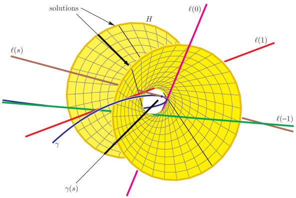
Positivity in real Schubert calculus

Slides available at `snkarp.github.io`



F. Sottile, "Frontiers of reality in Schubert calculus"

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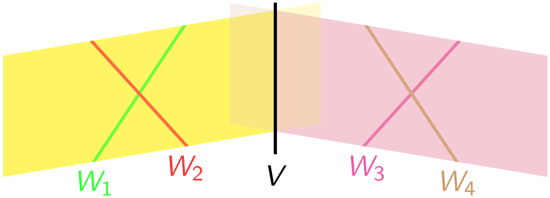
Schubert calculus (1886)

- Divisor Schubert problem: given subspaces $W_1, \ldots, W_{d(m-d)} \subseteq \mathbb{C}^m$ of dimension $m-d$, find all

$$d\text{-subspaces } V \subseteq \mathbb{C}^m \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- e.g. $d=2$, $m=4$ (projectivized). Given 4 lines $W_i \subseteq \mathbb{CP}^3$, find all lines $V \subseteq \mathbb{CP}^3$ intersecting all 4. Generically, there are 2 solutions.



1	2
3	4

1	3
2	4

We can see the 2 solutions explicitly when two pairs of the lines intersect.

- If the W_i's are generic, the number of solutions V is $f^{\square}$, the number of *standard Young tableaux* of rectangular shape $d \times (m-d)$.
- Fulton (1984): "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems."

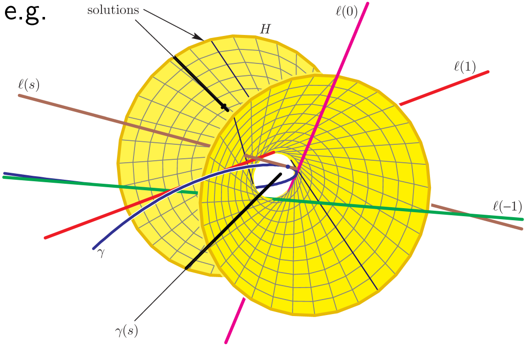
Shapiro–Shapiro conjecture

Shapiro–Shapiro conjecture (1993)

Let $W_1, \ldots, W_{d(m-d)} \subseteq \mathbb{R}^m$ be $(m-d)$-subspaces osculating the moment curve $\gamma(t) := \left(\frac{t^{m-1}}{(m-1)!}, \frac{t^{m-2}}{(m-2)!}, \ldots, t, 1\right)$ at real points. Then there exist $f^{\square}$

$$d\text{-subspaces } V \subseteq \mathbb{R}^m \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- e.g.



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- This Schubert problem arises in the study of linear series in algebraic geometry, differential equations, and pole placement problems in control theory.
- Bürgisser, Lerario (2020): a 'random' Schubert problem over $\mathbb{R}$ has $\approx \sqrt{f^{\square}}$ real solutions.

Shapiro–Shapiro conjecture and secant conjecture

- Sottile (1999) tested the conjecture and proved it asymptotically.
- Eremenko, Gabrielov (2002): cases $d \le 2$, $m - d \le 2$.
- Mukhin, Tarasov, Varchenko (2009): full conjecture via the *Bethe ansatz*.
- Levinson, Purbhoo (2021): topological proof of the full conjecture.

Secant conjecture, divisor form (Sottile (2003))

Let $W_1, \ldots, W_{d(m-d)} \subseteq \mathbb{R}^m$ be $(m-d)$-subspaces secant to the moment curve γ along non-overlapping intervals. Then there exist $f^{\square}$ real

$$d\text{-subspaces } V \subseteq \mathbb{R}^m \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- The Shapiro–Shapiro conjecture is a limiting case of this conjecture.
- Eremenko, Gabrielov, Shapiro, Vainshtein (2006): case $m - d \le 2$.

Theorem (Karp, Purbhoo (2023))

The divisor form of the secant conjecture is true.

Positive Shapiro–Shapiro conjecture

- The (projective) *Plücker coordinates* of a d-subspace $V \subseteq \mathbb{C}^m$ are the $d \times d$ minors $\Delta_J(V)$ of a $d \times m$ matrix whose rows form a basis of V.
- e.g. $V = \text{rowspan}\left(\begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix}\right) \rightsquigarrow$ $\begin{matrix} \Delta_{1,2}(V) = 1 & \Delta_{2,3}(V) = 4 \\ \Delta_{1,3}(V) = 3 & \Delta_{2,4}(V) = 3 \\ \Delta_{1,4}(V) = 2 & \Delta_{3,4}(V) = 1 \end{matrix}$

Positivity conjecture (Mukhin, Tarasov (2017); Karp (2021))

Let $W_1, \dots, W_{d(m-d)} \subseteq \mathbb{R}^m$ be $(m-d)$-subspaces osculating the moment curve $\gamma(t)$ at real points $t_1, \dots, t_{d(m-d)} > 0$. Then there exist f$^{\square}$ *d-subspaces $V \subseteq \mathbb{R}^m$ with all $\Delta_J(V) > 0$ such that $V \cap W_i \neq \{0\}$ for all i.*

- Karp (2023): the positivity conjecture is equivalent to a conjecture of Eremenko (2015), which implies the divisor form of the secant conjecture.
- Karp, Purbhoo (2023): the positivity conjecture is true. To prove it, we explicitly solve for all $\Delta_J(V)$ over $\mathbb{C}[\mathfrak{S}_{d(m-d)}]$.

Universal Plücker coordinates

- Shapiro–Shapiro problem: given $(m-d)$-subspaces $W_1, \ldots, W_{d(m-d)} \subseteq \mathbb{C}^m$ osculating $\gamma(t)$ at points $t_1, \ldots, t_{d(m-d)} \in \mathbb{C}$, find all

$$d\text{-subspaces } V \subseteq \mathbb{C}^m \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

Theorem (Karp, Purbhoo (2023))

There exist commuting linear operators $\beta_J = \beta_J(t_1, \ldots, t_{d(m-d)})$ indexed by d-subsets $J \subseteq \{1, \ldots, m\}$, satisfying:

(i) There is a bijection between the eigenspaces of the β_J's and the solutions V above, sending the eigenvalue of β_J to the $\Delta_J(V)$.

(ii) If $t_1, \ldots, t_{d(m-d)} > 0$, then the β_J's are positive definite.

$$\beta_J := \sum_{\substack{X \subseteq \{1,\ldots,d(m-d)\},\\ |X| = |\lambda(J)|}} \Big(\prod_{i \notin X} t_i\Big) \sum_{\pi \in \mathfrak{S}_X} \chi^{\lambda(J)}(\pi)\pi \in \mathbb{C}[\mathfrak{S}_{d(m-d)}].$$

- We show the β_J's satisfy the *Plücker relations* using the *KP hierarchy*.

Future directions

- Further explore the connection to the KP hierarchy.
- Find necessary and sufficient inequalities on the Plücker coordinates of V for all complex zeros of $\mathrm{Wr}(V)$ to be nonpositive, generalizing the Aissen–Schoenberg–Whitney theorem in the case $\dim(V) = 1$. (The positivity conjecture implies that the inequalities $\Delta_J(V) \geq 0$ are necessary.)
- Address generalizations and variations of the Shapiro–Shapiro conjecture: the discriminant conjecture, the general form of the secant conjecture, the monotone conjecture, the total reality conjecture for convex curves, ...

Thank you!