Analogues of two classical Pipedream results on Bumpless Pipedreams

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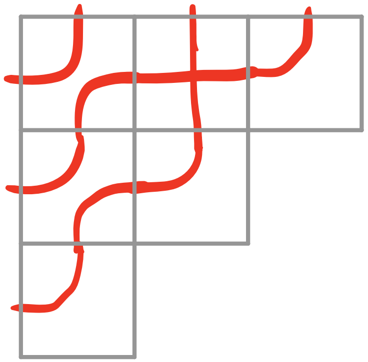
UCSD → UQAM.

Pipedream (PD)



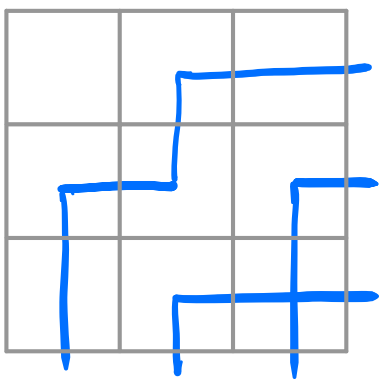
Fomin-Stanley generating function

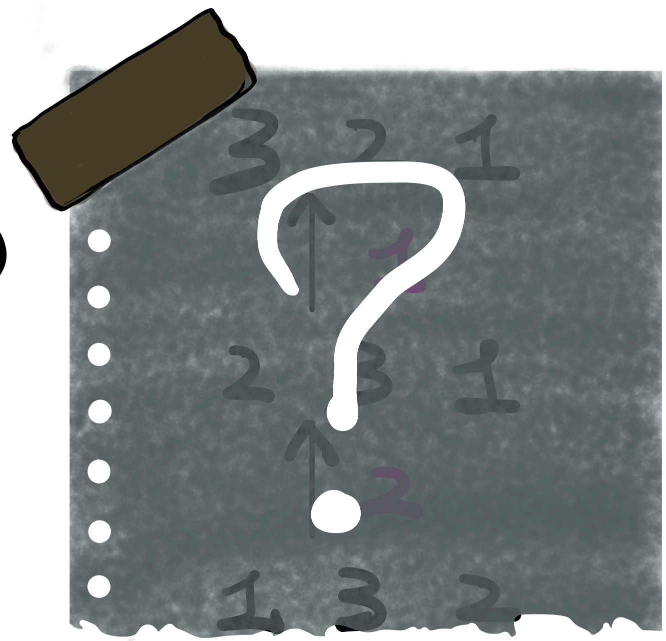
$$(1+x_1u_2)(1+x_1u_1)(1+x_2u_2)$$

Lenart-Sottile Bruhat Chain

$$\begin{array}{c} 3\ 2\ 1 \\ \uparrow {\color{magenta}2} \\ 3\ 1\ 2 \\ \uparrow {\color{magenta}1} \\ 1\ 3\ 2 \end{array}$$

Bumpless Pipedream (BPD)





Schubert Polynomial $\mathfrak{S}_w$.

Polynomials indexed by $w \in S_n$.

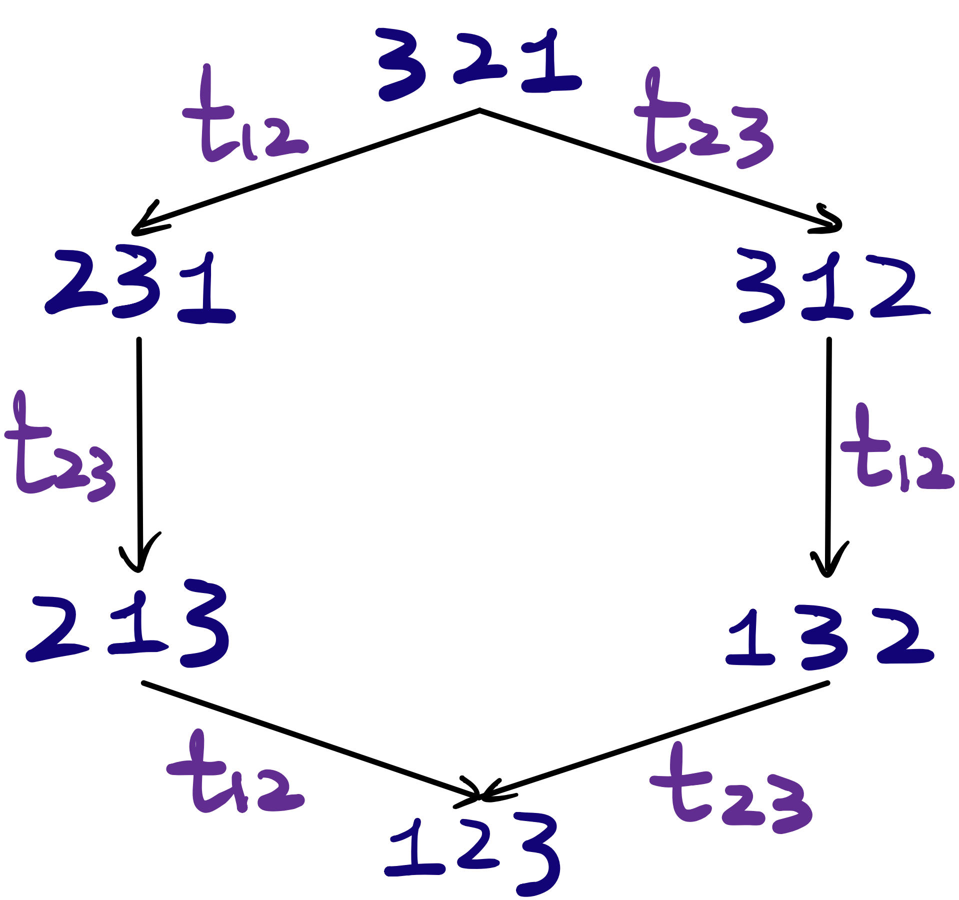
If $w_0 = [n, n-1, \dots, 2, 1]$,

$$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^2 x_{n-1}$$

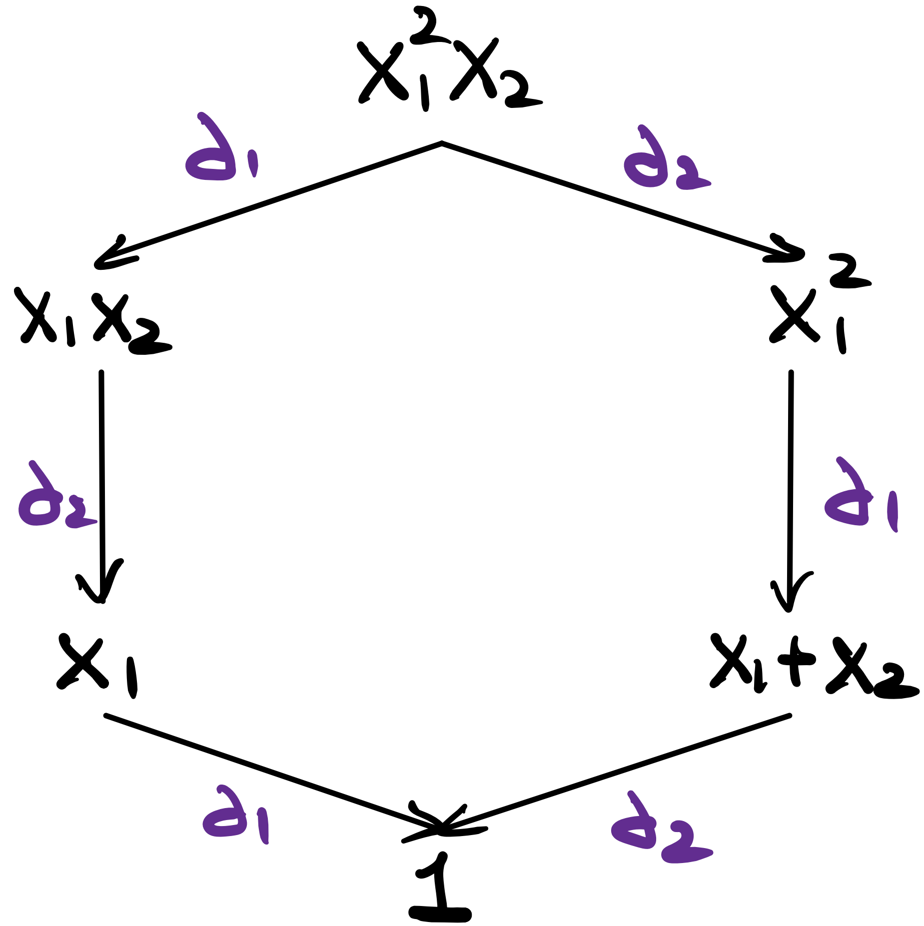
Define ∂_i on polynomials : $\partial_i(f) = \dfrac{f - s_i f}{x_i - x_{i+1}}$

$$\partial_1 (x_1^2 x_2) = \frac{x_1^2 x_2 - x_1 x_2^2}{x_1 - x_2} = x_1 x_2$$

Schubert Polynomial

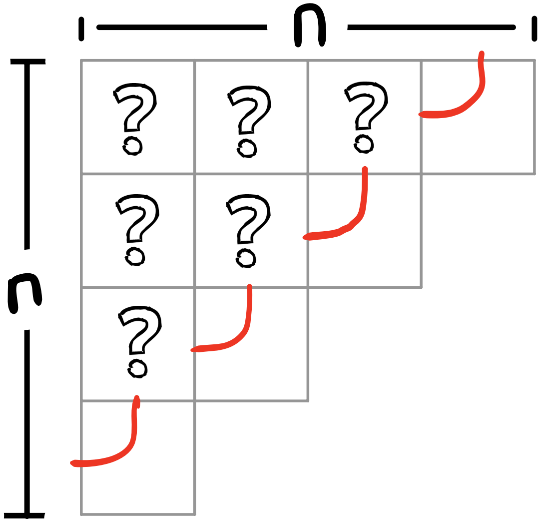


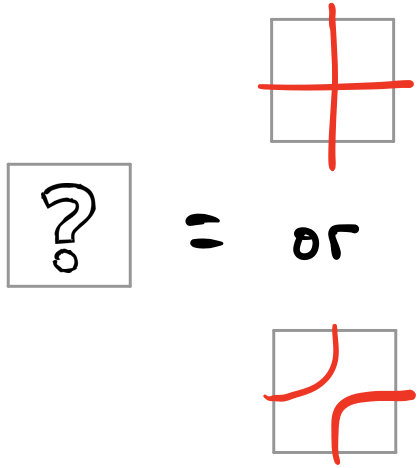
Take $t_{i,i+1}$ down

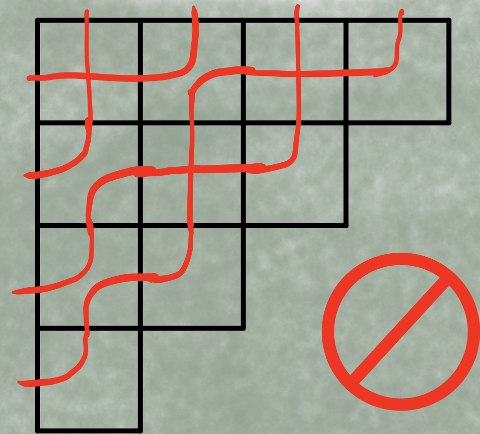


Apply ∂_i

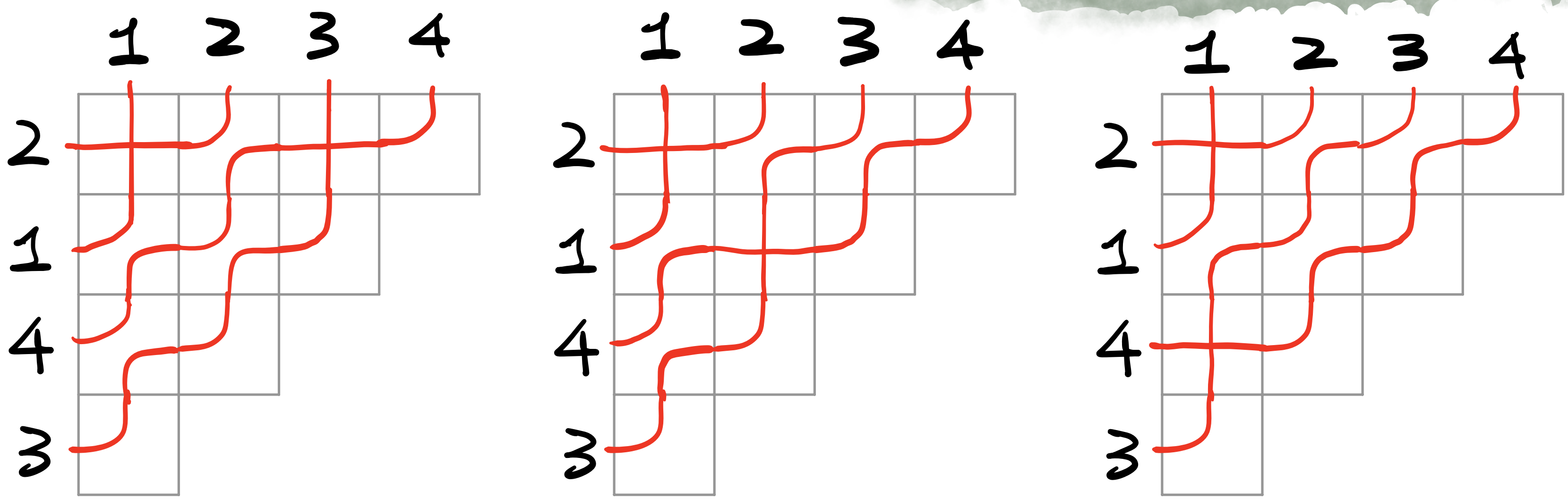
Pipedream (PD) [Bergeron-Billey]





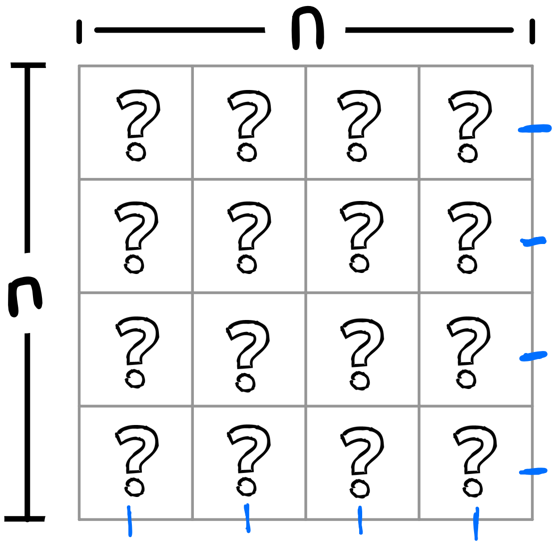


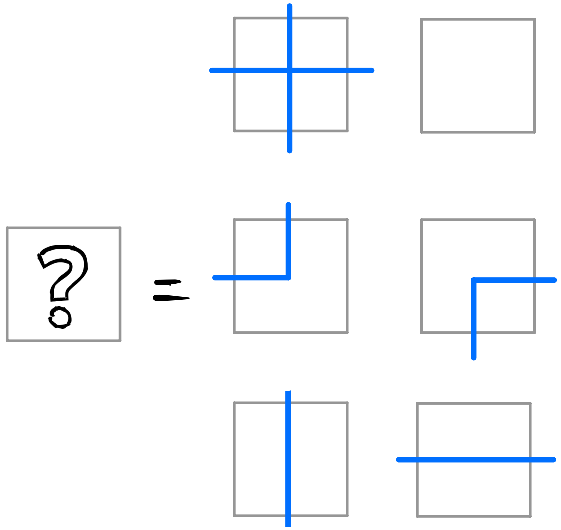
No Double Crossing

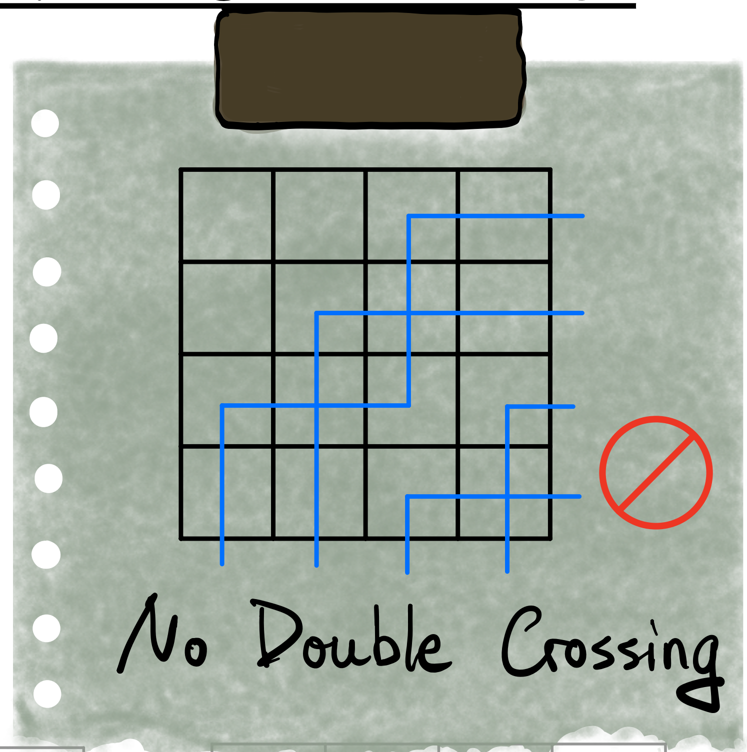


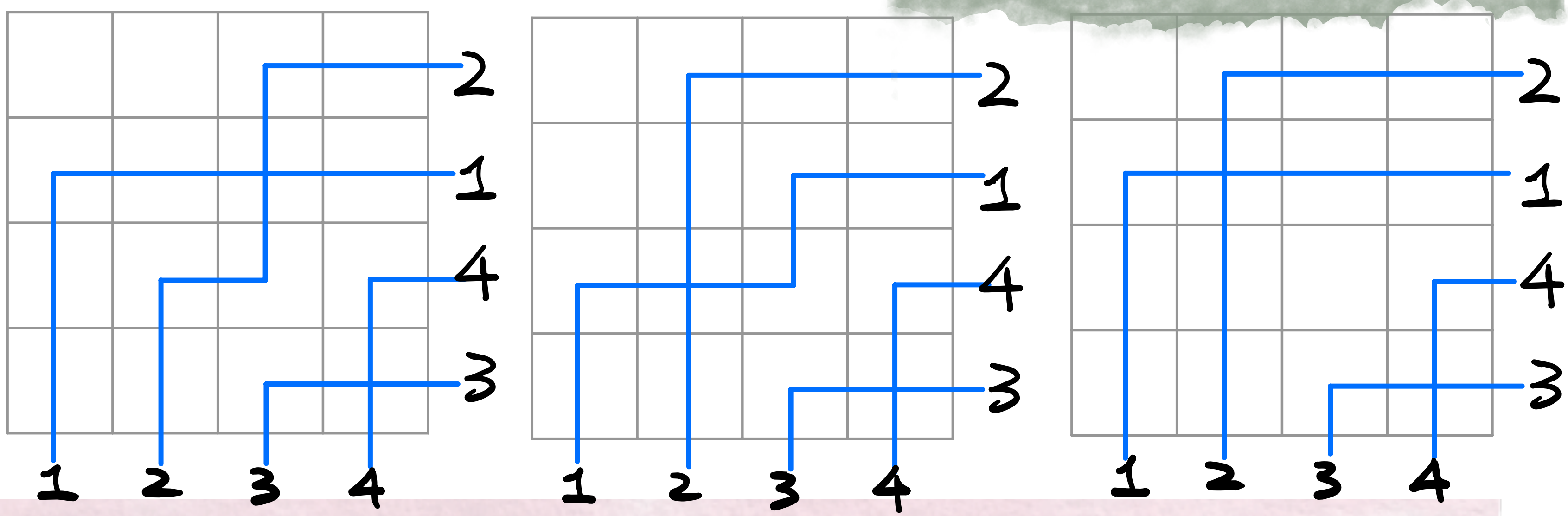
$$G_{2143} = x_1^2 + x_1x_2 + x_1x_3$$

Bumpless Pipedreams (BPD) [Lam-Lee-Shimozono]





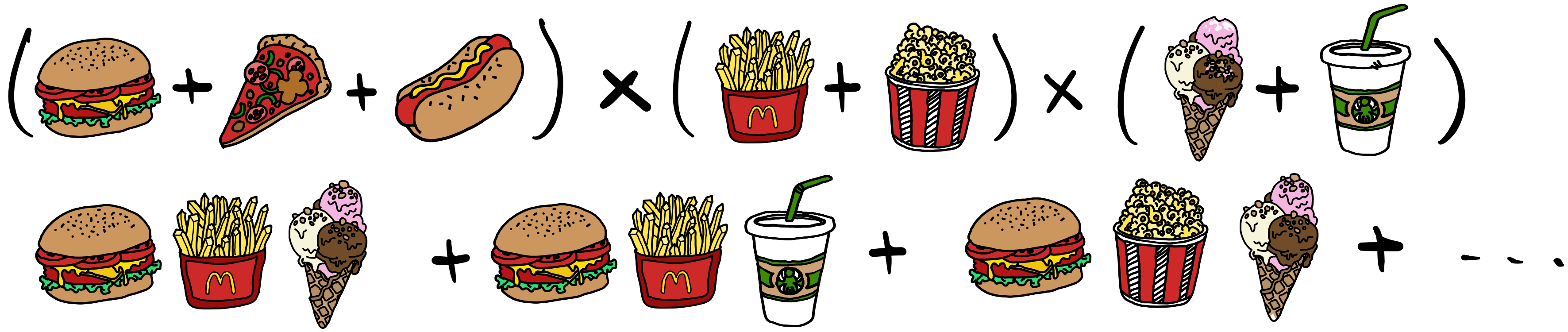


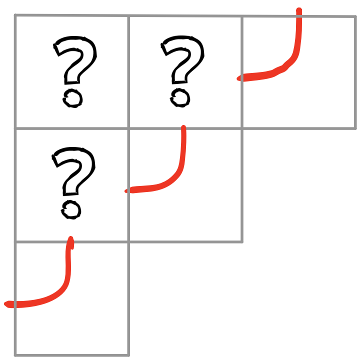


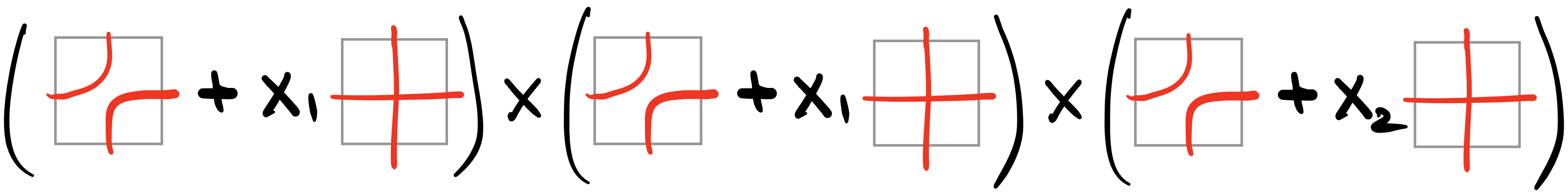
$$G_{2143} = x_1^2 + x_1 x_2 + x_1 x_3$$

Fomin - Stanley

- Use one expression to include all possible pipedreams.







- How to read permutations ?
- How to make sure no double crossings ?

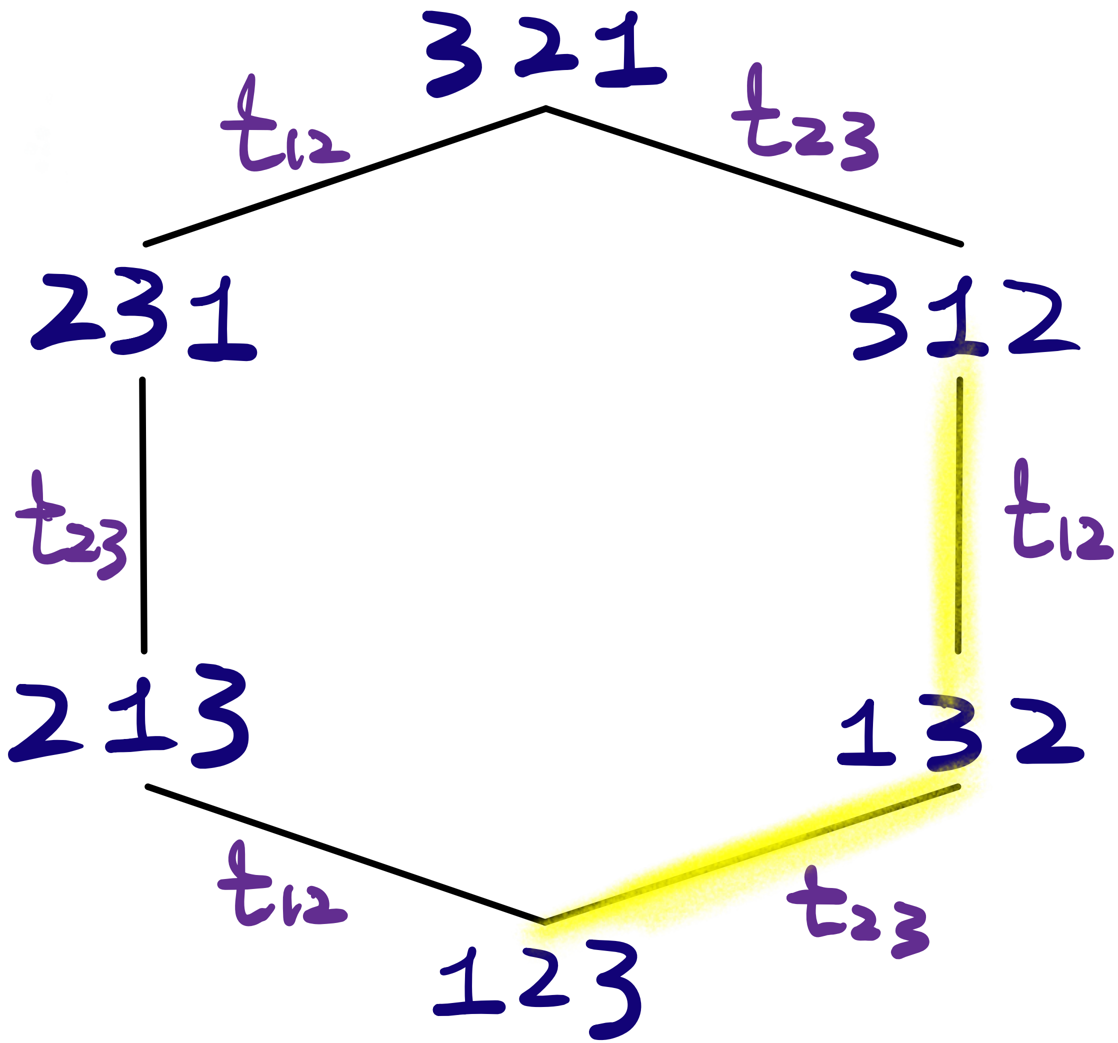
[Food stickers by gabby_scarba11]

Nil-Coxeter Algebra

Generated by $u_1, u_2, \ldots, u_{n-1}$

- $u_i u_i = 0$
- $u_i u_j = u_j u_i$ if $|i-j| > 1$.
- $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$

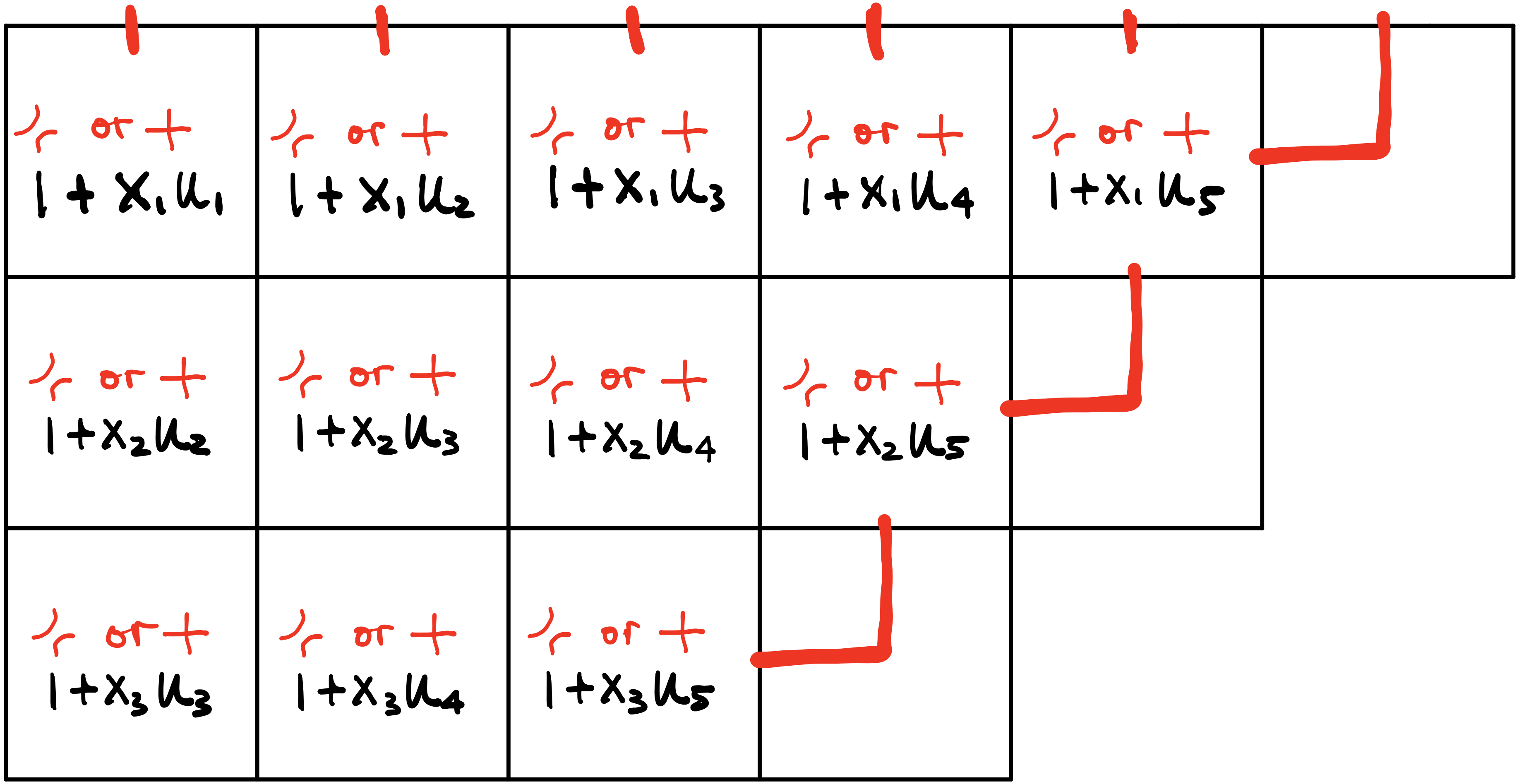
Each monomial is zero or corresponds to some $w \in S_n$



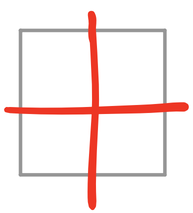
$u_2 u_1 \sim 312$

$u_2 u_2 = 0$

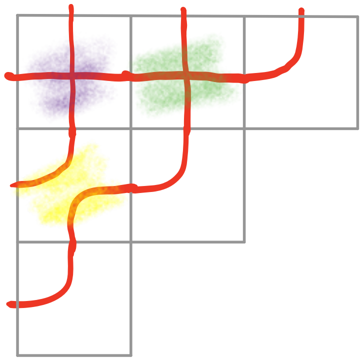
Fomin - Stanley



. . .

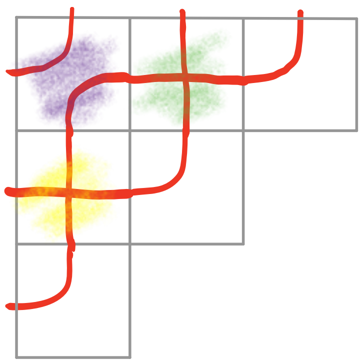
IDEA:  in diagonal j behaves as u_j

Fomin-Stanley



$$(1 + x_1 u_2) \times (1 + x_1 u_1) \times (1 + x_2 u_2)$$

$$x_1^2 u_2 u_1$$



$$(1 + x_1 u_2) \times (1 + x_1 u_1) \times (1 + x_2 u_2)$$

$$x_1 x_2 u_2 u_2 = 0$$

Fomin - Stanley

$$G := (1 + x_1 u_2) \times (1 + x_1 u_1) \times (1 + x_2 u_2)$$

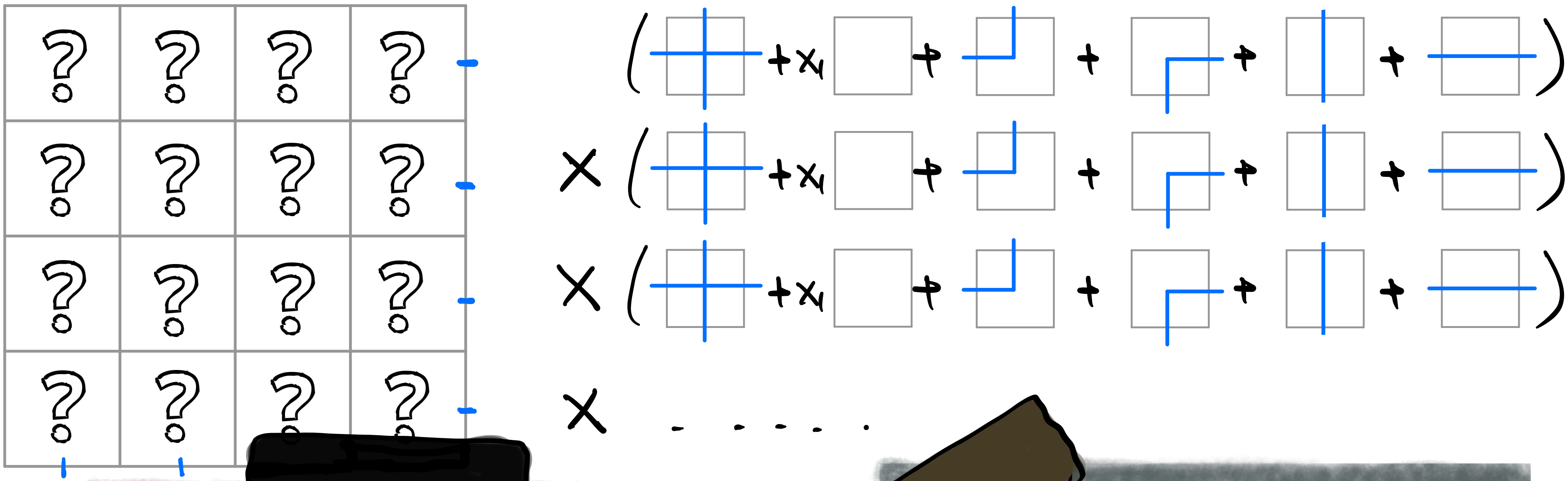
$$= \sum_{\text{Pipedream } P} x^{wt(P)} \, w(P)$$

To prove the PD formula for G_w, it's enough to show:

Thm [Fomin - Stanley]

$$\partial_i (G) = G \, u_i .$$

Fomin - Stanley on BPD ?



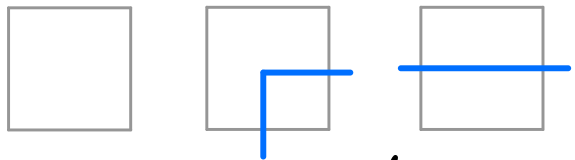
Challenges :

- Too many terms
- Too many "bad" terms
- Hard to read permutation

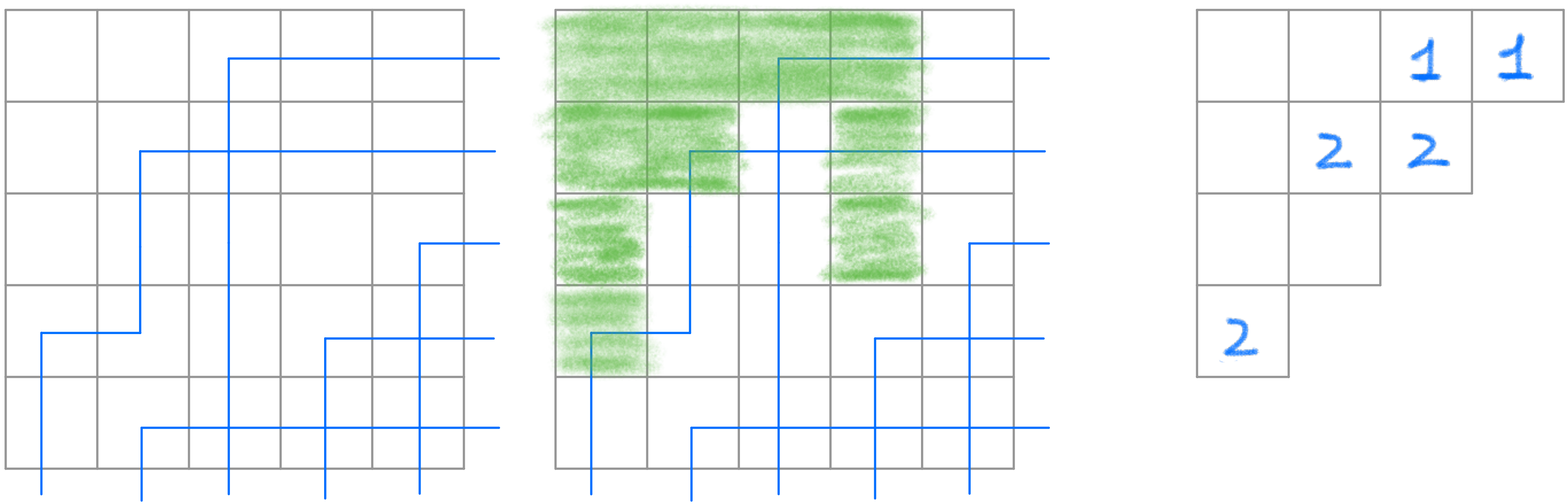
Solution :

Encode BPD in a more compact way!

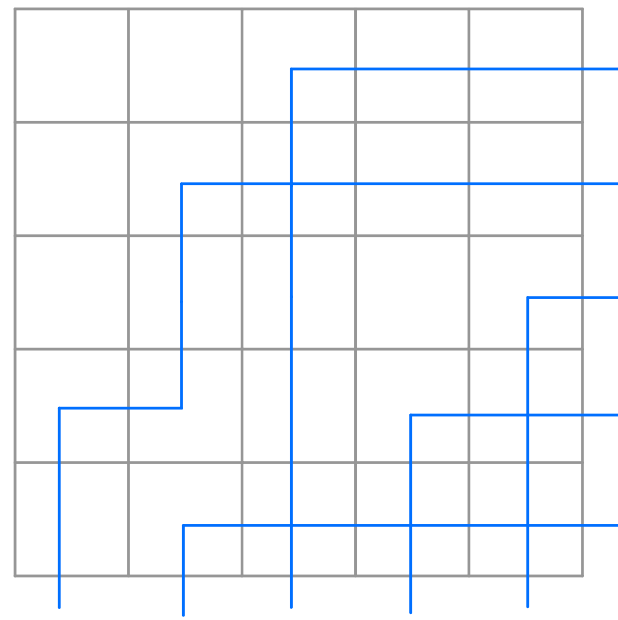
Encoding a BPD

Look at  , but ignore the rightmost such tile in each row.

Record the pipe in each such cell

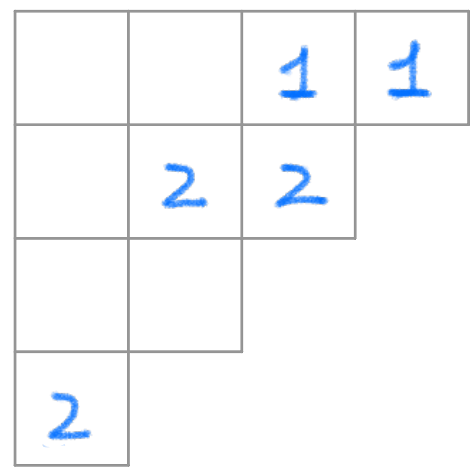


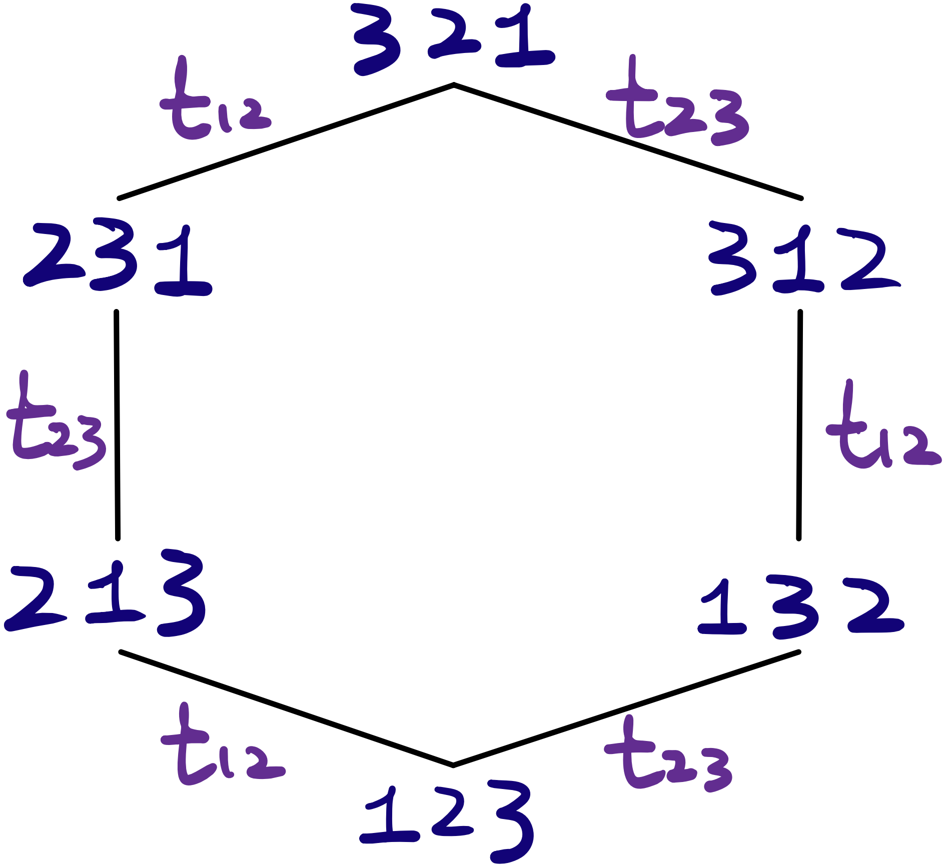
Encoding a BPD



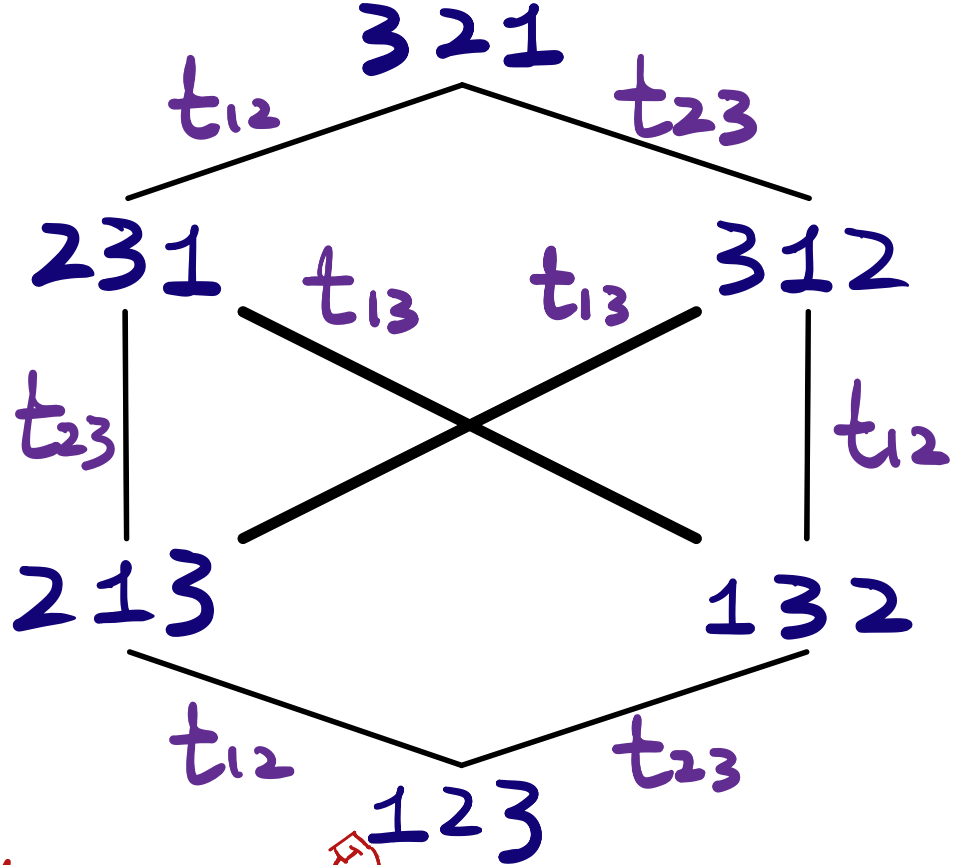
What fillings can appear?

- Row i filled w/ #s $\leq i$
- ? ? ?





weak Bruhat Order

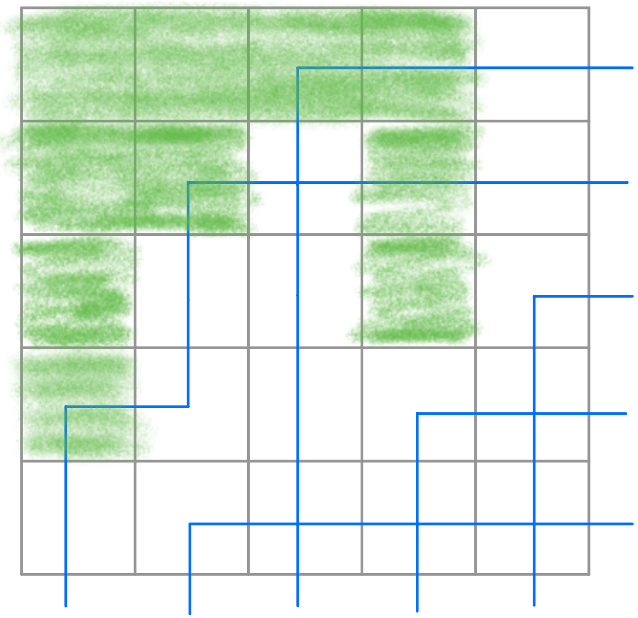


(Strong) Bruhat Order

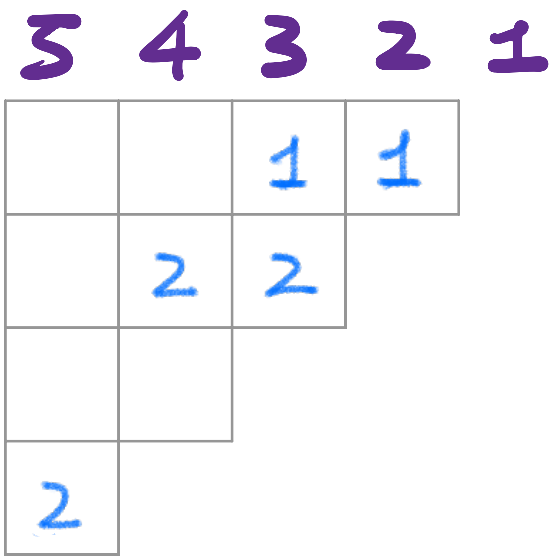
When you make a swap,

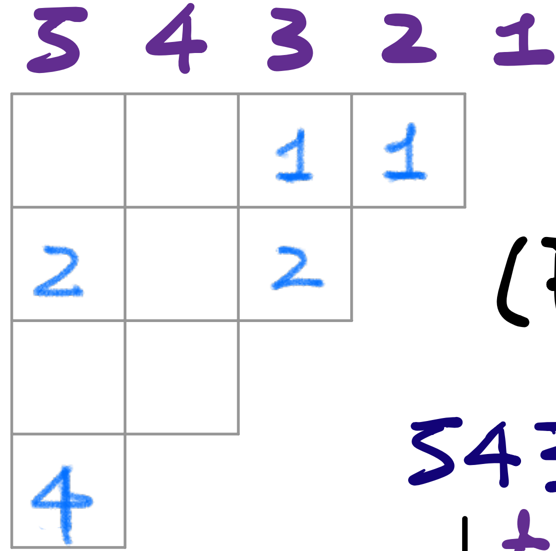
#s in between are not in between.

Chain from encoding



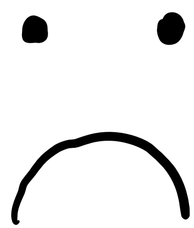
$$54321 \xrightarrow{t_{12}} 45321 \xrightarrow{t_{13}} 35421 \xrightarrow{t_{23}} 34521 \xrightarrow{t_{24}} 32541 \xrightarrow{t_{25}} 31542$$





(BAD)

$$54321 \xrightarrow{t_{12}} 45321 \xrightarrow{t_{13}} 35421 \xrightarrow{t_{23}} 34521 \xrightarrow{\cancel{t_{25}}} 31524$$

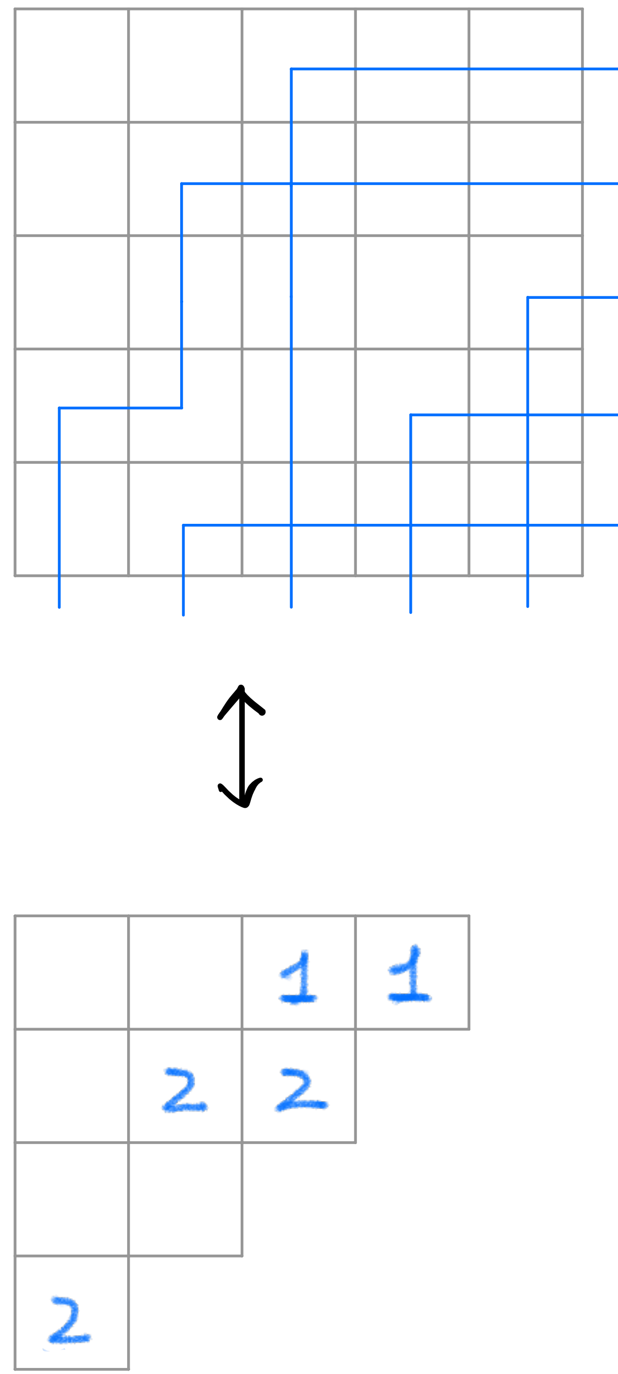


Encoding a BPD

Prop [Y]

This is a bijection between BPD of w and fillings of the "staircase" such that

- Row i filled w/ #s $\leq i$
- Correspond to a chain from w_0 to w in Bruhat order.



Fomin - Kirillov Algebra

$\mathcal{E}_n$ is generated by d_{ij}, $1 \le i < j \le n$.

- $d_{ij} d_{ij} = 0$
- $d_{ij} d_{jk} = d_{ik} d_{ij} + d_{jk} d_{ik}$
- $d_{jk} d_{ij} = d_{ij} d_{ik} + d_{ik} d_{jk}$
- $d_{ij} d_{kl} = d_{kl} d_{ij}$

  if i, j, k, l distinct.

Fomin-Kirillov Algebra

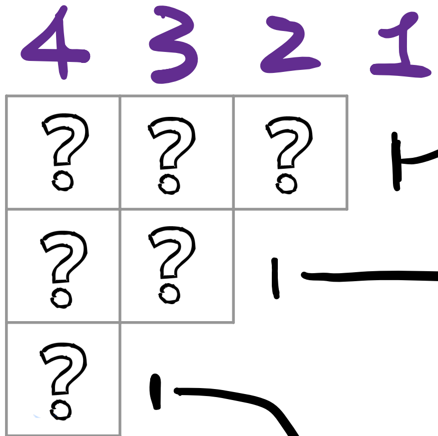
$\mathcal{E}_n$ is generated by d_{ij}, $1 \le i < j \le n$.

[redacted]

$\mathcal{E}_n$ acts on $\mathbb{Q}[S_n]$

$$w \odot d_{ij} = \begin{cases} wt_{ij} & \text{if } w > wt_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

BPD Analogue of Fomin – Stanley



$$(x_1 + \overset{2}{\boxed{1}})(x_1 + \overset{3}{\boxed{1}})(x_1 + \overset{4}{\boxed{1}})$$
$$= (x_1 + d_{12})(x_1 + d_{13})(x_1 + d_{14})$$

$$(x_2 + \overset{3}{\boxed{1}} + \overset{3}{\boxed{2}})(x_2 + \overset{4}{\boxed{1}} + \overset{4}{\boxed{2}})$$
$$= (x_2 + d_{13} + d_{23})(x_2 + d_{14} + d_{24})$$

$$(x_3 + \overset{4}{\boxed{1}} + \overset{4}{\boxed{2}} + \overset{4}{\boxed{3}})$$
$$= (x_3 + d_{14} + d_{24} + d_{34})$$

BPD Analogue of Fomin - Stanley

4	3	2	1
?	?	?	
?	?		
?			

Definition [Y]

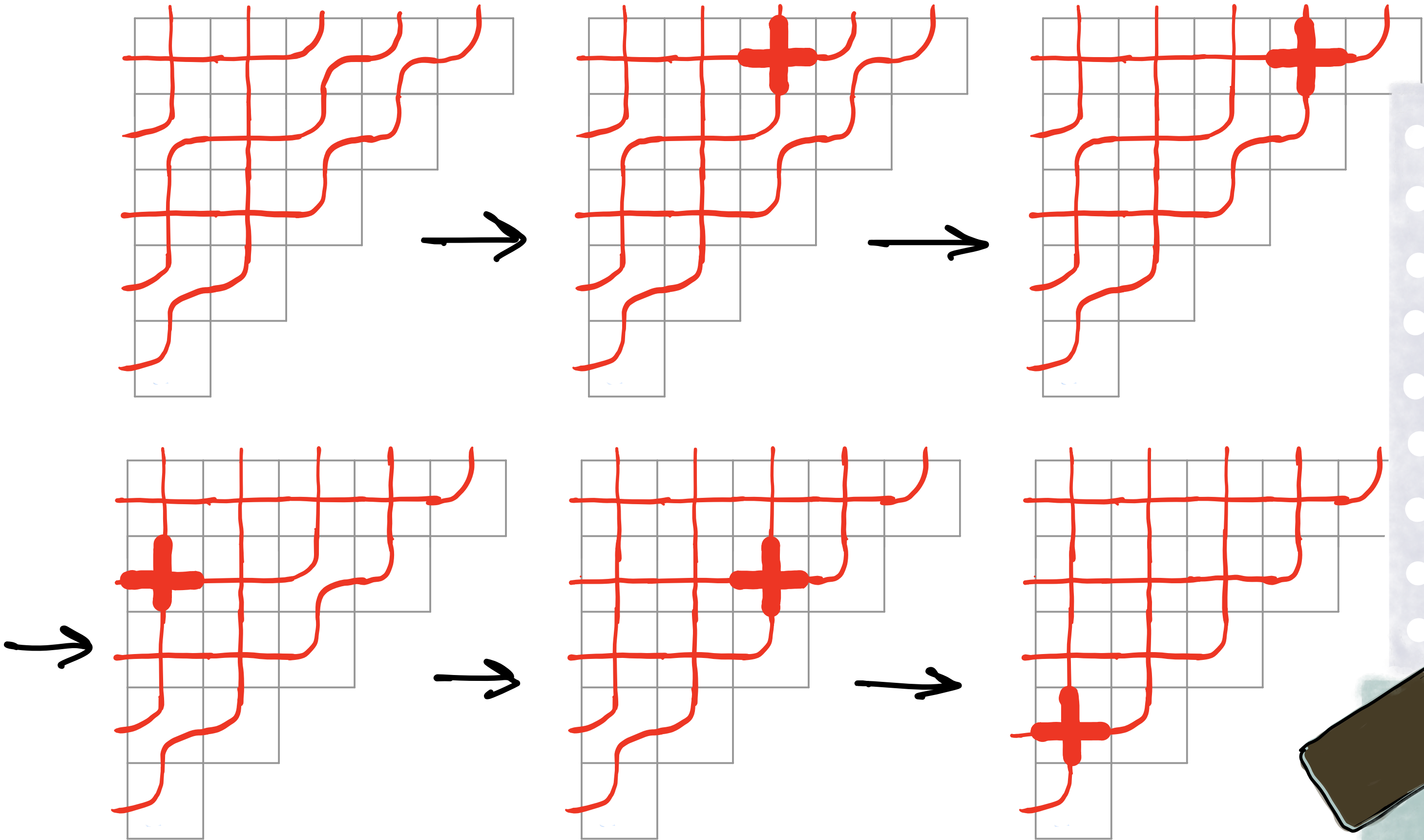
$$\mathfrak{G} = w_0 \odot (x_1+d_{12})(x_1+d_{13})(x_1+d_{14})(x_2+d_{13}+d_{23})(x_2+d_{14}+d_{24})(x_3+d_{14}+d_{24}+d_{34})$$

$$= \sum_{\text{BPD } D} x^{wt(D)} \, w(D)$$

As the PD case, showing the BPD formula for $\mathfrak{G}_w$ reduces to:

Theorem [Y] $d_i(\mathfrak{G}) = \mathfrak{G} \odot u_i$

Lenart - Sottile Chain

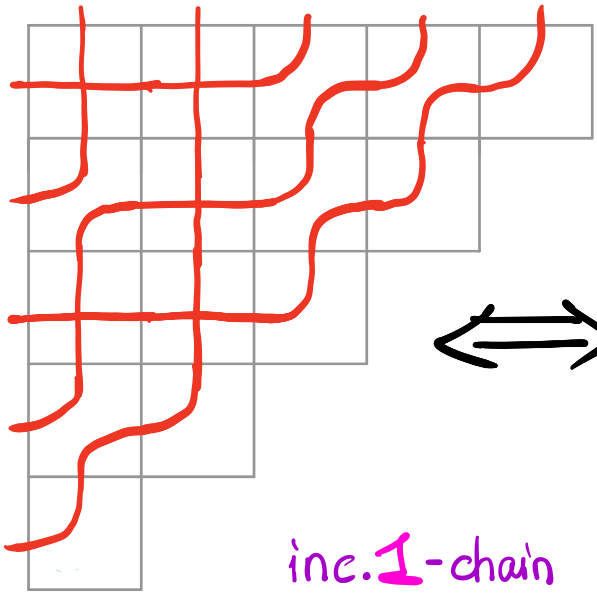


Change into + from top to bottom, left to right on each row.

Labels given by the number of the

$$31542 \xrightarrow{1} 41532 \xrightarrow{1} 51432 \xrightarrow{2} 53412 \xrightarrow{2} 54312 \xrightarrow{4} 54321$$

Lenart - Sottile Chain



$\Longleftrightarrow$

inc. 1-chain

$31542 \xrightarrow{1} 41532 \xrightarrow{1} 51432$

$\xrightarrow{2} 53412 \xrightarrow{2} 54312 \xrightarrow{4} 54321$

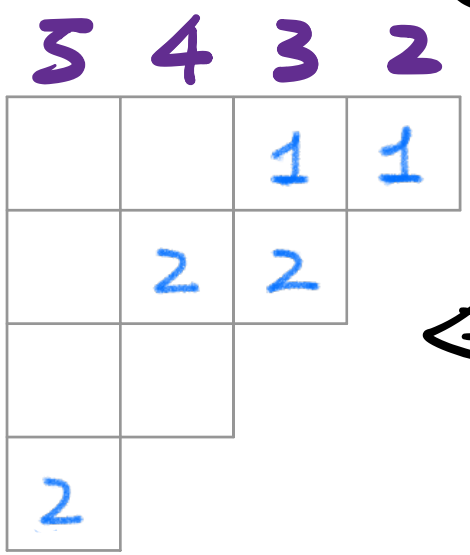
$31542 \xrightarrow{\text{inc. 1-chain}} 51432 \xrightarrow{\text{inc. 2-chain}} 54312 \xrightarrow{\text{inc. 3-chain}} 54312 \xrightarrow{\text{inc. 4-chain}} 54321$

Thm [Lenart – Sottile]

This is a bijection between PD(w) and

$w \xrightarrow{\text{inc. 1-chain}} \bullet \xrightarrow{\text{inc. 2-chain}} \cdots \xrightarrow{\text{inc. } n\text{-}2\text{-chain}} \bullet \xrightarrow{\text{inc. } n\text{-}1\text{-chain}} w_0$

BPD analogue of Lenart-Sottile



$\Longleftrightarrow$

$$31542 \xrightarrow{4} 32541 \xrightarrow{2} 34521$$
$$\xrightarrow{2} 35421 \xrightarrow{1} 45321 \xrightarrow{1} 54321$$

$$31542 \xrightarrow{\text{inc. } 4\text{-chain}} 32541 \xrightarrow{\text{inc. } 3\text{-chain}} 32541 \xrightarrow{\text{inc. } 2\text{-chain}} 35421 \xrightarrow{\text{inc. } 1\text{-chain}} 54321$$

Thm [Y]

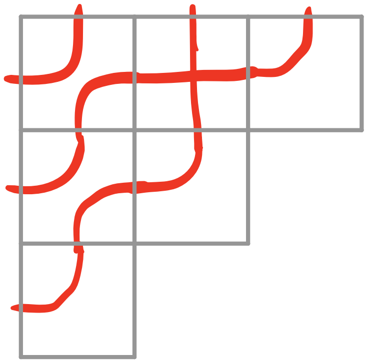
This is a bijection between BPD(w) and

$$w \xrightarrow{\text{inc. } n-1\text{-chain}} \bullet \xrightarrow{\text{inc. } n-2\text{-chain}} \cdots \xrightarrow{\text{inc. } 2\text{-chain}} \bullet \xrightarrow{\text{inc. } 1\text{-chain}} w_0$$

Pipedream (PD)

Fomin-Stanley Algebra

Lenart-Sottile Chain



$$(1+x_1u_2)(1+x_1u_1)(1+x_2u_2)$$

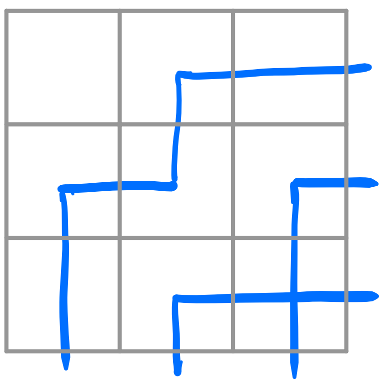
$$321 \xleftarrow{2} 312 \xleftarrow{1} 132$$

Bijections

[Gao-Huang]

[Knutson-Udell]

Bumpless Pipedream (BPD)



$$(x_1+d_{12})(x_1+d_{13})(x_2+d_{13}+d_{23})$$

$$321 \xleftarrow{1} 231 \xleftarrow{2} 132$$

New bijection between PDs and BPDs

$$\text{BPD} \overset{[Y]}{\Longrightarrow} w \xrightarrow{4} \cdot \xrightarrow{3} \cdot \xrightarrow{2} \cdot \xrightarrow{1} w_0$$

Lenart's Growth Diagram

$$\overset{[\text{Lenart}, Y]}{\Longrightarrow} w \xrightarrow{4} \cdot \xrightarrow{3} \cdot \xrightarrow{1} \cdot \xrightarrow{2} w_0$$

$$\overset{[\text{Lenart}, Y]}{\Longrightarrow} w \xrightarrow{1} \cdot \xrightarrow{4} \cdot \xrightarrow{3} \cdot \xrightarrow{2} w_0$$

$$\overset{[\text{Lenart}, Y]}{\Longrightarrow} w \xrightarrow{1} \cdot \xrightarrow{2} \cdot \xrightarrow{4} \cdot \xrightarrow{3} w_0$$

$$\overset{[\text{Lenart}, Y]}{\Longrightarrow} w \xrightarrow{1} \cdot \xrightarrow{2} \cdot \xrightarrow{3} \cdot \xrightarrow{4} w_0$$

$$\overset{[\text{Lenart-Sottile}]}{\Longrightarrow} \text{PD}.$$

Is this the same as Gao-Huang bijections?

Future Direction

PD: $w \xrightarrow[\text{inc.}]{1\text{-chain}} \xrightarrow[\text{inc.}]{2\text{-chain}} \xrightarrow[\text{inc.}]{3\text{-chain}} w_0$

Many interesting stories on the two ends!

- Crystal graphs
- Monk's move
- Double Schubert

......

How to generalize to the other chains?

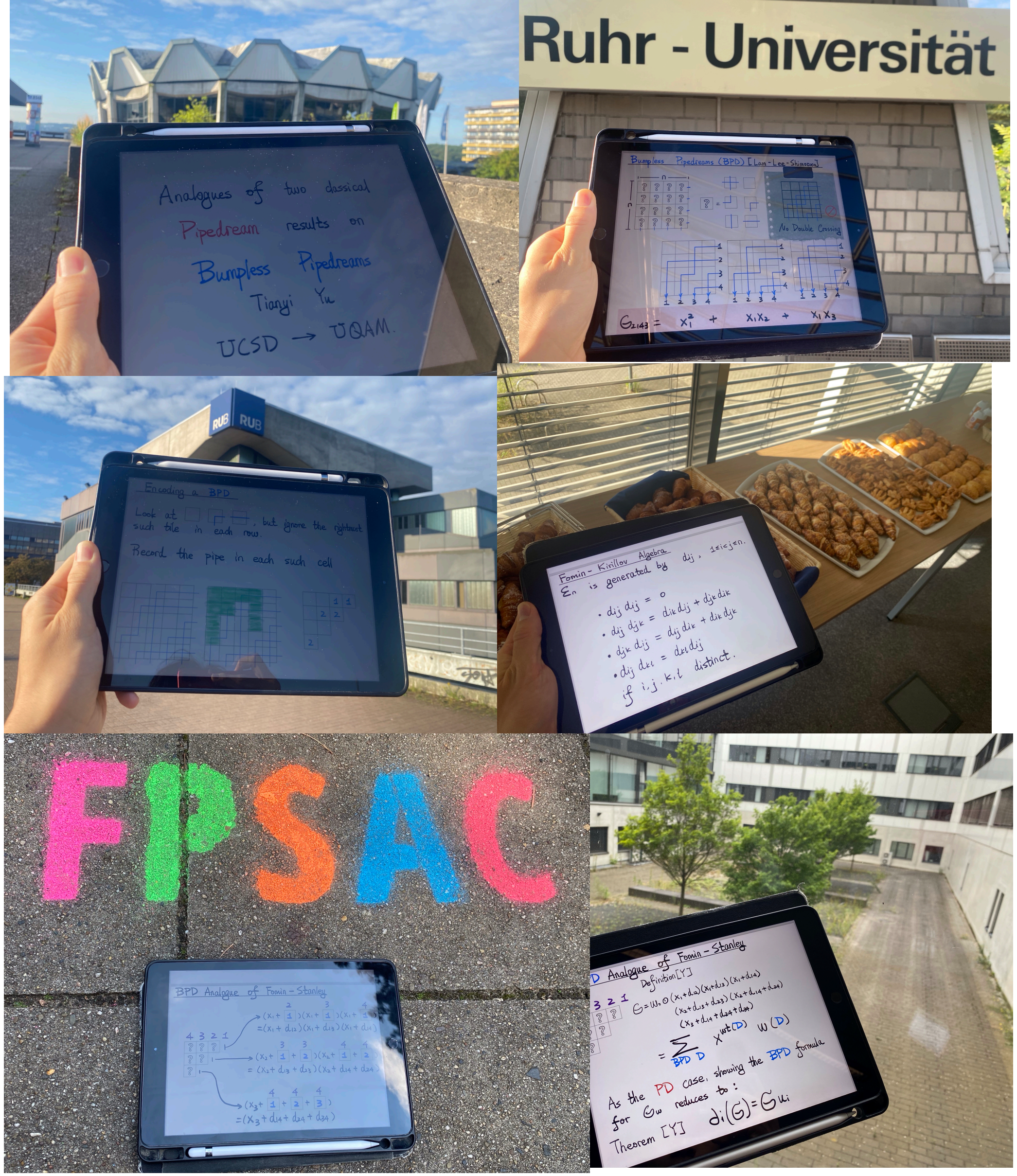
$w \xrightarrow[\text{inc.}]{1\text{-chain}} \xrightarrow[\text{inc.}]{3\text{-chain}} \xrightarrow[\text{inc.}]{2\text{-chain}} w_0$

$w \xrightarrow[\text{inc.}]{2\text{-chain}} \xrightarrow[\text{inc.}]{1\text{-chain}} \xrightarrow[\text{inc.}]{3\text{-chain}} w_0$

$w \xrightarrow[\text{inc.}]{2\text{-chain}} \xrightarrow[\text{inc.}]{3\text{-chain}} \xrightarrow[\text{inc.}]{1\text{-chain}} w_0$

$w \xrightarrow[\text{inc.}]{3\text{-chain}} \xrightarrow[\text{inc.}]{1\text{-chain}} \xrightarrow[\text{inc.}]{2\text{-chain}} w_0$

BPD: $w \xrightarrow[\text{inc.}]{3\text{-chain}} \xrightarrow[\text{inc.}]{2\text{-chain}} \xrightarrow[\text{inc.}]{1\text{-chain}} w_0$



Thank Yibo Gao for telling me this problem.

Thank Yibo Gao and Zachary Hamaker for valuable guidances.

Thank you for listening!