Plane partitions and rowmotion on rectangular and trapezoidal posets

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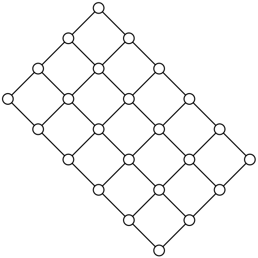
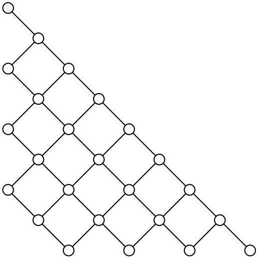
(joint work with Joseph Johnson)

Two posets



rectangle $R_{6,4}$

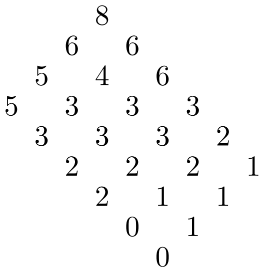


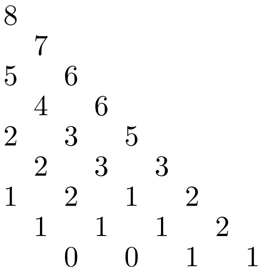
trapezoid $T_{6,4}$

Plane partitions

Definition

For a finite poset P, a plane partition of shape P (or P-partition) is an order-preserving map from P to $\mathbf{Z}_{\geq 0}$.
The height of a P-partition is the maximum number in its range.





Plane partitions

Theorem (Proctor '83)

The rectangular poset $R = R_{m,n}$ and the trapezoidal poset $T = T_{m,n}$ have the same number of plane partitions of height $\leq \ell$ for each ℓ.

For R, MacMahon showed that this number is

$$\prod_{i=1}^{\ell}\prod_{j=1}^{m}\prod_{k=1}^{n}\frac{i+j+k-1}{i+j+k-2}.$$

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Proctor's proof uses a branching rule from Lie algebra representation theory and is not bijective. He remarks: "...the question of a combinatorial correspondence for [this theorem] seems to be a complete mystery."

Plane partitions

Partial progress towards a combinatorial proof:

- Stembridge '86 and Reiner '97 for $\ell = 1$;
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- It extends to a continuous map on the order polytope.
- It intertwines with an action called rowmotion (which implies that rowmotion on T has order $m + n$).
- It is the tropicalization of a birational map that also respects birational rowmotion (so birational rowmotion on T has order $m + n$).

Order and chain polytopes

Definition

The order polytope $\mathcal{O}(P)$ of a poset P is the set of labelings $y = (y_p)_{p \in P}$ satisfying $0 \leq y_p \leq 1$ and $y_p \leq y_q$ when $p \leq q$ in P.

A P-partition of height $\leq \ell$ is a lattice point in $\ell\mathcal{O}(P)$.

Therefore, Proctor's theorem can be restated as saying that $\mathcal{O}(R)$ and $\mathcal{O}(T)$ have the same Ehrhart polynomial.

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The chain polytope $\mathcal{C}(P)$ of a poset P is the set of labelings $x = (x_p)_{p \in P}$ satisfying $0 \leq x_p$ and $\sum_{p \in C} x_p \leq 1$ for all chains $C \subseteq P$.

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Stanley showed that $\mathcal{O}(P)$ and $\mathcal{C}(P)$ have the same Ehrhart polynomial by exhibiting continuous, piecewise-linear and unimodular, bijective transfer maps that send $x \in \mathcal{C}(P) \longleftrightarrow y \in \mathcal{O}(P)$ via

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$$x_p = y_p - \max_{q \lessdot p} y_q$$

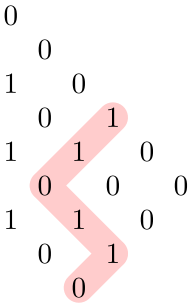
$$y_p = \max_{C} \sum_{p \in C} x_p,$$

where $\max \varnothing = 0$, and C ranges over chains with maximum element p.

Order and chain polytopes

$$\begin{array}{cccccc}
0 & & & & & \\
 & 0 & & & & \\
1 & & 0 & & & \\
 & 0 & & 1 & & \\
1 & & 1 & & 0 & \\
 & 0 & & 0 & & 0 \\
1 & & 1 & & 0 & \\
 & 0 & & 1 & & \\
 & & 0 & & &
\end{array}
\qquad\qquad
\begin{array}{cccccc}
4 & & & & & \\
 & 4 & & & & \\
4 & & 4 & & & \\
 & 3 & & 4 & & \\
3 & & 3 & & 2 & \\
 & 2 & & 2 & & 1 \\
1 & & 2 & & 1 & \\
 & 0 & & 1 & & \\
 & & 0 & & &
\end{array}$$

$$x \in 4\mathcal{C}(P) \qquad\qquad y \in 4\mathcal{O}(P)$$



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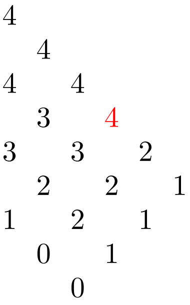

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Rowmotion

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Dualizing this construction, we can likewise give a correspondence $y \in \mathcal{O}(P) \longleftrightarrow z \in \mathcal{C}(P)$ via

$$z_p = \min_{q \gtrdot p} y_q - y_p.$$

(where $\min \varnothing = 1$).

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The map $\rho\colon \mathcal{C}(P) \to \mathcal{C}(P)$ sending $x \mapsto z$ is called (inverse antichain) rowmotion.

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Note: If x is the indicator for an antichain A, then y gives the order filter F generated by A, and z gives the maximal elements of $P \setminus F$.

Rowmotion

0					
	0				
1		0			
	0		1		
1		1		0	
	0		0		0
1		1		0	
	0		1		
		0			

$x \in 4\mathcal{C}(P)$

4					
	4				
4		4			
	3		4		
3		3		2	
	2		2		1
1		2		1	
	0		1		
		0			

$y \in 4\mathcal{O}(P)$

0					
	0				
0		0			
	1		0		
0		0		2	
	1		0		1
1		0		0	
	1		0		
		0			

$z \in 4\mathcal{C}(P)$

Rowmotion

- For most posets, rowmotion does not behave nicely (e.g. have small order).

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1	0	0	0	2
2 2	2 1	0 1	1 0	0 0
0 1	2 2	1 1	0 1	1 0
0	0	2	1	0

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$$
\begin{array}{cccc} & 1 & & \\ 2 & & 2 & \\ & 0 & & 1 \\ & & 0 & \end{array}
\qquad
\begin{array}{cccc} & 0 & & \\ 2 & & 1 & \\ & 2 & & 2 \\ & & 0 & \end{array}
\qquad
\begin{array}{cccc} & 0 & & \\ 0 & & 1 & \\ & 1 & & 1 \\ & & 2 & \end{array}
\qquad
\begin{array}{cccc} & 0 & & \\ 1 & & 0 & \\ & 0 & & 1 \\ & & 1 & \end{array}
\qquad
\begin{array}{cccc} & 2 & & \\ 0 & & 0 & \\ & 1 & & 0 \\ & & 0 & \end{array}
$$

- Williams conjectured that ρ_T for a trapezoid $T = T_{m,n}$ also has order $m+n$. (This was proved for the vertices of $\mathcal{C}(T)$ by Dao, Wellman, Yost-Wolff, and Zhang '22.)

The bijection

We construct a map $\zeta\colon \mathbf{R}^T \to \mathbf{R}^R$ that deforms T to R by applying rowmotion on subposets.

0							
	0						
1		0					
	0		1				
1		1		0			
	0		0		0		
1		1		0		1	
	0		1		2		0

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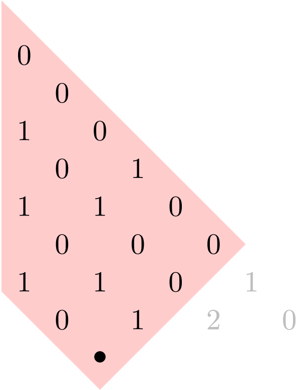

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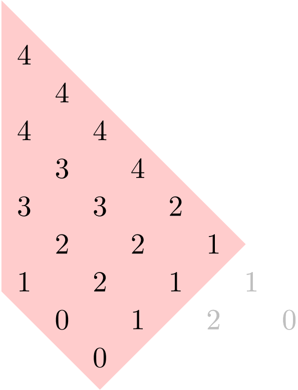

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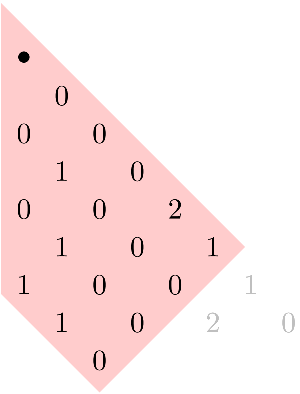

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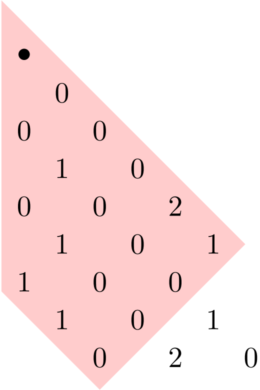

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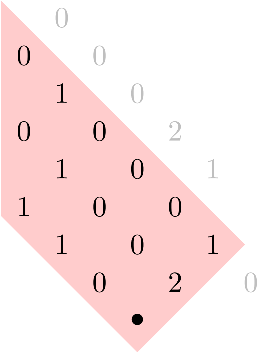

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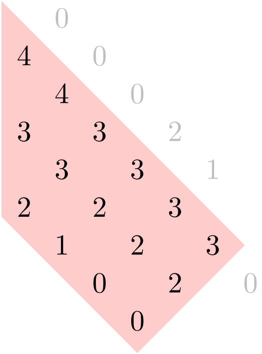

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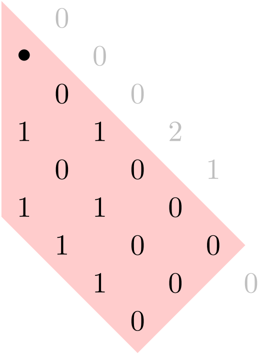

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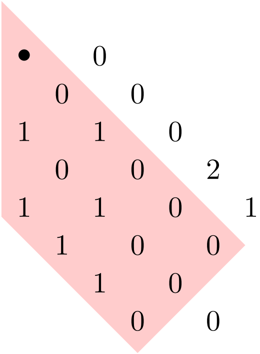

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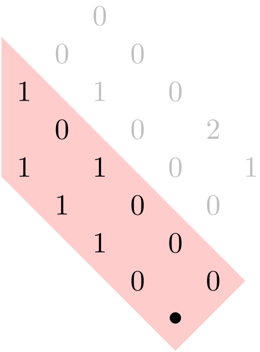

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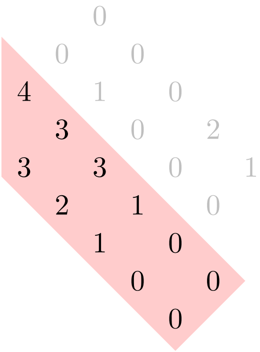

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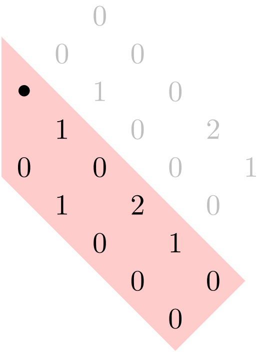

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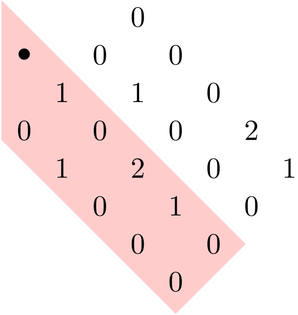

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 & & 0 & & 0 & & & \\
 & 1 & & 1 & & 0 & & \\
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 & & 0 & & 1 & & 0 & \\
 & & & 0 & & 0 & & \\
 & & & & 0 & & &
\end{array}
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The bijection

Theorem (Johnson-L.)

The map ζ is a continuous, piecewise-linear and unimodular map that gives a bijection from $\ell\mathcal{C}(T)$ to $\ell\mathcal{C}(R)$ for all nonnegative integers ℓ, so in particular $\mathcal{C}(T)$ and $\mathcal{C}(R)$ have the same Ehrhart polynomial.

Conjugating ζ by the transfer map then gives a bijection between plane partitions of R and T of the same height.

Birational maps

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Equivalently:

$$x_p^{-1} = \sum_{q \lessdot p} \frac{y_q}{y_p},$$

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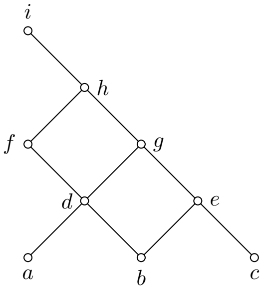
Example of the birational map $\zeta\colon \mathbf{R}_+^T \to \mathbf{R}_+^R$



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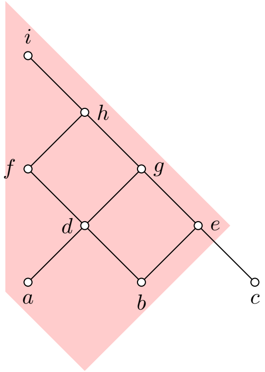

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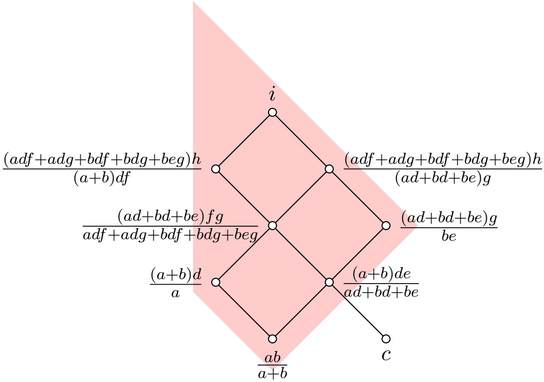

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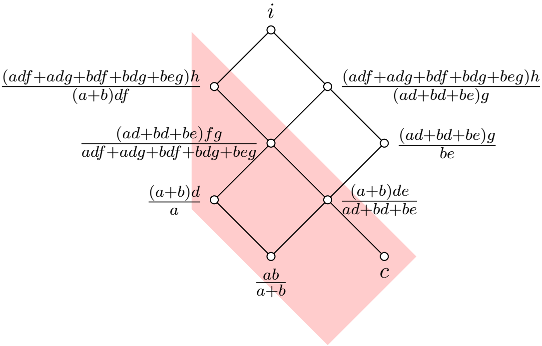

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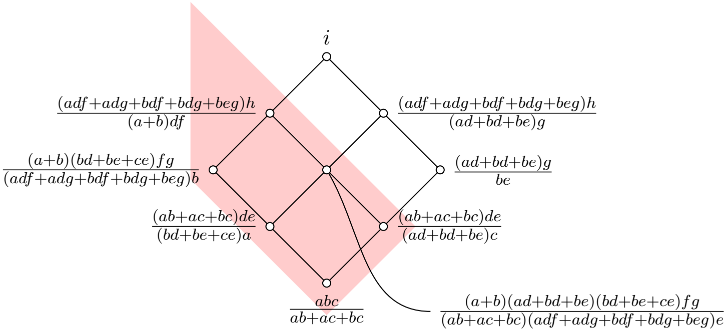

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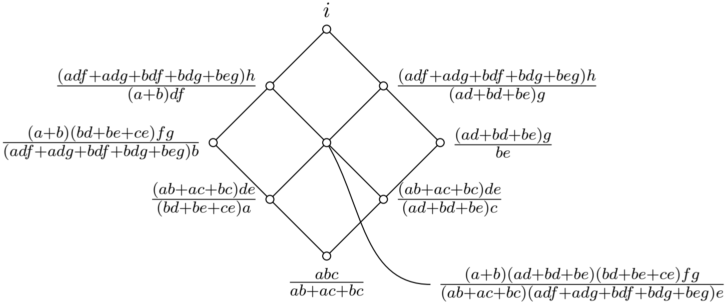

The total weight of all maximal chains is the same as in the original labeling, $adfhi + adghi + bdfhi + bdghi + beghi + ceghi$.

Theorem (Johnson-L.)

The birational map $\zeta\colon \mathbf{R}_+^T \to \mathbf{R}_+^R$ preserves $\sum_C \prod_{p \in C} x_p$, where C ranges over all maximal chains (of T or R).

This implies the corresponding piecewise-linear result since tropicalizing implies that $\max_C \sum_{p \in C} x_p$ is unchanged.

Proof idea

- We relate weights of chains in points $x, z \in \mathbf{R}_+^P$ where $z = \rho(x)$ for a general skew shape poset P.

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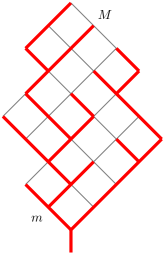
- Since $x_p^{-1} = \sum_{q \lessdot p} \frac{y_q}{y_p}$, we can express the products of x_p in terms of upward arborescences (weighted by Laurent monomials in y).



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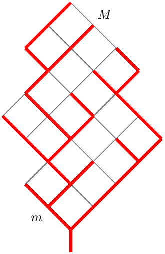

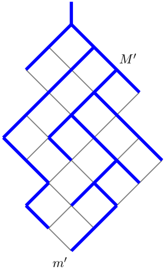

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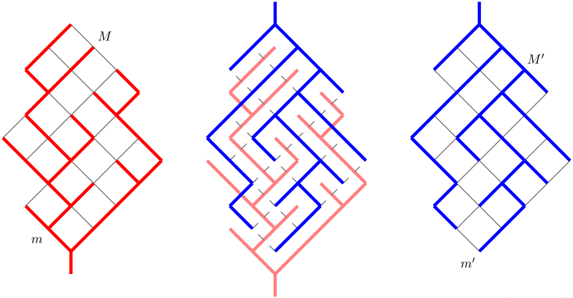
- These can then be related via a form of duality for spanning trees.



Birational rowmotion

Theorem (Johnson-L.)

The birational map ζ intertwines with birational rowmotion:

$$\zeta \circ \rho_T = \rho_R \circ \zeta.$$

Since Grinberg–Roby showed that birational rowmotion on the rectangle $R_{m,n}$ has order $m+n$, we get the following corollary.

Corollary

(Birational) rowmotion on the trapezoid $T_{m,n}$ has order $m+n$.