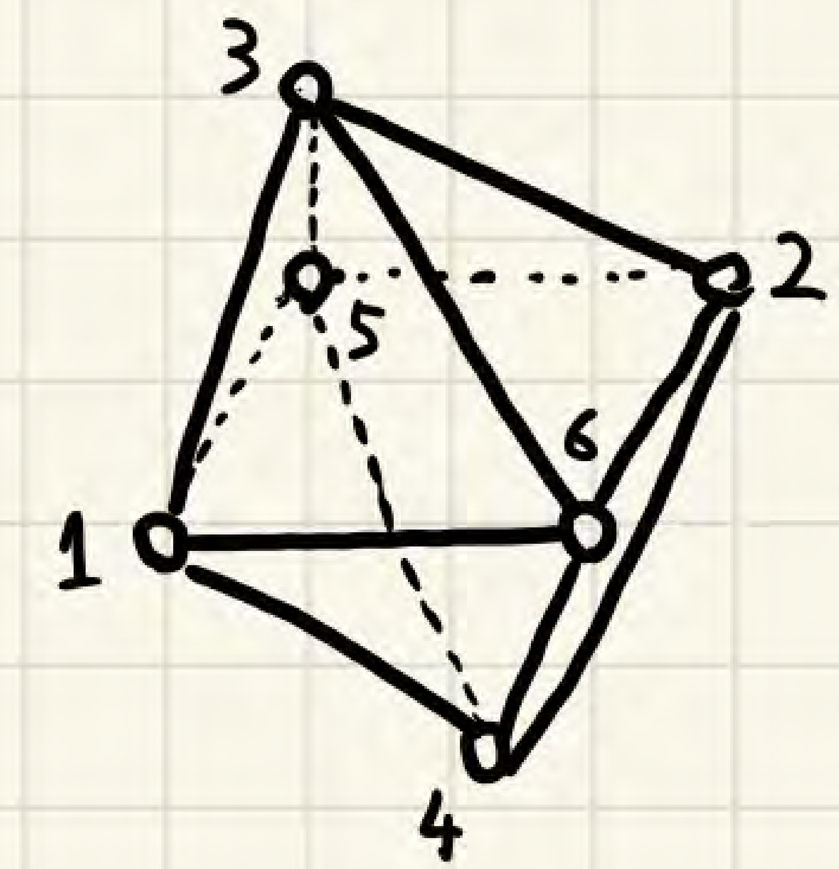
ex① Dehn-Sommerville equation

① Define $h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta)$ by

$$h_k(\Delta) \overset{\text{def}}{=} \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(\Delta)$$

Rem $d = \dim \Delta + 1$

② P : simplicial d-polytope



Theorem (Dehn–Sommerville equation, h-version)

$h_i(P) = h_{d-i}(P)$ for $i = 0, 1, \dots, d$.

This can be explained algebraically

$h_0 = 1$

$h_1 = 3$

$h_2 = 3$

$h_3 = 1$

symmetric

Some commutative algebra: PDA

- A graded $\mathbb{R}$-algebra $R = R_0 \oplus R_1 \oplus \cdots \oplus R_d$ is a **Poincaré duality algebra (PDA)** of socle degree d

$\overset{\text{def}}{\iff}$ $R_d \cong \mathbb{R}$ and multiplication map

$$\begin{array}{ccc} R_i \times R_{d-i} & \longrightarrow & R_d \cong \mathbb{R} \\ \uplus & & \uplus \\ (a, b) & \longmapsto & a \cdot b \end{array}$$

is non-degenerated for all i

Rem

If R is a PDA of deg d with $\mathrm{Hilb}(R,t) = \sum_{k=0}^{d} h_k t^k$

$\Rightarrow$ $h_i = h_{d-i}$ for all i

$\mathrm{Hilb}(R,t)$

$= \sum_{k \geq 0} (\dim_{\mathbb{R}} R_k) t^k$

Dehn-Sommerville eq

$h_i(P) = h_{d-i}(P)$

Some commutative algebra: Artinian reduction

Fact If $R = S/I$ is a graded $\mathbb{R}$-algebra of Krull dim d,
$\exists$ linear forms $\theta = \theta_1, \dots, \theta_d$ s.t.
$R/\theta R = S/I+(\theta)$ has Krull dim 0 ($\Leftrightarrow \dim_{\mathbb{R}} R/\theta R < \infty$)

θ is called a linear system of parameters (l.s.o.p)

$R/\theta R$ is called an Artinian reduction of R

① R is Cohen-Macaulay (CM)
$\Leftrightarrow \forall \theta = \theta_1, \dots, \theta_d$: lsop, θ_i is a NZD of $R/(\theta_1, \dots, \theta_{i-1})R$

② R is Gorenstein
$\Leftrightarrow R$ is CM & $R/\theta R$ is a PDA ($\forall \theta$: l.s.o.p)

ex ① Dehn-Sommerville equation

① P : Simplicial d-polytope, $R = \mathbb{R}[\Delta]$

Theorem (Dehn–Sommerville equation, h-version)

$h_i(P) = h_{d-i}(P)$ *for* $i = 0, 1, \ldots, d$.

Theorem (Stanley 1975)

If Δ *is CM, then Hilb*$(R/\Theta R, t) = h_0(\Delta) + h_1(\Delta)t + \cdots + h_d(\Delta)t^d$.

Theorem (Reisner 1976)

$R = \mathbb{R}[P]$ *is Gorenstein, in particular,* $R/\Theta R$ *is a PDA.*

These imply
Dehn-Sommervill eq

ex② UBT & LBT

① P : simplicial d-polytope with n vertices

Upper Bound Theorem (McMullen 1970)

$f_i(P) \le f_i(C(n,d))$ *for all* i.

Lower Bound Theorem (Barnette 1973)

$f_i(P) \ge f_i(S(n,d))$ *for all* i.

C(n,d)
= cyclic d-polytope
with n vertices

S(n,d)
= stacked d-polytope
with n vertices

h-versions

Upper Bound Theorem (h-version)

$h_i(P) \le \binom{n-d-1+i}{i}$ *for all* $i \le \frac{d}{2}$.

Lower Bound Theorem (h-version)

$h_2(P) \ge h_1(P)$.

$\boxed{d \ge 4}$

Hard Lefschetz property

- $R = \bigoplus_{k=0}^{d} R_k$: PDA of socle degree d
- R satisfies Hard Lefschetz property (HL)

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists \ell \in R_1$ s.t.

$$\times \ell^{d-2i} : R_i \to R_{d-i}$$

is a bijection for all $i \le \frac{d}{2}$

Rem $\ell \in R_1$ is called a Lefschetz element

Rem R: PDA with $\mathrm{Hilb}(R,t) = h_0 + h_1 t + \cdots + h_d t^d$.

R has HL $\Rightarrow$ $\mathrm{Hilb}(R/\ell R, t) = h_0 + (h_1 - h_0)t + (h_2 - h_1)t^2 + \cdots$

This in particular says $h_0, h_1, \ldots, h_d$ is unimodal

Theorem (Stanley 1980)

Let $R = \mathbb{R}[P]$. $R/\Theta R$ satisfies HL for a certain l.s.o.p. Θ.

ex② UBT & LBT

① P : Simplicial d-polytope with n vertices

Theorem (UBT & LBT)

$f_i(S(n,d)) \leq f_i(P) \leq f_i(C(n,d))$ for all i.

Theorem (UBT (g-version) & generalized LBT)

$0 \leq h_i(P) - h_{i-1}(P) \leq \binom{n-d+i-2}{i}$ for all $i \leq \frac{d}{2}$.

Quick Proof Let $R = \mathbb{R}[P] = S/I_P$. Then

$$0 \leq \underbrace{\dim_{\mathbb{R}}(R/(\theta,\ell)R)_i}_{= h_i - h_{i-1}} \leq \underbrace{\dim_{\mathbb{R}}(S/(\theta,\ell)S)_i}_{= \binom{n-d+i-2}{i}}$$

UBT (h-version)

$h_i(P) \leq \binom{n-d-1+i}{i}$

LBT (h-version)

$h_1(P) \leq h_2(P)$

$\theta = \theta_1, \dots, \theta_d$: lsop

ℓ : Lefschetz element

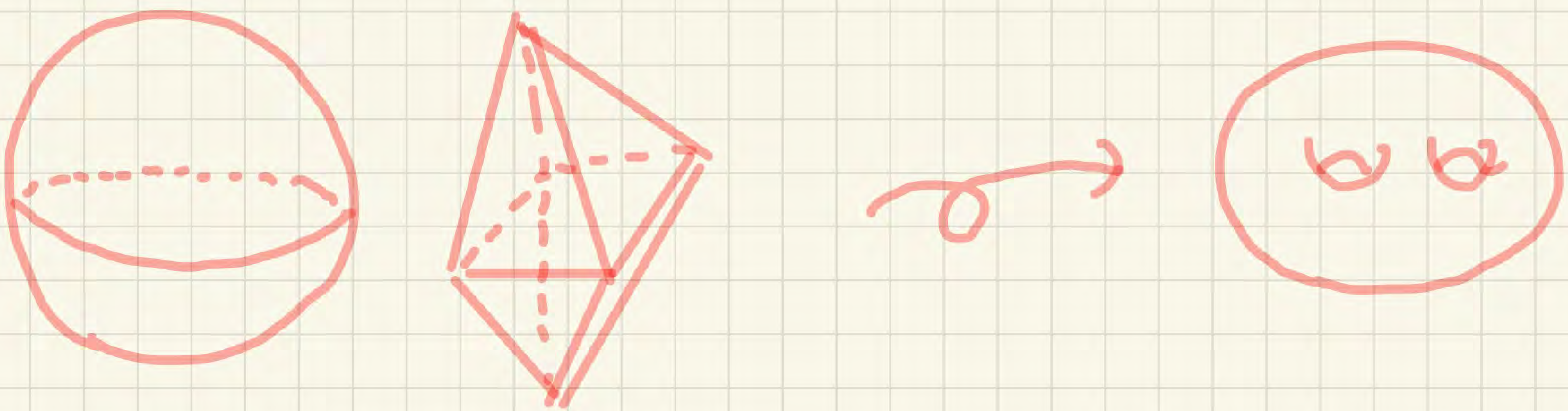
$S = \mathbb{R}[x_1, \dots, x_n]$

Recent breakthrough: Hard Lefschetz for spheres

Theorem (Adiprasito, Papadakis-Petrotou)

Let Δ be a simplicial sphere and $R = \mathbb{R}[\Delta]$.
$R/\Theta R$ satisfies HL for a generic l.s.o.p. Θ.

SR rings of triangulated mfds



Klee's Dehn-Sommervill equation

- Δ : triangulation of a connected closed $(d-1)$-mfd

Theorem (Klee 1964)

$$h_{d-i}(\Delta) = h_i(\Delta) + (-1)^{d-1}\binom{d}{i}\left(\chi(\Delta) - \chi(\mathbb{S}^{d-1})\right)$$

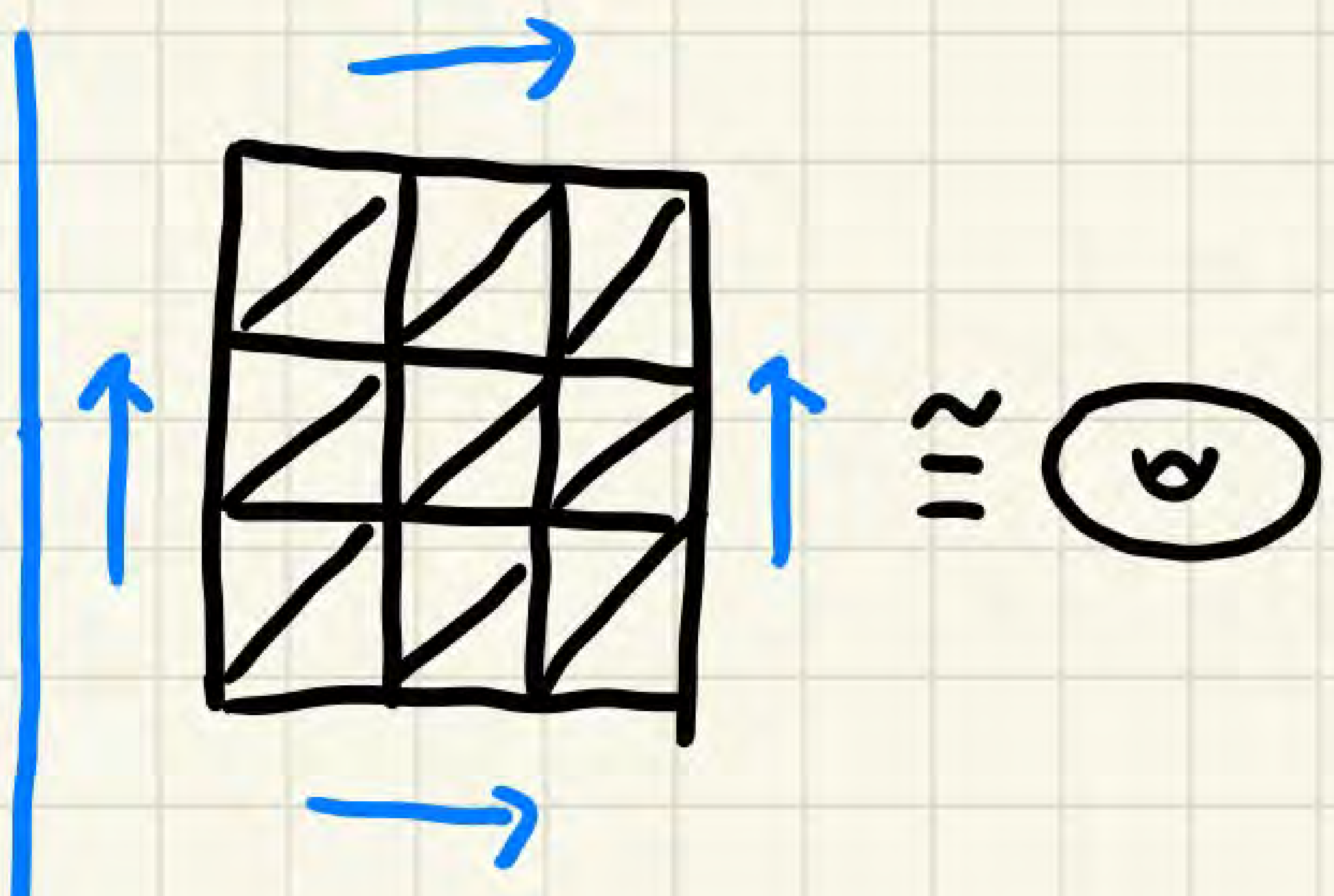
Q Can this be explained using PDA?

difficulty

- $\mathbb{R}[\Delta]$ is not CM
- $\mathbb{R}[\Delta]/(\text{l.s.o.p})$ is not a PDA.

good point

- $\mathbb{R}[\Delta]$ is a Buchsbaum ring
- Comb. formula of $\mathrm{Hilb}\left(\mathbb{R}[\Delta]/(\text{l.s.o.p}), t\right)$ when Δ is Buchsbaum (Schenzel 1981)



For Stanley-Reisner rings,

Buchsbaum = locally CM
(every link is CM)

Schenzel's formula

$$\dim_{\mathbb{R}}\left(\mathbb{R}[\Delta]/(\text{l.s.o.p})\right)_i$$
$$= h_i(\Delta) - \binom{d}{i}\sum_{k=2}^{i-1}(-1)^{i-k}\beta_{k-1}(\Delta)$$

Goto's Σ-ideal

Def (Goto 1983) $R = S/I$: Buchsbaum graded $\mathbb{R}$-alg
$\Theta = \theta_1, \dots, \theta_d$: (linear) s.o.p. for R

$$\Sigma(\theta) \overset{\text{def}}{=} \sum_{i=1}^{d} (\theta_1, \dots, \hat{\theta}_i, \dots, \theta_d) : \theta_i$$

Point For a Buchsbaum algebra R,
$R/\Sigma(\theta)$ usually behave nicer than $R/\theta R$

Rem

- $I : f = \{ g \in R \mid f \cdot g \in I \}$
- $\Sigma(\theta) \supset (\theta) \quad (d \geq 2)$
- R is CM
  $\Leftrightarrow \Sigma(\theta) = (\theta)$

Note $\Sigma(\theta)$ is the kernel of

$$R/(\theta_1, \dots, \theta_d)R \xrightarrow{\times \theta_1 \cdots \theta_d} R/(\theta_1^2, \dots, \theta_d^2)R \xrightarrow{\times \theta_1 \cdots \theta_d} R/(\theta_1^3, \dots, \theta_d^3)R \to \cdots$$

Klee's Dehn-Sommervill equation

- Δ: triangulation of a connected closed $(d-1)$-mfd M

Theorem (Novik–Swartz 2009)

$R = \mathbb{R}[\Delta]$, Θ: *l.s.o.p. of* R.
If M *is orientable, then* $R/\Sigma(\Theta)$ *is a PDA.*

※ Recall we know Hilb fc of $R/\Theta R$ (Schenzel's formula)

Theorem (Novik–Swartz 2009)

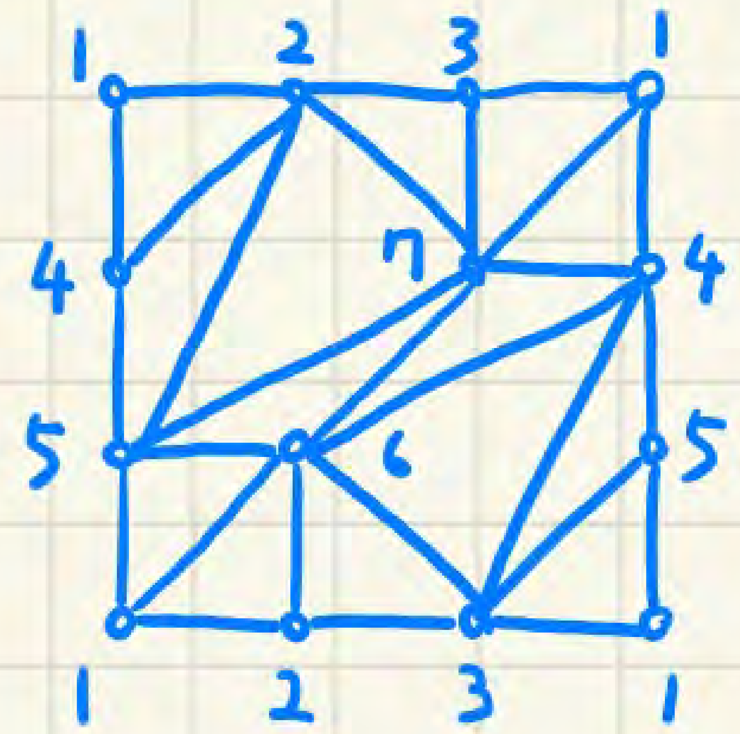
If $R = \mathbb{R}[\Delta]$ *is Buchsbaum and* Θ *is an l.s.o.p. of* R*, then*

$$\dim_{\mathbb{R}} \left(\Sigma(\Theta)/\Theta R\right)_i = \binom{d}{i} \beta_{i-1}(\Delta)$$

Rem These imply klee's Dehn-Sommerville eq.

Rem Goto (1983) gave a result for general Buchsbaum rings

ex



- (h_0, h_1, h_2, h_3)
  $= (1, 4, 10, -1)$
- $\mathrm{Hilb}(\mathbb{R}[\Delta]/\Sigma(\Theta), t)$
  $= \underline{1 + 4t + 4t^2 + t^3}$
- $\mathrm{Hilb}(\mathbb{R}[\Delta]/(\mathrm{lsop}), t)$
  $= 1 + 4t + 7t^2 + t^3$

Hard Lefschetz property (Recent result)

Ⓓ Δ: triangulation of a connected orientable closed mfd, $R = \mathbb{R}[\Delta]$

Theorem (Adiprasito–Papadakis–Petrotou)

The algebra $R/\Sigma(\Theta)$ satisfies HL for a generic l.s.o.p. Θ.

application to $f_0^{min}(M)$: general mfd

- Δ : triangulation of a connected $(d-1)$-mfd (not necessary closed)

Claim $\binom{f_0(\Delta)-d+i-1}{i} \geq \binom{d}{i} \beta_{i-1}(\Delta)$

Proof Let $R = \mathbb{R}[\Delta]$ and Θ : lsop of R.

$$\dim_{\mathbb{R}}\left(\Sigma(\Theta)/\Theta R\right)_i \leq \dim_{\mathbb{R}}\left(R/\Theta R\right)_i \leq \dim_{\mathbb{R}}\left(S/\Theta R\right)_i$$

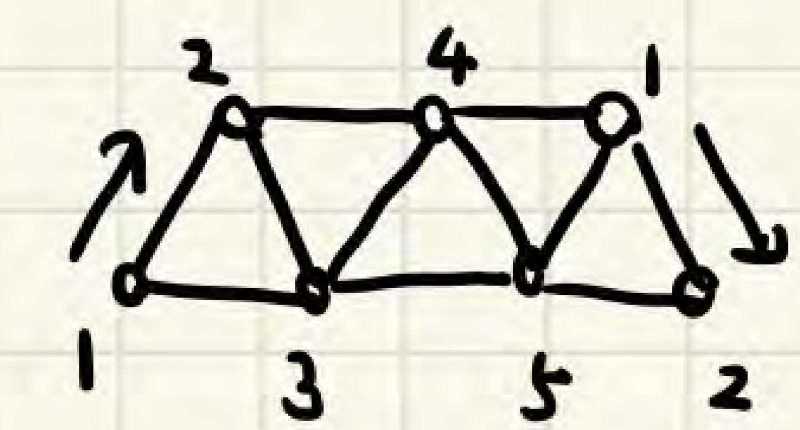
$$\dim_{\mathbb{R}}\left(\Sigma(\Theta)/\Theta R\right)_i = \binom{d}{i}\beta_{i-1}, \qquad \dim_{\mathbb{R}}\left(S/\Theta R\right)_i = \binom{f_0-d+i-1}{i}$$

Example

- If $d=3$, $i=2$, $\beta_1=1$,

$\binom{f_0-2}{2} \geq 3 \Leftrightarrow f_0 \geq 5$

$\rightsquigarrow$ f_0^{min}(Mobius band) $= 5$



- Applying $d=4$, $i=2$, $\beta_1=1$,

$f_0^{min}(\underbrace{D^2 \times S^1}_{\text{solid torus}}) = 7$

Application to $f_0^{\min}(M)$ for closed mfds

- In the previous statement we compare

$$\Sigma(\Theta)/\Theta R, \quad R/\Theta R, \quad S/\Theta S$$

- If we want to say something special for closed mfd, we need something similar for $(\Theta, \underset{\sim}{\ell}) R$

lefschetz element

Theorem (Nevo-M 2013 (+ Adiprasito's HL))

Δ: triangulation of connected orientable mfd, $R = \mathbb{R}[\Delta]$. There is an ideal J of $R/(\Theta, \ell)R$ such that

$$\dim_{\mathbb{R}}(J) = \binom{d+1}{i}\beta_{i-1}(\Delta) \quad (i < d/2)$$

$\boxed{\text{idea}}$

$\Theta R \to \Sigma(\Theta)$

$(\Theta, \ell)R \to ? := J$

Application to $f_0^{min}(M)$ for closed mfds

Theorem (Novik-Swartz, Adiprasito(Conjectured by Kühnel))

If Δ is a triangulation of a closed $(d-1)$-mfd then

$$\binom{f_0(\Delta)-d-2+r}{r} \geq \binom{d+1}{r}\beta_{r-1}(\Delta) \quad (r < \frac{d-1}{2})$$

$$\binom{f_0(\Delta)-d-1+r}{r+1} \geq \binom{d}{r}\beta_{r-1}(\Delta) \quad (r = \frac{d-1}{2})$$

Proof of the first part. $R = \mathbb{R}[\Delta]$, θ, ℓ : generic

$$\underset{\binom{d+1}{r}}{\overset{}{\dim_{\mathbb{R}} J}} \leq \dim_{\mathbb{R}} R/(\theta,\ell)R \leq \dim_{\mathbb{R}} S/(\theta,\ell)S$$

$$\dim_{\mathbb{R}} J = \binom{d+1}{r}, \qquad \dim_{\mathbb{R}} S/(\theta,\ell)S = \binom{f_0-d-2+r}{r}$$

analogous to LBT & UBT!

Further Problems

- More results on $f_0^{min}(M)$?

  Conj $f_0^{min}(S^1 \times S^1 \times S^1) \geq 15$.

- Refine upper bounds & lower bounds of $f_i(\Delta)$?

  Q What is $\max\{f_2(\Delta) \mid \Delta \text{ is an } n \text{ vertex triangulation of } S^1 \times S^3\}$?

- Currently we only consider $\beta_i(\Delta)$. Other topological invariants?

  Thm (Novik-M 2017) If Δ is a conn. closed $(d-1)$-mfd
  $$h_2(\Delta) - h_1(\Delta) \geq \binom{d+1}{2} \times (\text{minimal num. of gens of } \pi_1(\Delta))$$

- Does $\mathbb{R}[\Delta]/\Sigma(\Theta)$ relate to geometric object?