Triangular (q,t)-Schröder polynomials & Khovanov-Rozansky homology

arxiv: 2407.18123

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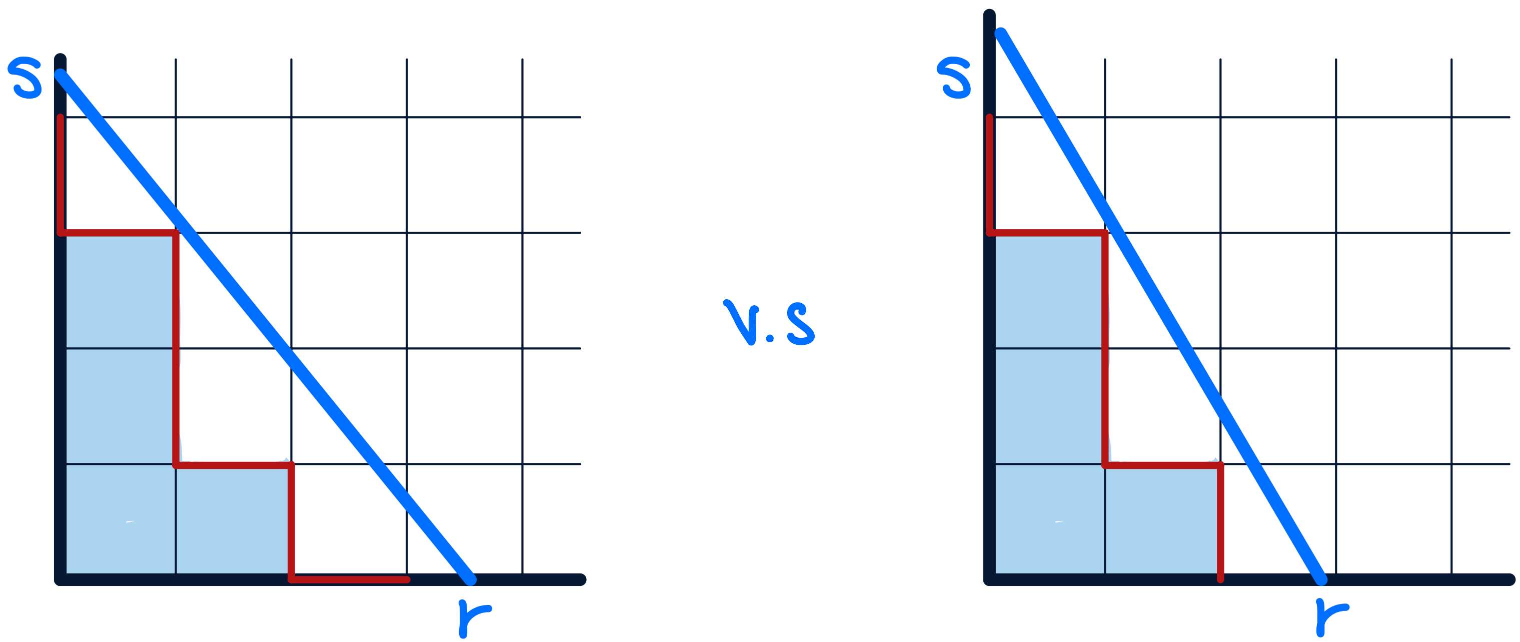
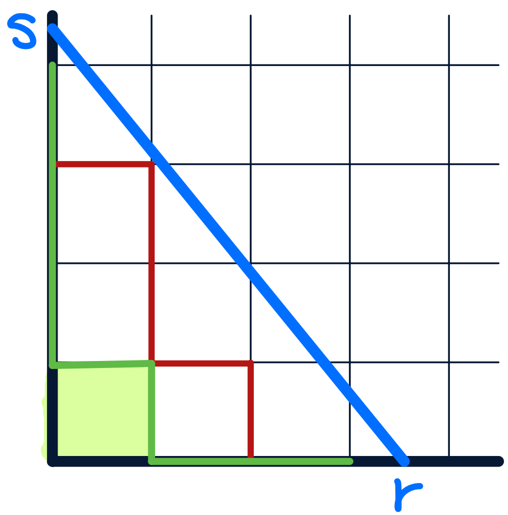
Triangular Partitions & Dyck Paths

A partition τ is **triangular** if $\exists$ a line $L_{r,s} = \{ry + sx = sr\}$ s.t. τ consists of all the boxes beneath $L_{r,s}$. $(r, s \in \mathbb{R}_{>0})$

If so, we write $\tau = \tau_{r,s}$.



An (r,s)-Dyck path is any subpartition $\lambda \leq \tau_{r,s}$.



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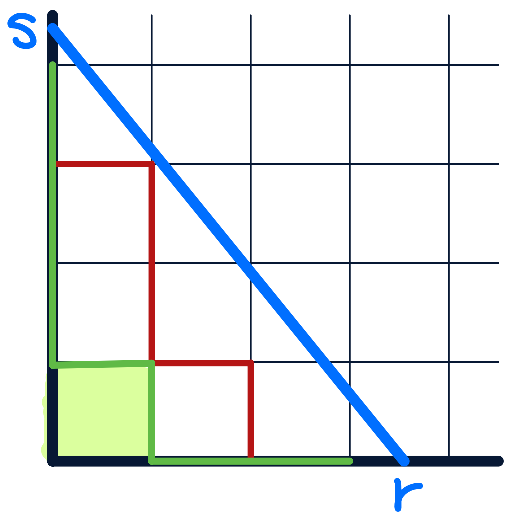

When $n \in \mathbb{Z}_{>0}$:

The (q,t)-Catalan polynomial is:

$$C_n(q,t) = \sum_{\lambda \leq \tau_{n,n+1}} q^{\mathrm{area}(\lambda)} t^{\mathrm{dinv}(\lambda)}$$

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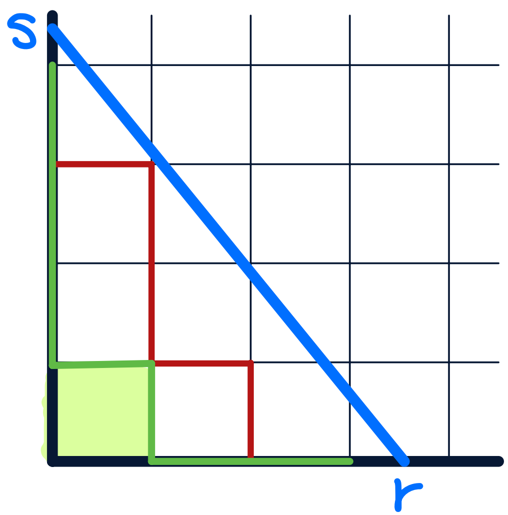

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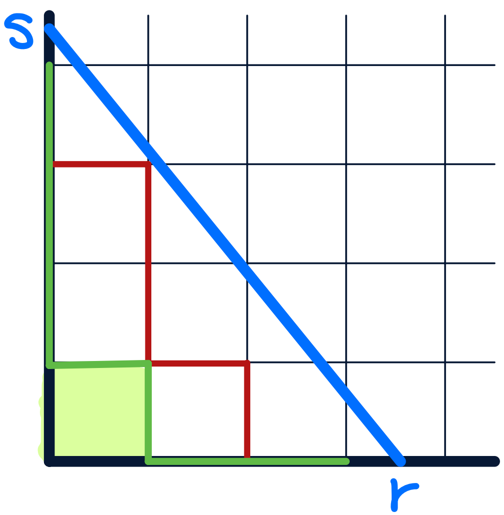

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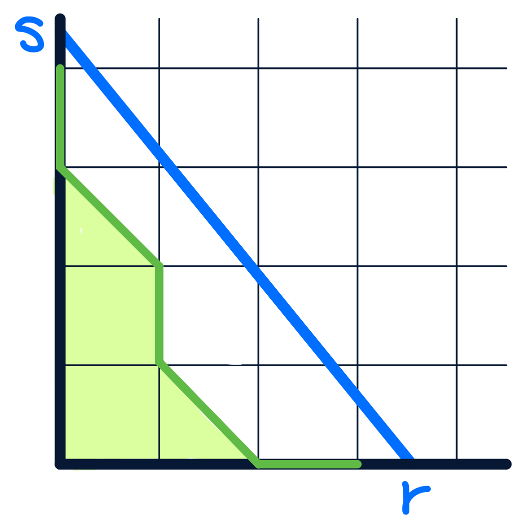
$$S_{n,k}(q,t) = \sum q^{\mathrm{area}(\pi)} t^{\mathrm{bounce}(\pi)}$$

↑ # diag. steps



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Invariant Subsets

Let $m,n > 0$ coprime integers.

A 0-normalized (m,n)-invariant subset is a subset $\Delta \subseteq \mathbb{Z}_{\geq 0}$ s.t. $\Delta + n \subseteq \Delta$ & $\Delta + m \subseteq \Delta$.

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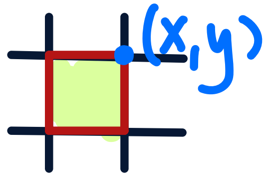
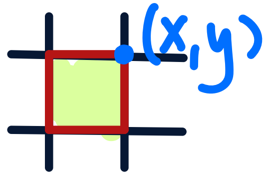
Fact: (Gorsky-Mazin-Vazirani) $D(m,n) \longleftrightarrow \mathrm{Inv}^{0}_{m,n}$

(m,n)-Dyck paths — 0-normalized inv. subsets.

Identify each cell 

by its NE corner (x, y)

Identify each cell 

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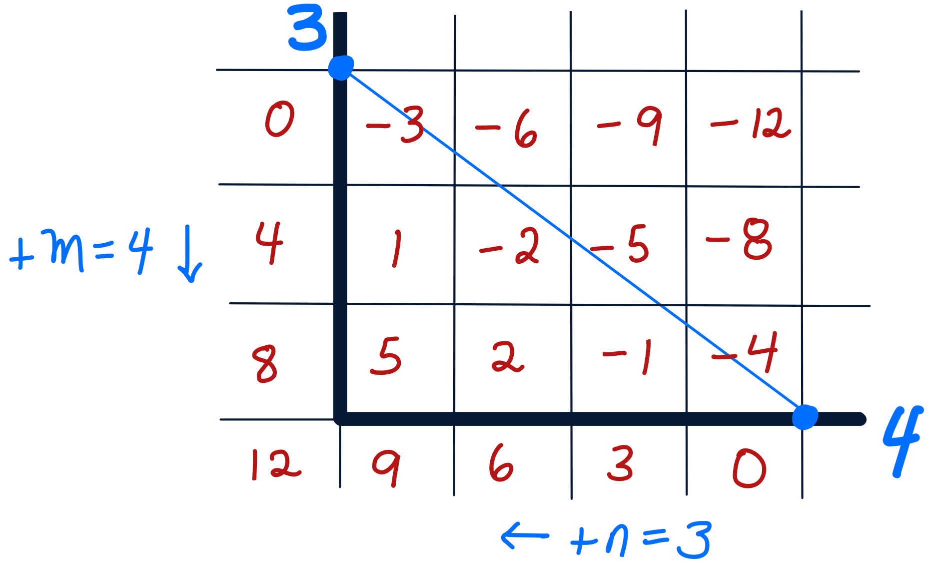
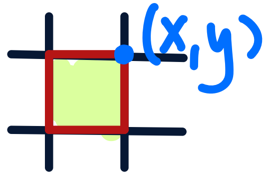
The Anderson labelling:

$$\square^{(x,y)} \longmapsto \gamma(x,y) = mn - my - nx$$



Identify each cell  by its NE corner (x,y)

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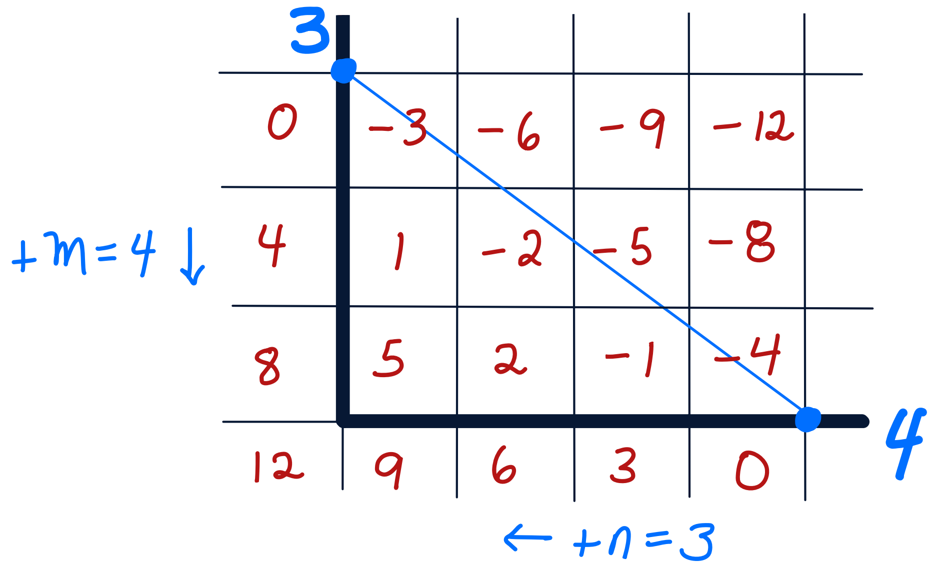
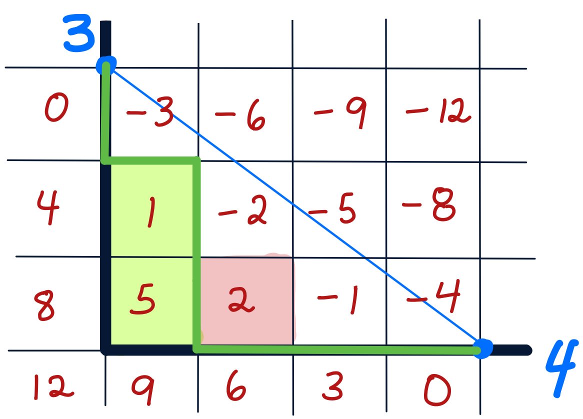


For $\lambda \in D(m,n)$:

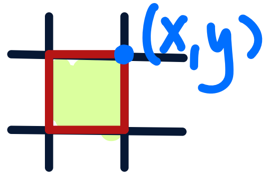
$$\mathrm{Gaps}(\lambda) = \{\gamma(\square) \mid \square \in \tau_{m,n} \setminus \lambda\}$$



$\tau_{3,4} = (2,1)$

$\lambda = (1,1)$

$\mathrm{Gaps}(\lambda) = \{2\}$.

Identify each cell 

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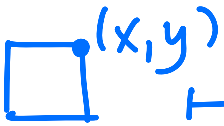

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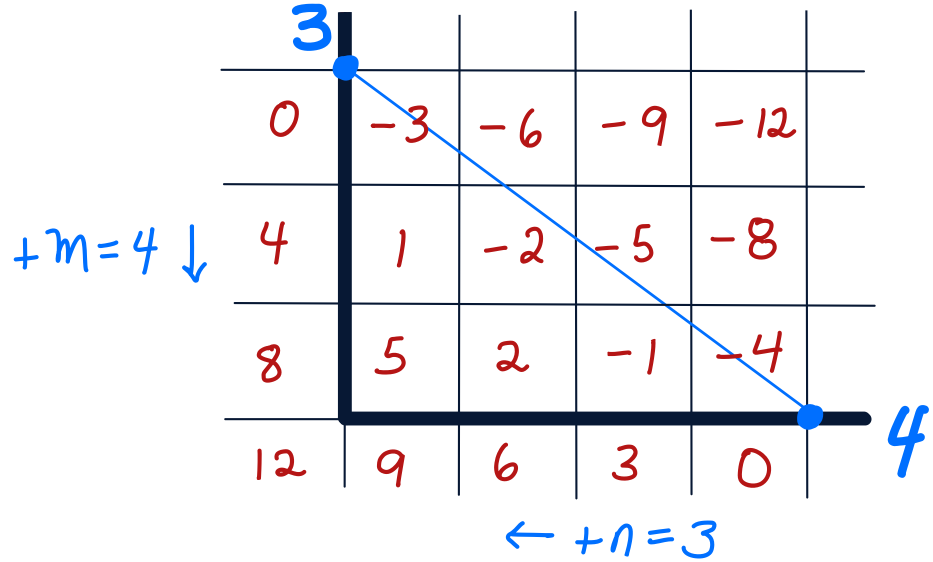

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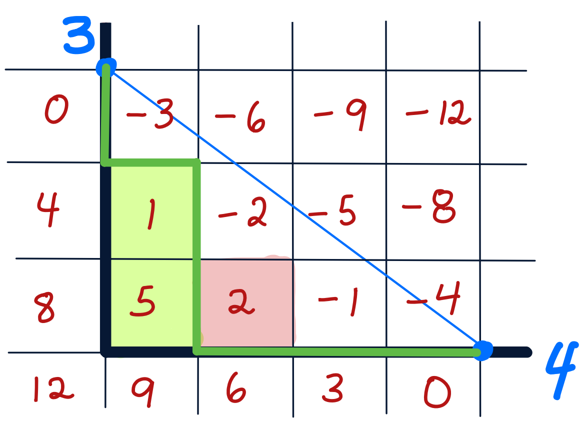

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Bijection: dinv & area preserving

$$\lambda \in D(m,n)$$

$$\big\downarrow \mathcal{A}$$

$$\Delta = \mathbb{Z}_{\geq 0} \setminus \text{Gaps}(\lambda)$$

$$\in \mathrm{Inv}^{0}_{m,n}$$

Claim: For any $r, s \in \mathbb{R}_{>0}$ $\exists\, m, n > 0$ coprime w/

$n > s$ & $\frac{s}{r} = \frac{n}{m}$.

Then $\tau_{r,s} \subseteq \tau_{m,n} \Rightarrow D(r,s) \subseteq D(m,n)$.

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Fix r,s,m,n:

$$\mathrm{Inv}^{\circ}_{r,s} := \{ \Delta \in \mathrm{Inv}^{\circ}_{m,n} \mid \Delta \cap \mathrm{Gaps}(\tau_{r,s} \subseteq \tau_{m,n}) = \phi \}$$

let $$\widetilde{\mathrm{Inv}}^{\circ}_{r,s} := \left(\mathrm{Inv}^{\circ}_{m,n} - \lceil (m-s)n - (n+m) \rceil \right) \cap \mathbb{Z}_{\geq 0}.$$

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Prop: $\exists$ an area & dinv preserving bijection:

$$D(r,s) \xleftrightarrow{\mathcal{A}} \widetilde{\mathrm{Inv}}^{\circ}_{r,s}$$

$$\lambda \subseteq \tau_{r,s} \longmapsto (\mathbb{Z}_{\geq 0} \setminus \mathrm{Gaps}(\lambda \subseteq \tau_{m,n}) - \lceil (m-s)n-(n+m) \rceil) \cap \mathbb{Z}_{\geq 0}$$

The triangular (q,t)-Schröder polynomials are

$$S_{\tau_{r,s}}(q,t,a) := t^{|\tau_{r,s}|} \sum_{\Delta \in \widetilde{\mathrm{Inv}}^{\circ}_{r,s}} q^{\mathrm{area}(\Delta)} t^{-\mathrm{codinv}(\Delta)} \prod_{k \in \mathrm{cogen}_{\geq 0}(\Delta)} \left(1 + a t^{-\xi_k(\Delta)}\right)$$

$\mathrm{cogen}(\mathcal{A}(\lambda)) = \mathcal{A}$. labels of addable boxes of λ

$\mathrm{ngen}(\mathcal{A}(\lambda)) = \mathcal{A}$. labels of vertical steps of λ

$\xi_k(\Delta) = \left| \mathrm{ngen}(\Delta) \cap [n+1+k, n+m+k] \right|$

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Thm (CGHM)

$$S_{\tau_{r,s}}(q,t,a) = \sum_{k \geq 0} a^k \left(\overbrace{\sum_{\lambda \in D(r,s)} \sum_{\substack{L \subseteq AB(\lambda) \\ |L| = k}}}^{\text{Dyck paths w/ k-addable boxes} \leftrightarrow \text{Schröder paths w/ k diag. steps}} q^{\mathrm{area}(\lambda)} t^{\mathrm{dinv}(\lambda) - \sum_{\square \in L} \xi(\lambda, \square)} \right)$$

where: $\xi(\lambda, \square) = \{ \square' \in \mathrm{vert}(\lambda) \mid \gamma(\square) + n < \gamma(\square') \leq \gamma(\square) + n + m \}$

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Example:

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$r = 5.6$

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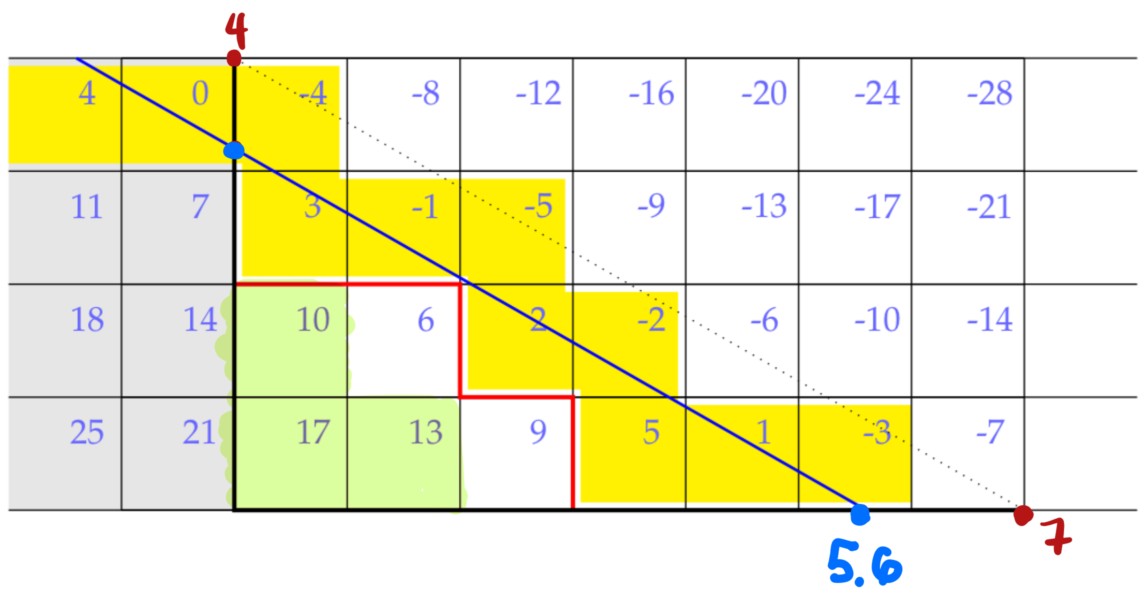
$m = 7$

$n = 4$



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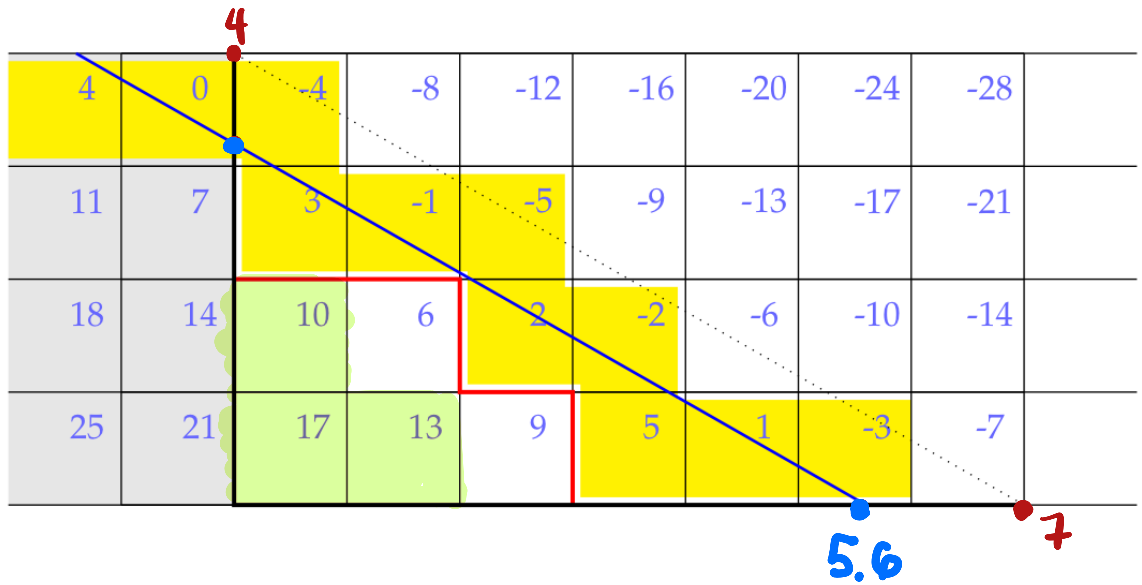

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$$\mathrm{Gaps}(\lambda \subseteq \tau_{m,n}) = \{0,1,2,3,5\} \cup \{6,9\} \xrightarrow[\text{by } 5]{\text{shift}} \{6,7,8,10\} \cup \{11,14\}$$

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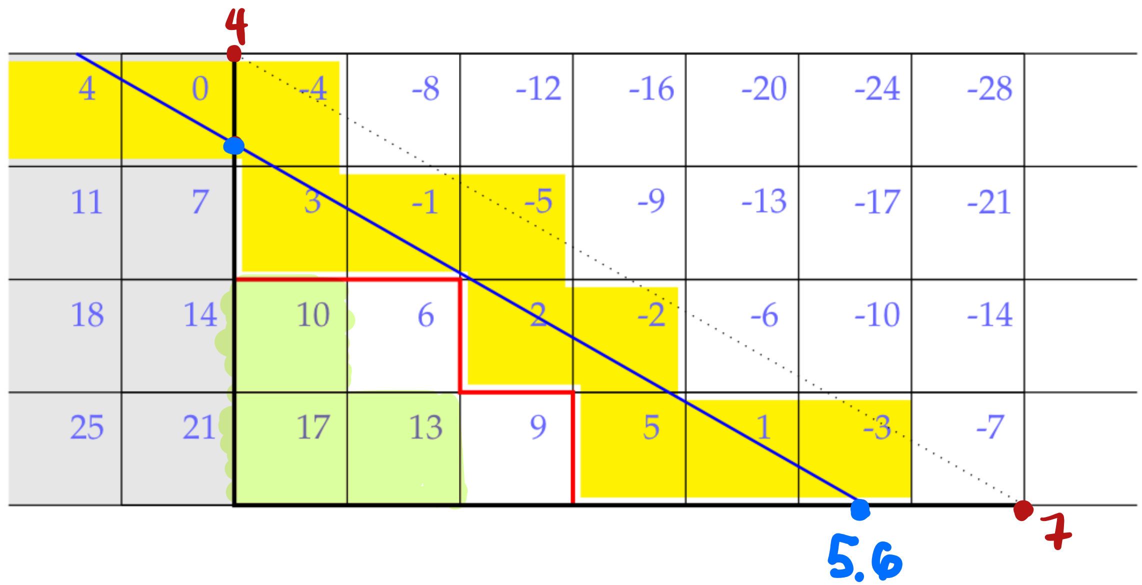

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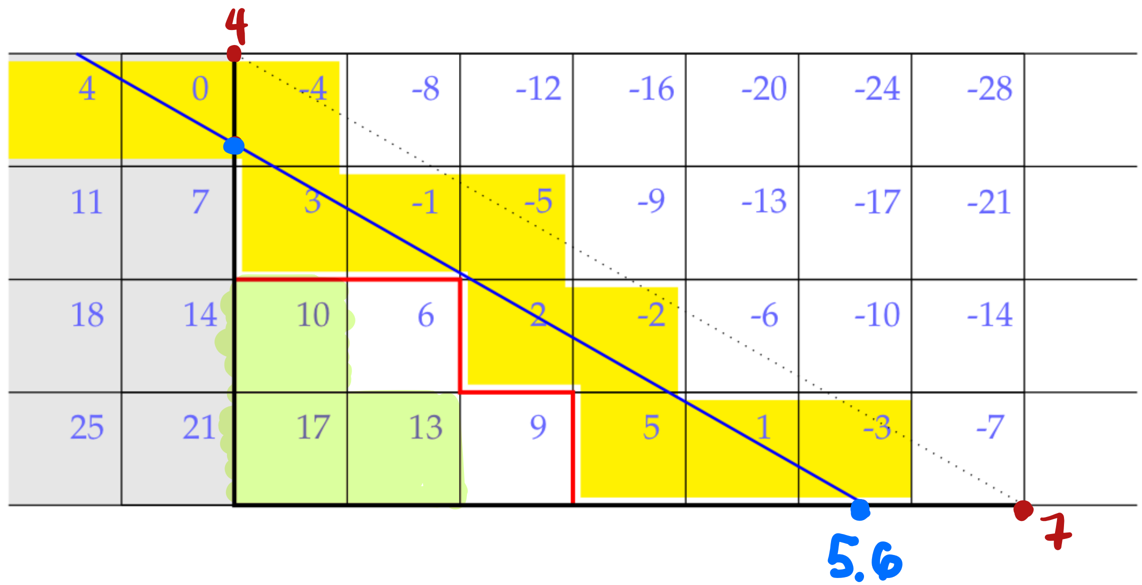

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$$\Longrightarrow \quad q^2 t^2 (1+at^{-2})(1+at^{-1})(1+at^{0})$$

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- $S_{\tau_{r,s}}(q,t,0) =$ triangular (q,t)-Catalan polyn. of Blasiak et.al.

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The (q,t)-Schröder theorem (Haglund):

$$S_n(q,t,a) = \sum_{k\geq 0} S_{n,k}(q,t)a^k = \sum_{k\geq 0} \langle \nabla e_n, h_k e_{n-k} \rangle a^k$$

i.e: $S_{n,k}(q,t) =$ hook components in the classical shuffle theorem.

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Thm (CGHM):

$$S_{\tau_{r,s}}(q,t,a) = \sum_{k\geq 0} \langle \underbrace{\mathcal{H}_\tau(X;q,t)}, h_k e_{\lfloor s \rfloor - k} \rangle a^k$$

Combinatorial side of shuffle thm under any line of Blasiak et. al.

Khovanov-Rozansky homology

KR-homology is a triply graded link homology theory that categorifies the HOMFLY-PT polynomial.

Knot/Link $\overset{\text{cat.}}{\rightsquigarrow}$ KR-homology $\overset{\text{grdim}}{\rightsquigarrow}$ KR-series

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There are alot of conjectures relating KR-homology to AG, RT &, combinatorics.

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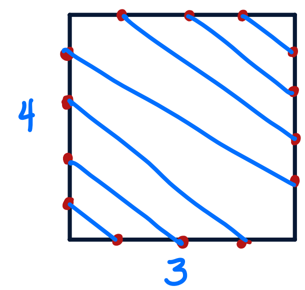
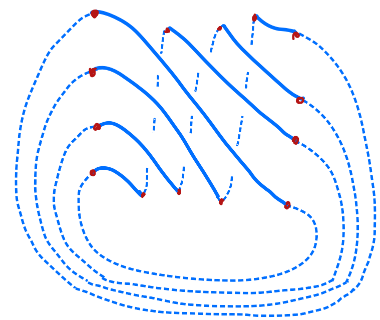
(Mellit) $S_{m,n}(q,t,a) = P^{KR}_{T(m,n)}(q,t,a)$

$T(3,4)$





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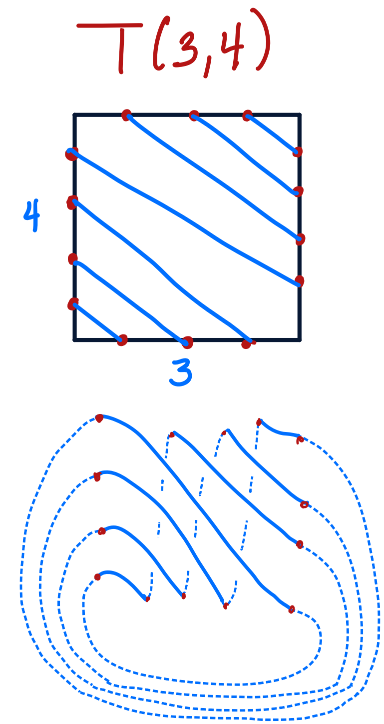
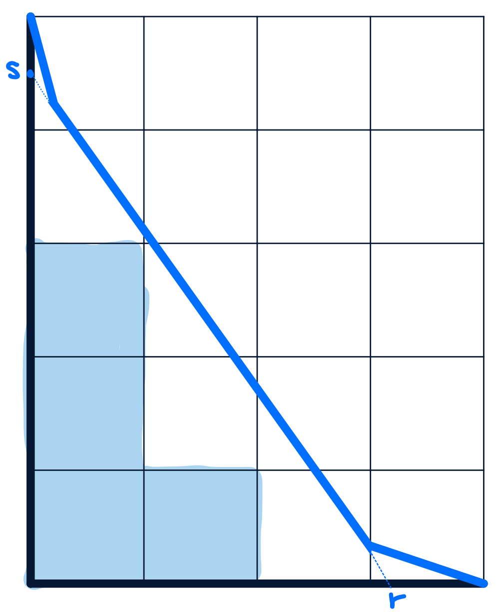


Monotone & Coxeter Knots

Given τ = triangular partition



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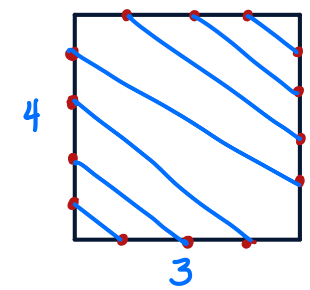

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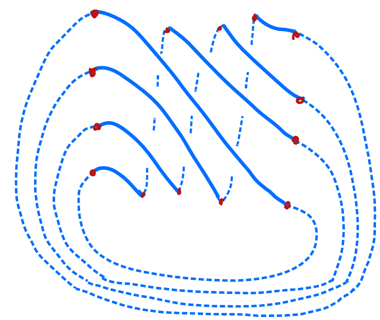

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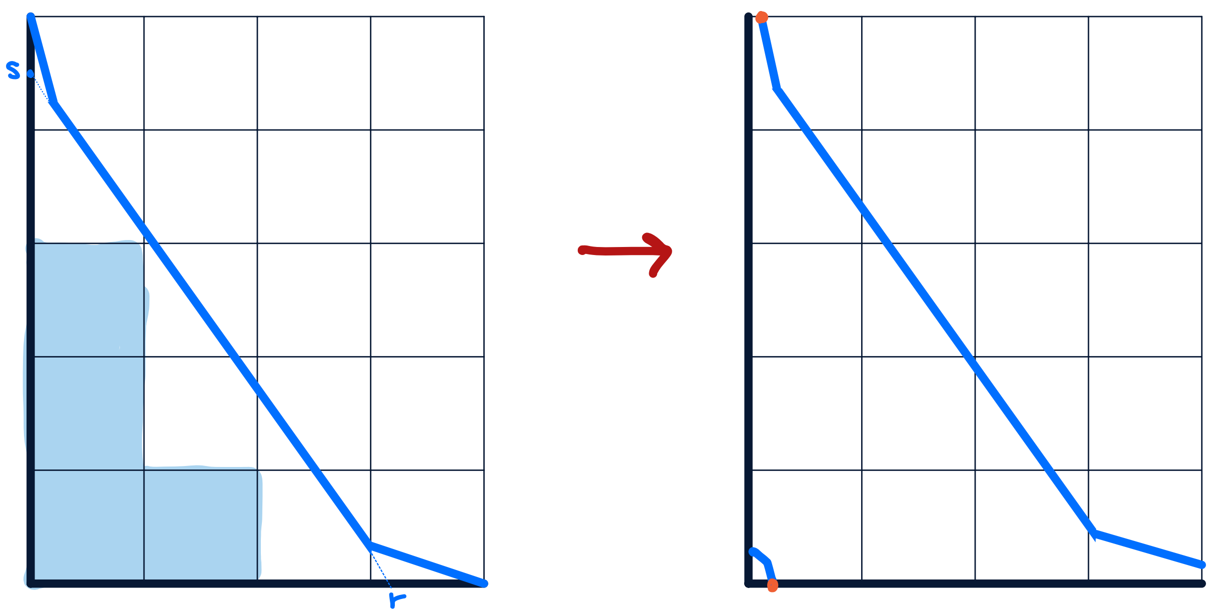

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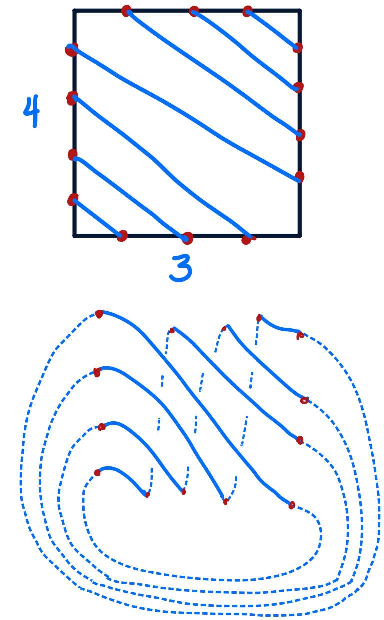

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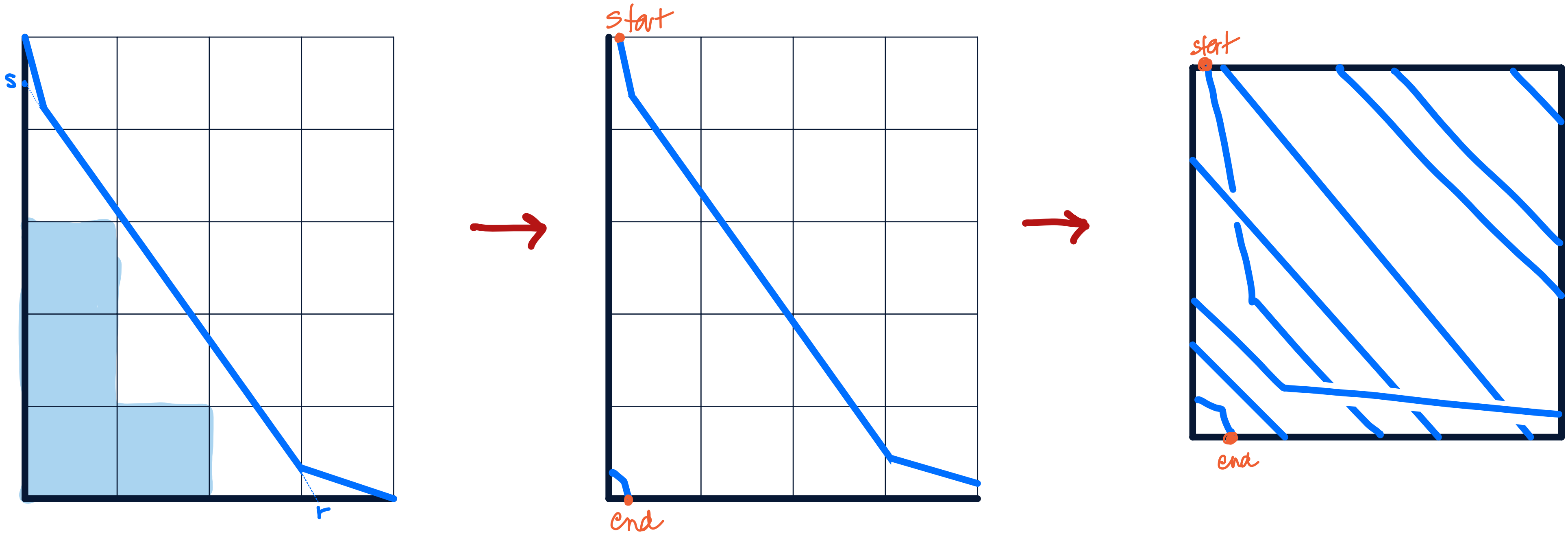

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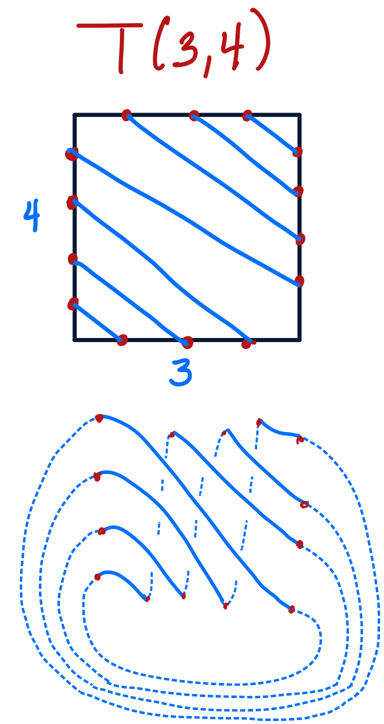
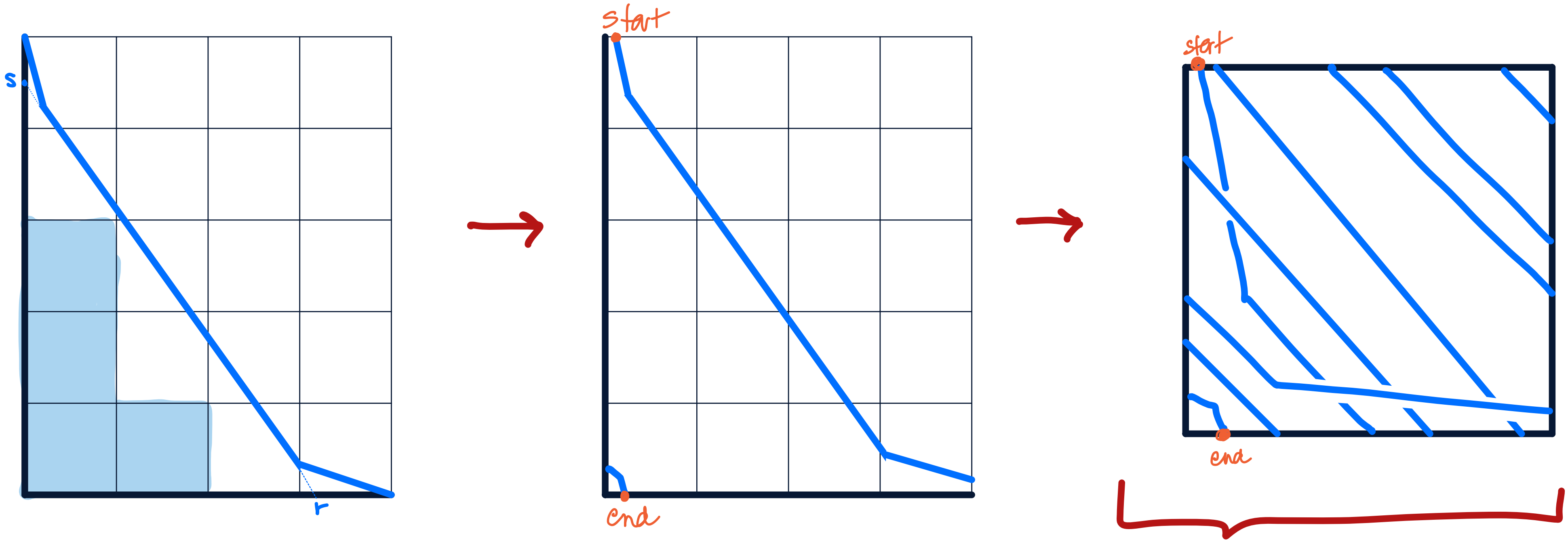


Monotone / Coxeter Knots

Given τ = triangular partition



$\tau \rightsquigarrow$ Coxeter knot K_τ

Thm (CGHM) For any triangular partition $\tau = \tau_{r,s}$

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Idea of proof: (why invariant subsets?)

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$\rightsquigarrow$ isotopic to K_τ.

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Corollary (CGHM) The ORS conjecture is true for K_τ w/ $\tau = \tau_{md,nd}$.

Thank You!

Full paper: Khovanov-Rozansky homology of Coxeter Knots & Schröder polyn for paths under any line.

arxiv: 2407.18123