Unimodular triangulations of matroid polytopes

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Joint w/ Spencer Backman

Matroid: A collection of subsets (**bases**) of $[n]$ satisfying **basis exchange**: For any two bases A, B, and $i \in A \setminus B$, there exists $j \in B \setminus A$ such that $A \setminus \{i\} \cup \{j\}$ is a basis.

Example: Given a finite set of vectors spanning a vector space, the collection of all vector space bases among these vectors is a matroid.

Theorem (Edmonds, Gelfand–Goresky–MacPherson–Serganova)

A collection $\mathcal{B}$ of subsets of $[n]$ is a matroid if and only if the polytope

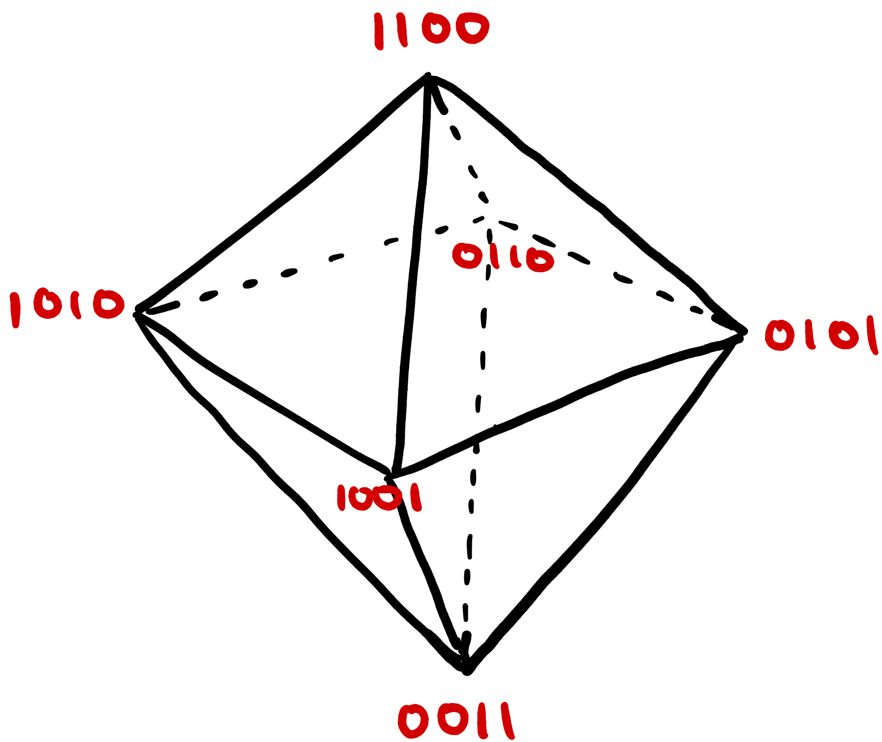
$$\operatorname{conv}\left(\sum_{i \in A} e_i \; : \; A \in \mathcal{B} \right)$$

(e_i is a standard basis vector of $\mathbb{R}^n$)

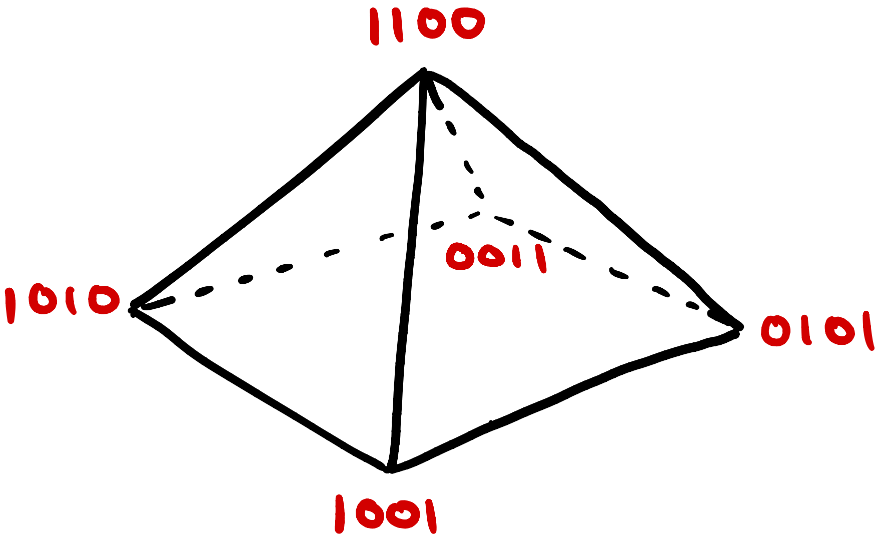
has each edge parallel to $e_i - e_j$ for some i, j.

Such a polytope is called a **matroid polytope**.

Example: $U_{4,2} = \binom{[4]}{2}$ = all 2-element subsets of $[4]$.



$\subset \mathbb{R}^4$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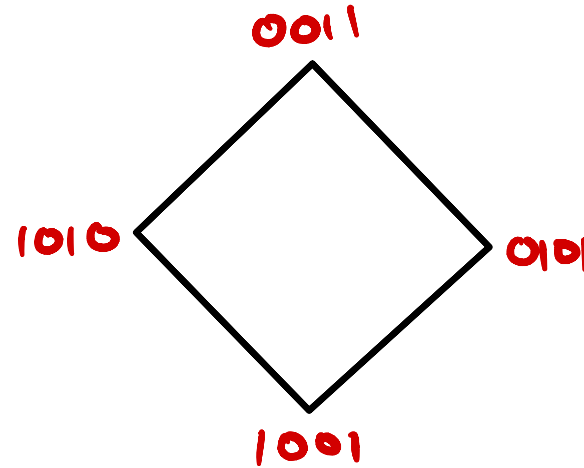
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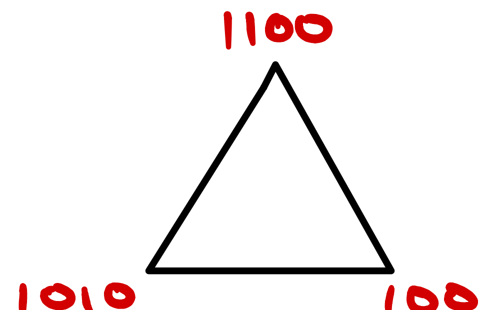
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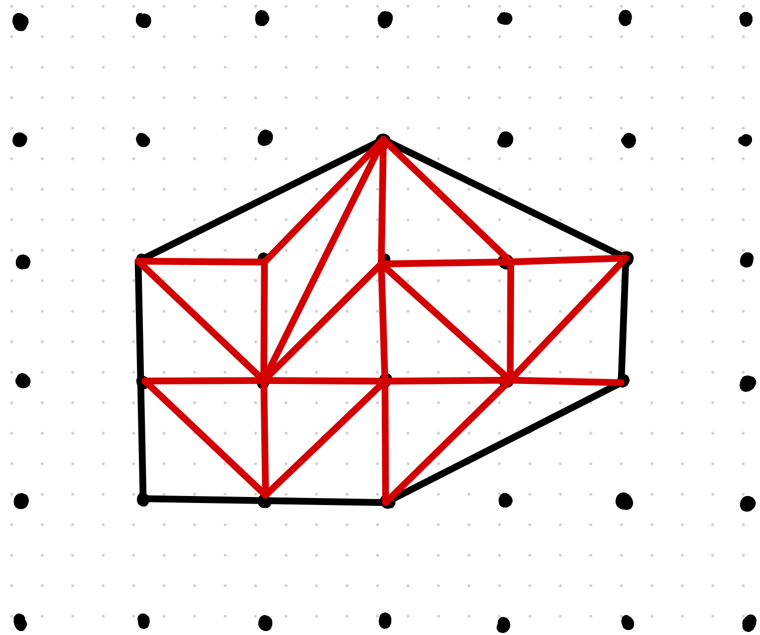
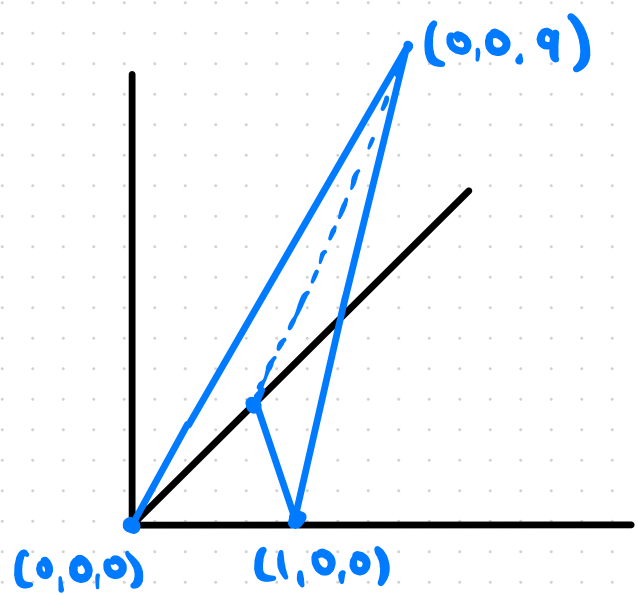
A unimodular triangulation is a lattice triangulation where every simplex has volume 1/dimension!.



Every lattice polygon has a unimodular triangulation.

However, lattice polytopes of dimension ≥ 3 do not necessarily have unimodular triangulations.

Example:



Unimodular triangulations have connections to enumerative combinatorics, algebra, and algebraic geometry.

- Ehrhart theory
- Gröbner bases
- Resolutions of singularities

In general, it is hard to construct unimodular triangulations or prove their existence.

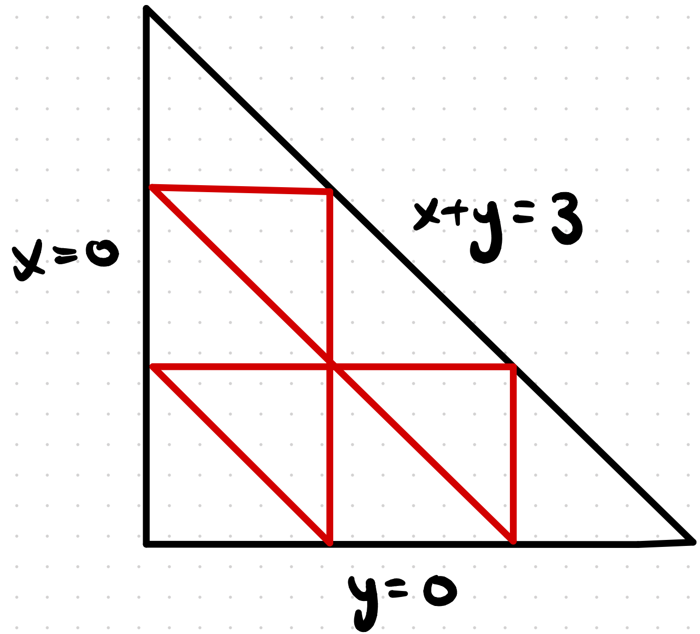
- Do 3-dimensional lattice parallelepipeds have unimodular triangulations?

Systematic ways to construct unimodular triangulations?

Alcoved polytope: Polytope whose facet-defining equations are of the form $x_i + x_{i+1} + x_{i+2} + \cdots + x_j = k$.

(Alternate definition: Polytope whose facet-defining equations are of the form $y_i - y_j = k$ and $y_i = k$.)

Alcoved polytopes can be unimodularly triangulated by **dicing** with the hyperplanes $x_i + x_{i+1} + \cdots + x_j = k$.



Matroids whose polytopes are alcoved are called positroids.
Positroid polytopes therefore have unimodular triangulations.

Theorem (Backman, L.): All matroid polytopes have regular unimodular triangulations.

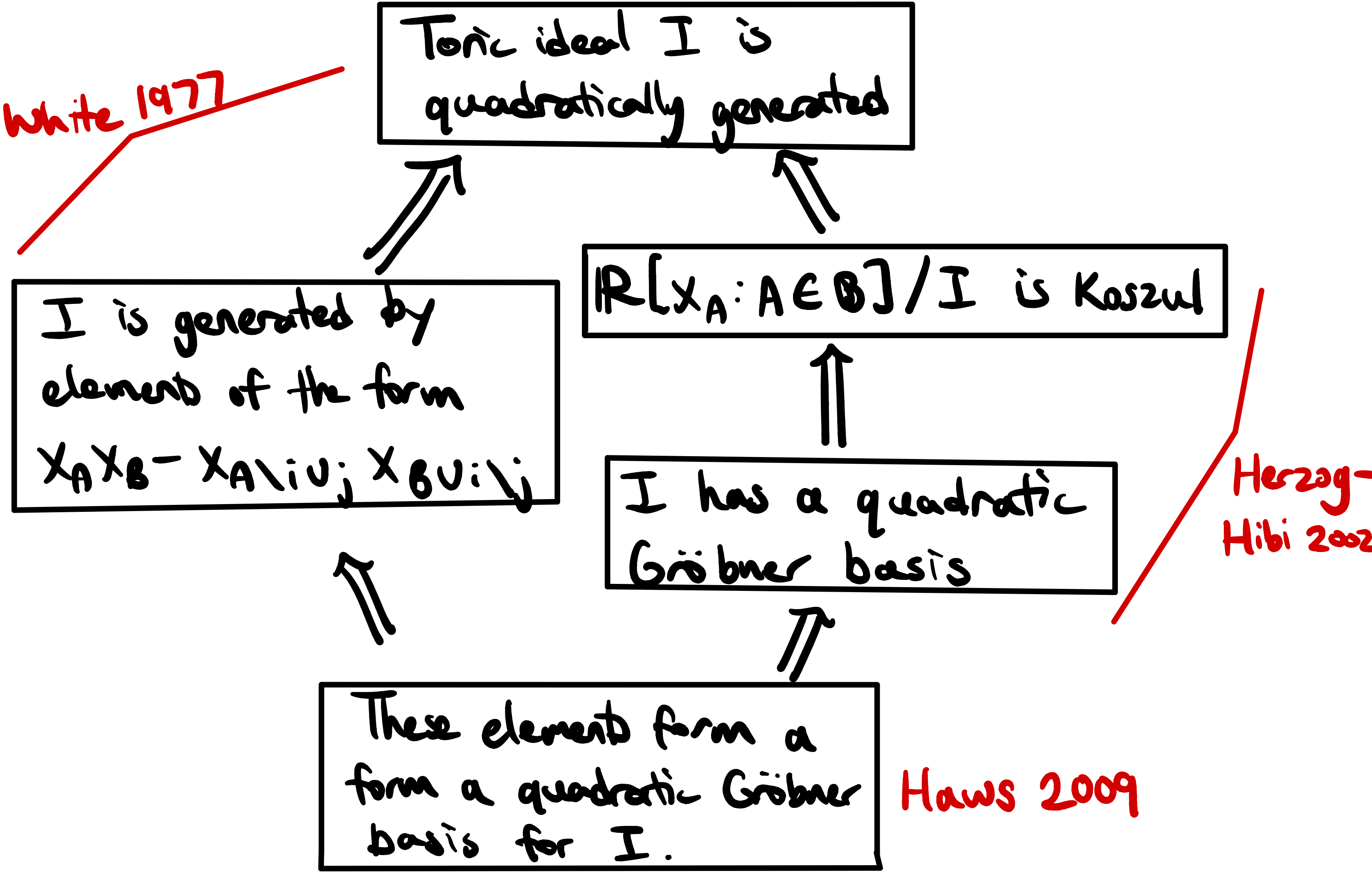
Extends to integral generalized permutahedra, which includes matroid independence polytopes.

Conjectured by Haws 2009.

Given a matroid $\mathcal{B}$ on $[n]$, define its toric ideal to be the kernel of the ring map

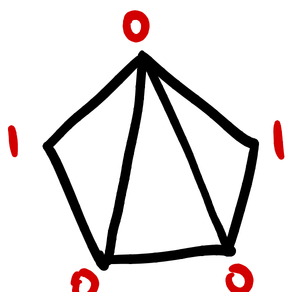
$$\mathbb{R}[x_A : A \in \mathcal{B}] \longrightarrow \mathbb{R}[x_1, \dots, x_n]$$
$$x_A \longmapsto \prod_{i \in A} x_i$$

Conjecture (White): The toric ideal of a matroid is quadratically generated.

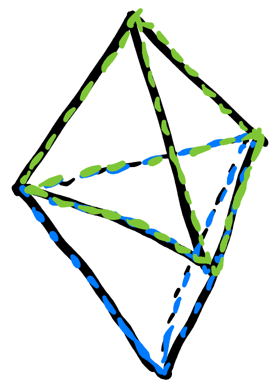

Toric ideal I is quadratically generated
White 1977
I is generated by elements of the form $x_A x_B - x_{A \setminus i \cup j} x_{B \cup i \setminus j}$
$\mathbb{R}[x_A : A \in \mathcal{B}]/I$ is Koszul
I has a quadratic Gröbner basis
Herzog-Hibi 2002
These elements form a form a quadratic Gröbner basis for I.
Haws 2009

I has a quadratic Gröbner basis $\overset{\text{Sturmfels}}{\Longleftrightarrow}$ The matroid polytope has a unimodular, regular, flag triangulation.

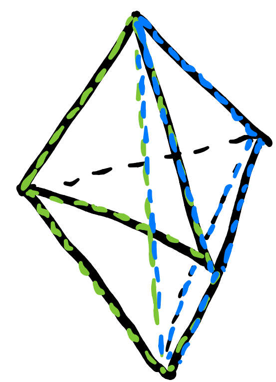
Regular: The triangulation is induced by a real-valued function on its vertices.



Flag: Every clique in the 1-skeleton of the triangulation is the 1-skeleton of a cell of the triangulation.



Flag

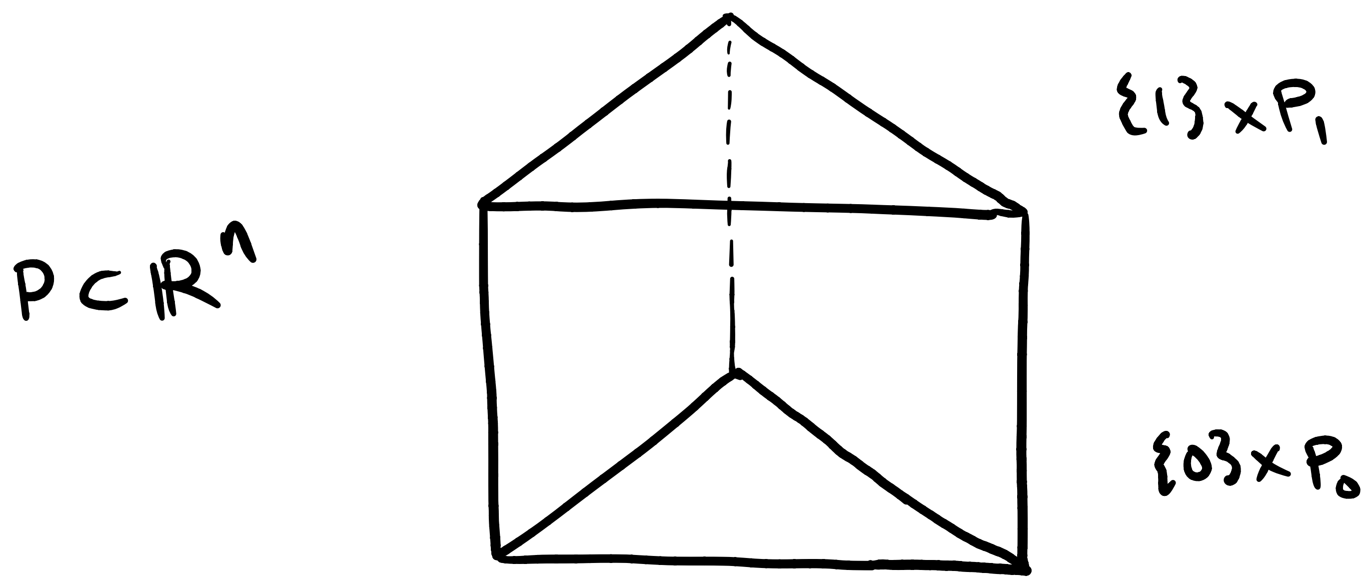


Not flag

$\boxed{\text{$I$ has a quadratic Gröbner basis}} \iff \boxed{\text{The matroid polytope has a unimodular, regular, flag triangulation.}}$

With Matt Larson and Samuel Payne, we used our construction to computationally search for regular unimodular triangulations which are also flag. However, we were unable to find any for the Fano matroid.

__Sketch of proof__: Partition the vertices of the matroid polytope into those with a 0 in the first coordinate and those with a 1. Call these polytopes P_0 and P_1.

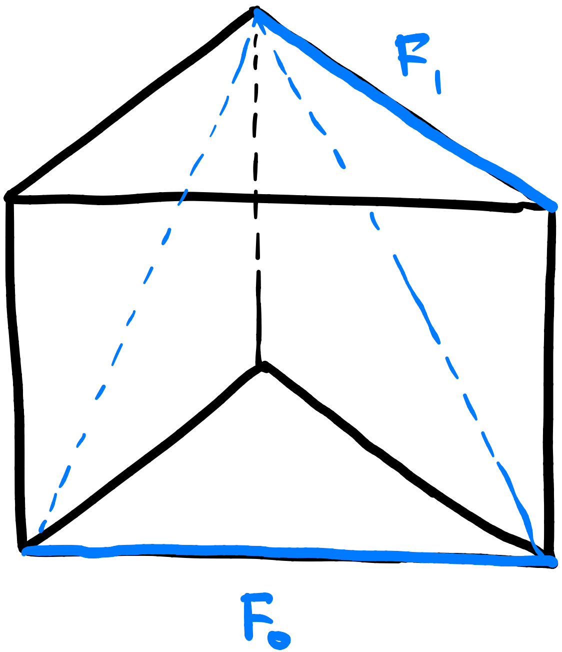


P_0 and P_1 are matroid polytopes, so by induction there are functions f_0, f_1 on the vertices of P_0, P_1 which induce unimodular triangulations.

Let $\ell : \mathbb{R}^{n-1} \to \mathbb{R}$ be a generic linear functional. Define a function f on the vertices of P by

$$f(x_1, \dots, x_n) = \begin{cases} \varepsilon f_0(x_2, \dots, x_n) & \text{if } x_1 = 0 \\ \ell + \varepsilon f_1(x_2, \dots, x_n) & \text{if } x_1 = 1 \end{cases} \qquad 0 < \varepsilon << 1$$

Claim: The subdivision induced by f is unimodular.



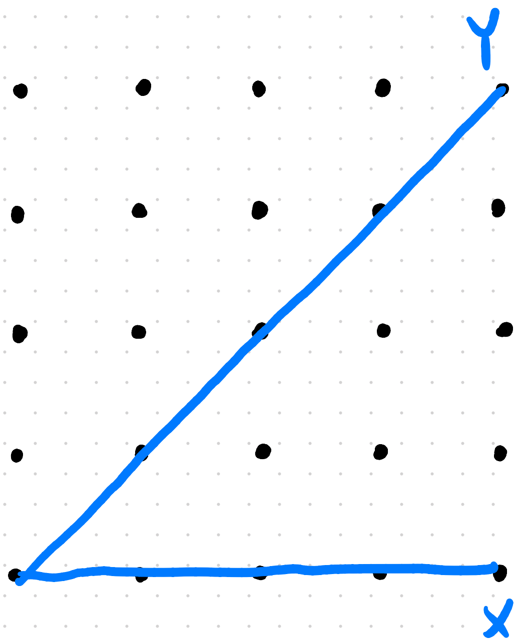
Every cell of the subdivision is contained in a polytope of the form $\mathrm{conv}(F_0, F_1)$, where F_0 and F_1 are affinely independent faces of P_0 and P_1.

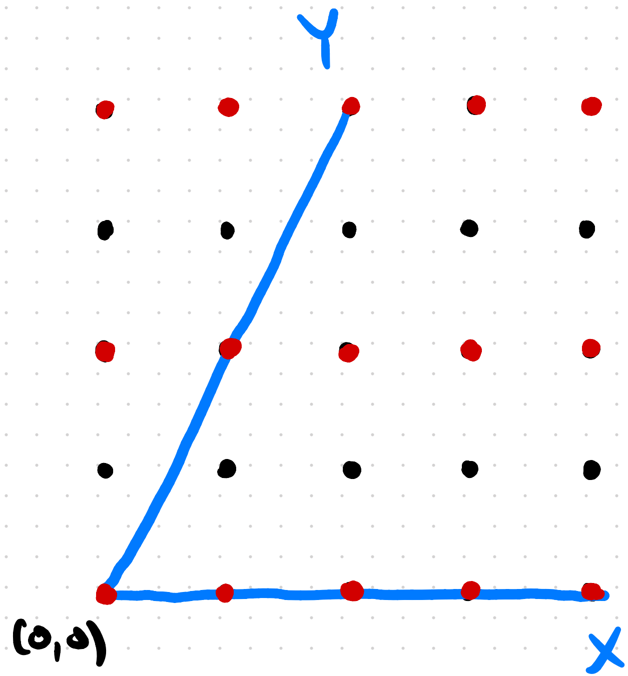
Let $\mathrm{lin}\, F_0 = \mathrm{span}_{\mathbb{R}}\{u-v : u,v \in F_0\}$

$\mathrm{lin}\, F_1 = \mathrm{span}_{\mathbb{R}}\{u-v : u,v \in F_1\}$

<u>Key observation:</u>

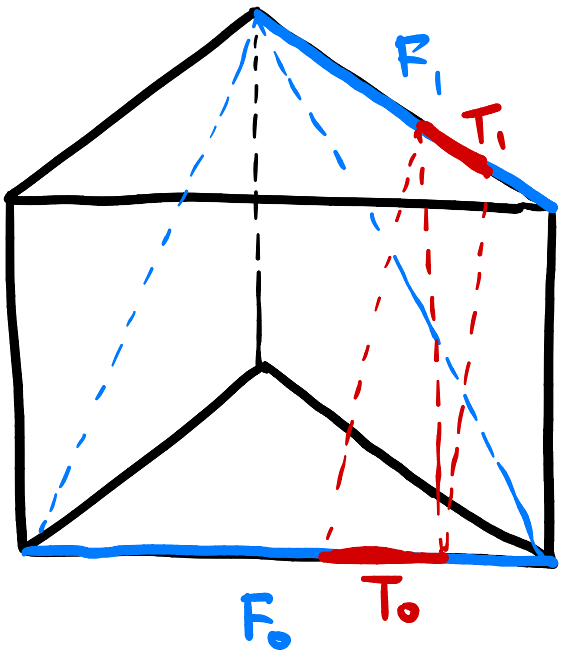
$$(\mathrm{lin}\, F_0 \cap \mathbb{Z}^{n-1}) + (\mathrm{lin}\, F_1 \cap \mathbb{Z}^{n-1}) = (\mathrm{lin}\, F_0 + \mathrm{lin}\, F_1) \cap \mathbb{Z}^{n-1}$$





$X\cap\mathbb{Z}^2 + Y\cap\mathbb{Z}^2 = (X+Y)\cap\mathbb{Z}^2$

$X\cap\mathbb{Z}^2 + Y\cap\mathbb{Z}^2 \neq (X+Y)\cap\mathbb{Z}^2$



Let $T = \mathrm{conv}(T_0, T_1)$ be a cell in the subdivision induced by f.

By induction, T_0 and T_1 are unimodular within their respective lattices. Also,

$$(\mathrm{lin}\, T_0 \cap \mathbb{Z}^{n-1}) + (\mathrm{lin}\, T_1 \cap \mathbb{Z}^{n-1})$$
$$= (\mathrm{lin}\, T_0 + \mathrm{lin}\, T_1) \cap \mathbb{Z}^{n-1}$$

which implies T is unimodular.

Vielen Dank!