Asymptotics of Bounded Lecture Hall Tableaux

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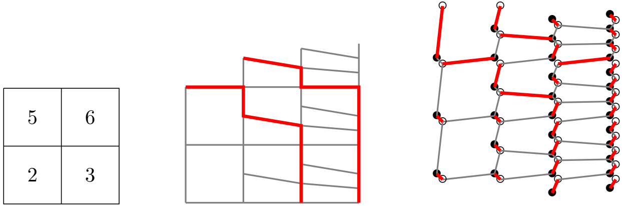

Figure: Tableau, non-intersecting paths, and dimers (Figure by Corteel, Keating and Nicoletti). The left graph represents a lecture hall tableaux L of shape $\lambda = (2,2)$ with $L(1,1) = 5$, $L(1,2) = 6$, $L(2,1) = 3$, $L(2,2) = 3$ and $n = 2$. Then $\frac{L(1,1)}{n+1-1} = \frac{5}{2}$; $\frac{L(2,1)}{n+1-2}$=2;$\frac{L(1,2)}{n+2-1} = 2$;$\frac{L(2,2)}{n+2-2} = \frac{3}{2}$. The lecture hall tableaux is bounded by $t = 3$. The middle graph represents the corresponding non-intersecting path configuration. The right graph represents a dimer configuration on a graph which is not doubly-periodic.

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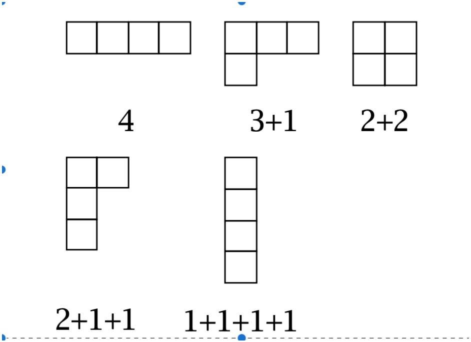

Figure: partitions and corresponding Young diagram

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- An n-lecture hall tableau of shape λ/μ is a tableau L of shape λ/μ satisfying the following conditions

$$\frac{L(i,j)}{n+c(i,j)} \geq \frac{L(i,j+1)}{n+c(i,j+1)}, \qquad \frac{L(i,j)}{n+c(i,j)} > \frac{L(i+1,j)}{n+c(i+1,j)}.$$

where $c(i,j) = j - i$ is the content of the cell (i,j). The set of n-lecture hall tableaux is denoted by $LHT_n(\lambda/\mu)$.

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- If

$$L(i,j) < t(n+j-i)$$

  We say these tableaux are bounded by $t > 0$.

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2. Given a positive integer t and a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, a non-intersecting path configuration is a system of n paths on the graph $\mathcal{G}_t$. For each integer i satisfying $1 \leq i \leq n$, the ith path starts at $\left(n-i, t - \frac{1}{n-i+1}\right)$, ends at $(n-i+\lambda_i, 0)$ and moves only downwards and rightwards. The paths are said to be not intersecting if they do not share a vertex.

Limit Shape

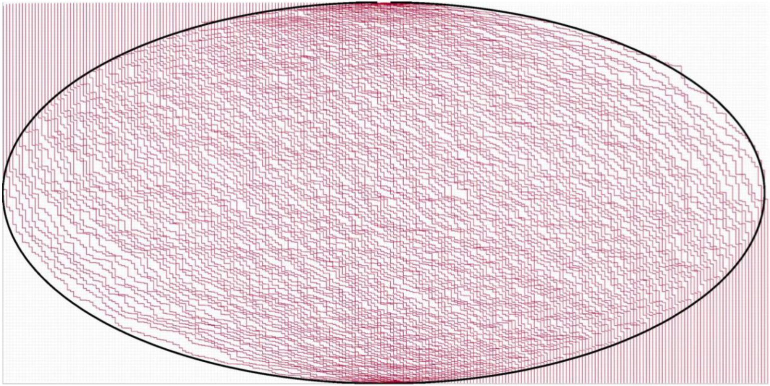

Figure: Limit shape when $\lambda = (n, \ldots, n)$,$n = t = 120$; by Corteel, Keating and Nicoletti

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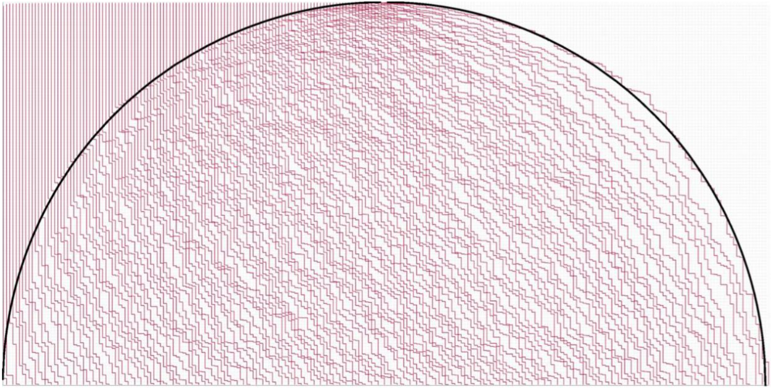

Figure: Limit shape when $\lambda = (n, \ldots, 1), n = t = 120$; by Corteel, Keating and Nicoletti

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- (Corteel and Kim 2019) $L^n_{\lambda/\mu}(\mathbf{x}) \neq |\mathbf{x}|^{|\lambda/\mu|} s_{\lambda/\mu}(1^n)$.

Limit shape and complex Burgers equation: a conjecture by Corteel, Keating and Nicoletti

Conjecture

(Corteel, Keating and Nicoletti, 2019) The function $u(x, y)$ solution of the Burgers equation

$$uu_x + u_y = 0$$

satisfies

$$\Im(u) = \pi \frac{\partial h}{\partial y} \text{ and } \arg(u) = \pi \frac{\partial h}{\partial x} + 1$$

for some branch of the arg.

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- (Pittel and Romik 2004, Sun, W. 2018) The solution complex Burgers equation is also known to be related to the scaling limit of the standard Young tableaux, whose corresponding particle configurations form a Gelfand-Testlin scheme.

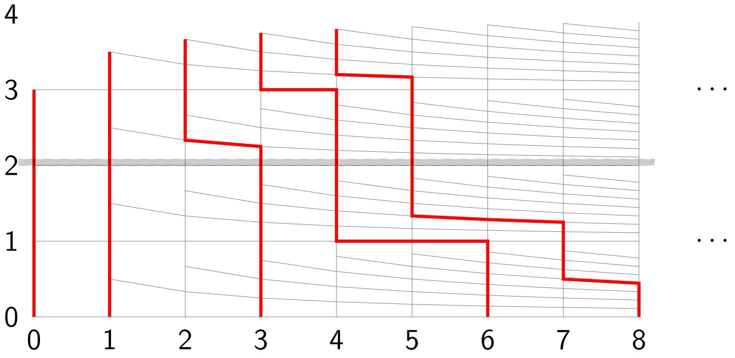
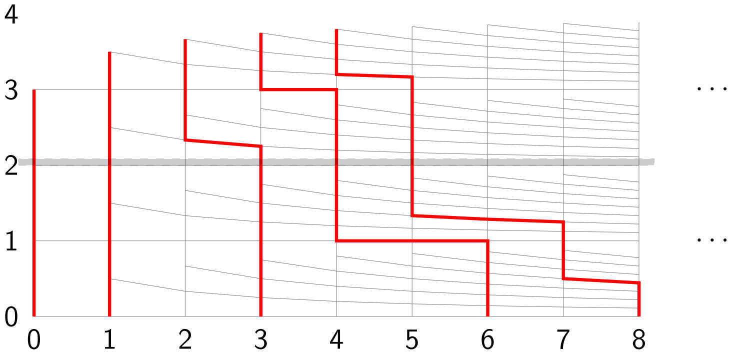


Figure: Non-intersecting lattice paths on $\mathcal{G}_4$ for $n = 5$. We have $\lambda^{(3)} = (1, 0, 0, 0, 0)$, $\lambda^{(2)} = (1, 1, 1, 0, 0)$, $\lambda^{(1)} = (3, 3, 1, 0, 0)$ and $\lambda^{(0)} = (4, 3, 1, 0, 0)$. The sequence of partitions $(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ do not form a Gelfand-Tsetlin scheme.

- the particle configuration does not satisfy the interlacing condition of Gelfand-Tsetlin scheme;



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4. $H_{\mathbf{m}}(u) = \int_0^{\ln u} R_{\mathbf{m}}(t)dt + \ln\left(\frac{\ln u}{u-1}\right)$.

Main Result: Moment of the Limit Measure

Theorem

Let n be the the total number of non-interacting paths in $\mathcal{G}$, and let t be the height of $\mathcal{G}$. Let $\rho_\kappa(n)$ be the probability distribution of $\lambda^{(\kappa)}$. Assume

$$y := \lim_{n\to\infty} \frac{\kappa}{n}; \qquad s := \lim_{n\to\infty} \frac{|\mathbf{x}_\kappa|}{|\mathbf{x}|}; \qquad \alpha := \lim_{n\to\infty} \frac{t}{n}; \tag{1}$$

such that

$$s \in (0,1); \qquad y \in (0,\alpha).$$

Then random measures $\mathbf{m}_{\rho_\kappa(n)}$ converge as $n \to \infty$ in probability, in the sense of moments to a deterministic measure $\mathbf{m}_y$ on $\mathbb{R}$, whose moments are given by

$$\int_{\mathbb{R}} x^j \mathbf{m}_y(dx) = \frac{1}{2(j+1)\pi\mathbf{i}} \oint_1 \frac{dz}{z-1+s} \left((z-1+s)H'_{\mathbf{m}_0}(z) + \frac{z-1+s}{z-1} \right)^{j+1}$$

Here $\mathbf{m}_0$ is the limit counting measure for the boundary partition $\lambda^{(0)} \in \mathbb{Y}_n$ as $n \to \infty$.

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- ρ_κ be the probability distribution of $\lambda^{(\kappa)}$;
- $\mathbf{u} = (u_1, u_2, \ldots, u_n)$; $\mathbf{x} = (x_1, x_2, \ldots, x_t)$; $|\mathbf{x}| = x_1 + x_2 + \ldots + x_t$.

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- Schur generating function: $\mathcal{S}_{\rho_\kappa}(|\mathbf{x}|, \mathbf{u}) = \sum_{\lambda \in \mathbb{Y}} \rho_\kappa(\lambda) \frac{s_\lambda(|\mathbf{x}| + \mathbf{u})}{s_\lambda(|\mathbf{x}|)}$.

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- Analyzing the leading terms,

$$\mathbb{E}\left(\int_{\mathbb{R}} x^p m\left[\lambda_{\rho_\kappa}\right] dx\right) \approx I_p$$

$$\mathbb{E}\left(\int_{\mathbb{R}} x^p m\left[\lambda_{\rho_\kappa}\right] dx\right)^2 \approx I_p^2$$

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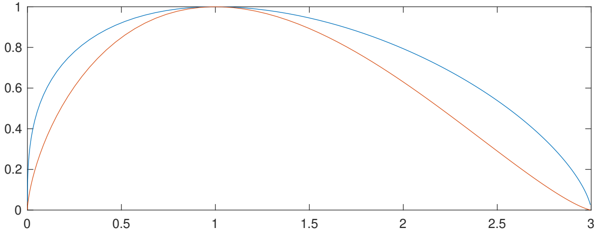

Figure: Frozen boundary for the scaling limit of weighted non-interaction paths. The blue curve is for the uniform weight; the red curve is when the limit weight function s satisfies $y = (1-s)^2$. Boundary partition given by $(2n, 2n-2, \ldots, 2, 0)$

Rescaled Height Function and Complex Burgers Equation

Theorem

(Li, Keating and Prause 2023)Assume $\mathcal{G}$ is uniformly weighted such that $s = 1 - y$. Let

$$u = \frac{1}{\mathbf{z}_+(\chi, y) S_{\mathbf{m}_0}^{(-1)}(\ln \mathbf{z}_+(\chi, y))}$$

Then

$$\frac{\partial h}{\partial x} = \frac{1}{\pi}\left(2 - \mathrm{Arg}(u)\right); \qquad \frac{\partial h}{\partial y} = \frac{1}{\pi}\Im u \tag{2}$$

where $\mathrm{Arg}(\cdot)$ is the branch of the argument function taking values in $[0, 2\pi)$. Moreover, u satisfies the complex Burgers equation

$$u_x - u u_y = 0. \tag{3}$$

Height Fluctuations and GFF

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- Gaussian fluctuation obtained by verifying the Wick's theorem for Gaussian distribution
- The covariance can be expressed as an integral of the Green's function.
- The liquid region is mapped to $\mathbb{H}$ by an explicit homeomorphism; the height fluctuation is the pull-back of GFF in $\mathbb{H}$.

Thank you!