Real matroid Schubert varieties, zonotopes, and virtual Weyl groups

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Matroid Schubert varieties

Let V be a finite-dimensional $\mathbb{C}$-vector space, E a finite set, and $\mathcal{A} = \{\alpha_e\}_{e \in E}$ a spanning set for V^*.

Definition

The *matroid Schubert variety* $Y_{\mathcal{A}}(\mathbb{C})$ is the closure of V in $(\mathbb{P}^1(\mathbb{C}))^E = (\mathbb{C} \cup \{\infty\})^E$ under the embedding $v \mapsto (\alpha_e(v))_{e \in E}$.

Goal

Understand the topology of $Y_{\mathcal{A}}(\mathbb{C})$ in terms of the matroid combinatorics of $\mathcal{A}$.

- compute topological invariants of $Y_{\mathcal{A}}(\mathbb{C})$
- use complicated algebro-geometric machinery to deduce/motivate combinatorial results
  - Dowling–Wilson top-heavy conjecture, matroidal Kazhdan–Lusztig theory...

Matroid Schubert varieties over $\mathbb{R}$

Let V be a finite-dimensional $\mathbb{R}$-vector space, E a finite set, and $\mathcal{A} = \{\alpha_e\}_{e \in E}$ a spanning set for V^*.

Definition

The *matroid Schubert variety* $Y_{\mathcal{A}}(\mathbb{R})$ is the closure of V in $(\mathbb{P}^1(\mathbb{R}))^E = (\mathbb{R} \cup \{\infty\})^E$ under the embedding $v \mapsto (\alpha_e(v))_{e \in E}$.

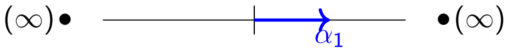
Goal

Understand the topology of $Y_{\mathcal{A}}(\mathbb{R})$ in terms of the oriented matroid combinatorics of $\mathcal{A}$.

- compute topological invariants of $Y_{\mathcal{A}}(\mathbb{R})$
- use methods of combinatorial topology to deduce/motivate combinatorial results
  - ???

Visualising matroid Schubert varieties

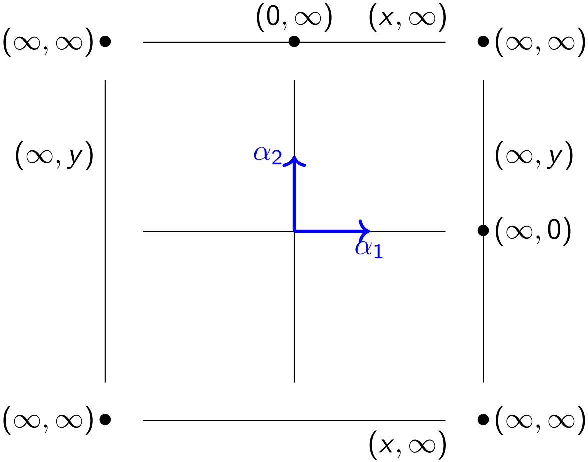
$V \cong V^* \cong \mathbb{R}$, $\mathcal{A} = \{\alpha_1 = (1)\}$, $Y_{\mathcal{A}}(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{R})$



$$Y_{\mathcal{A}}(\mathbb{R}) = \mathbb{P}^1(\mathbb{R}) \cong S^1$$

Visualising matroid Schubert varieties

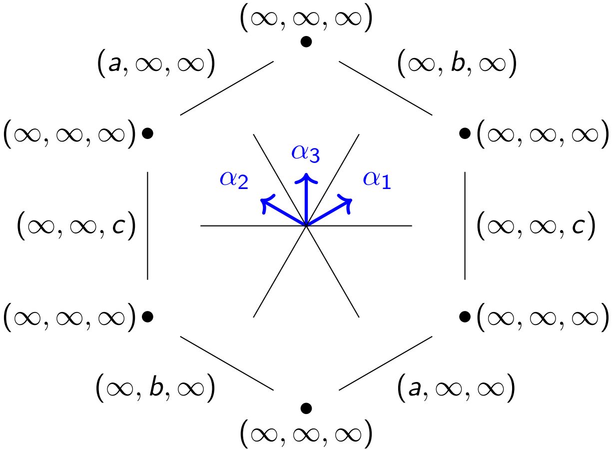
$V \cong V^* \cong \mathbb{R}^2$, $\mathcal{A} = \{\alpha_1 = (1,0), \alpha_2 = (0,1)\}$, $Y_{\mathcal{A}}(\mathbb{R}) \subset (\mathbb{P}^1(\mathbb{R}))^2$



$$Y_{\mathcal{A}}(\mathbb{R}) = \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \cong S^1 \times S^1$$

Visualising matroid Schubert varieties

$$V \cong V^* \cong \mathbb{R}^2,\ \mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\},\ Y_{\mathcal{A}}(\mathbb{R}) \subset (\mathbb{P}^1(\mathbb{R}))^3$$



$$Y_{\mathcal{A}}(\mathbb{R}) = \{(a, b, c) \in (\mathbb{P}^1)^3 \colon a + b = c\}$$

A combinatorial model for $Y_{\mathcal{A}}(\mathbb{R})$

Theorem (J.–Li)

$Y_{\mathcal{A}}(\mathbb{R})$ is homeomorphic to the zonotope $Z_{\mathcal{A}} = \sum_{e \in E} [-1, 1]\alpha_e$ with parallel faces identified.

Previous work

Bartholdi–Enriquez–Etingof–Rains: considered $Z_{\mathcal{A}}/\sim$ for $Z_{\mathcal{A}} =$ the (type A) permutohedron

Ilin–Kamnitzer–Li–Przytycki–Rybnikov: $Y_{\mathcal{A}}(\mathbb{R}) \cong Z_{\mathcal{A}}/\sim$ for $\mathcal{A} =$ Weyl arrangement

Example

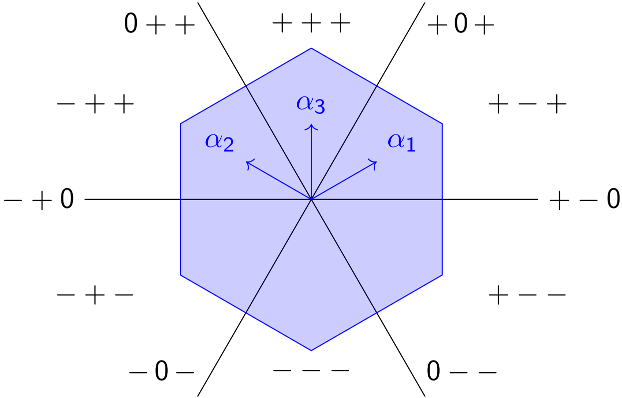
If $\dim V = \dim V^* = 2$, then $Z_{\mathcal{A}}$ is a $2n$-gon and $Y_{\mathcal{A}}(\mathbb{R})$ is homeomorphic to either Σ_g (if $n = 2g$ is even) or Σ_g with two points identified (if $n = 2g + 1$ is odd).

A combinatorial model for $Y_{\mathcal{A}}(\mathbb{R})$

The faces of $Z_{\mathcal{A}}$ are in bijection with covectors of $\mathcal{A}$:

$$C \in \{+,-,0\}^E \leftrightarrow F_C = \sum_{C(e)=+} \alpha_e - \sum_{C(e)=-} \alpha_e + \sum_{C(e)=0} [-1,1]\alpha_e.$$

Identify F_C, F_D (by translation) if $C^{-1}(0) = D^{-1}(0)$.



A combinatorial model for $Y_{\mathcal{A}}(\mathbb{R})$

Proof sketch.

$$
\begin{array}{ccc}
(\mathbb{P}^1(\mathbb{R}))^E & \xrightarrow{\;\cong\;} & [-1,1]^E/\sim \\
\big\uparrow & & \big\downarrow \times \\
Y_{\mathcal{A}}(\mathbb{R}) & \overset{???}{\dashrightarrow} & Z_{\mathcal{A}}/\sim
\end{array}
$$

A combinatorial model for $Y_{\mathcal{A}}(\mathbb{R})$

Proof sketch.

$$\begin{array}{ccccccc}
(\mathbb{P}^1(\mathbb{R}))^E & \hookleftarrow & \mathbb{R}^E & \xrightarrow{\cong} & (-1,1)^E & \subset & [-1,1]^E/\sim \\
\uparrow & & \uparrow & \circlearrowright & \downarrow & & \downarrow \\
Y_{\mathcal{A}}(\mathbb{R}) & \hookleftarrow & V & \longrightarrow & Z^{\circ}_{\mathcal{A}} & \hookrightarrow & Z_{\mathcal{A}}/\sim
\end{array}$$

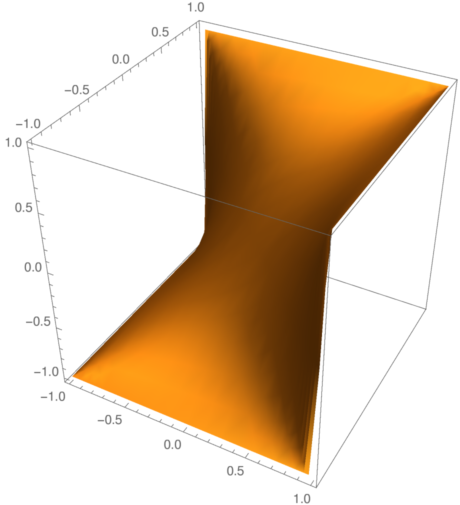
Check this extends to a well-defined bijection $Y_{\mathcal{A}}(\mathbb{R}) \to Z_{\mathcal{A}}/\sim$. □

A combinatorial model for $Y_{\mathcal{A}}(\mathbb{R})$

Example

$\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\} \subset V^* \cong \mathbb{R}^2$

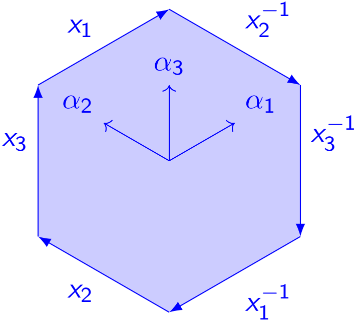
$V \hookrightarrow \mathbb{R}^3 \cong (-1, 1)^3$



Fundamental groups

Corollary

$\pi_1(Y_{\mathcal{A}}(\mathbb{R}))$ has a presentation with generators indexed by rank 1 flats and relations indexed by rank 2 flats.



$$\pi_1(Y_{A_2}(\mathbb{R})) \cong \langle x_1, x_2, x_3 \mid x_2 x_3 x_1 x_2^{-1} x_3^{-1} x_1^{-1} = 1 \rangle$$

Virtual Weyl groups

Definition

The *virtual braid group* VB_n is the free product $S_n * B_n$ modulo "mixed relations" $s_i s_{i+1} \sigma_i = \sigma_{i+1} s_i s_{i+1}$, $s_i \sigma_j = \sigma_j s_i$ if $|i-j| > 1$. The *virtual symmetric group* is $VS_n = VB_n / \langle \sigma_i^2 = 1 \rangle$.

Example (BEER, IKLPR)

$\pi_1(Y_{A_{n-1}}(\mathbb{R})) = PVS_n = \ker(VS_n \xrightarrow{\sigma_i, s_i \mapsto s_i} S_n)$ and $\pi_1^{S_n}(Y_{A_{n-1}}(\mathbb{R})) = VS_n$.

Theorem (J.–Li)

When $\mathcal{A}$ is a Weyl arrangement,

$$\pi_1(Y_{\mathcal{A}}(\mathbb{R})) \cong PVW \text{ and } \pi_1^W(Y_{\mathcal{A}}(\mathbb{R})) \cong VW,$$

where VW is the quotient of Bellingeri–Paris–Thiel's virtual Artin group $VA = W * A/\sim$ by $\sigma_i^2 = 1$.