Pattern heights and the minimal power of q in a Kazhdan–Lusztig polynomial

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joint work with
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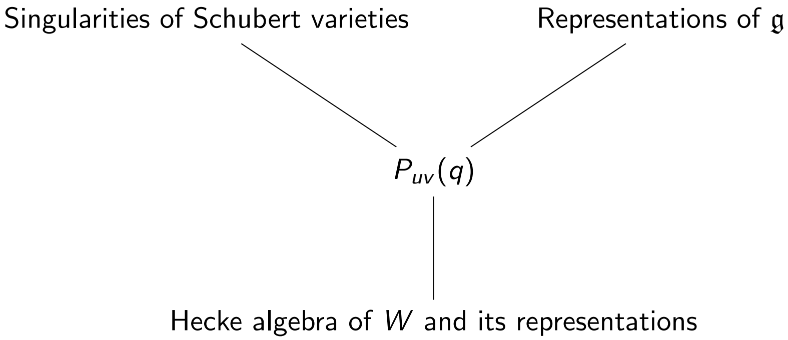
1. Kazhdan–Lusztig polynomials

2. The minimal power of q in P_{ew}

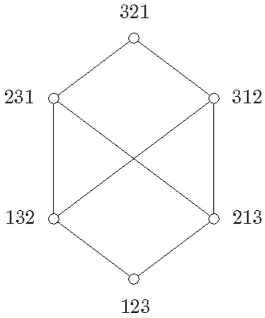
3. An upper bound in simply-laced type

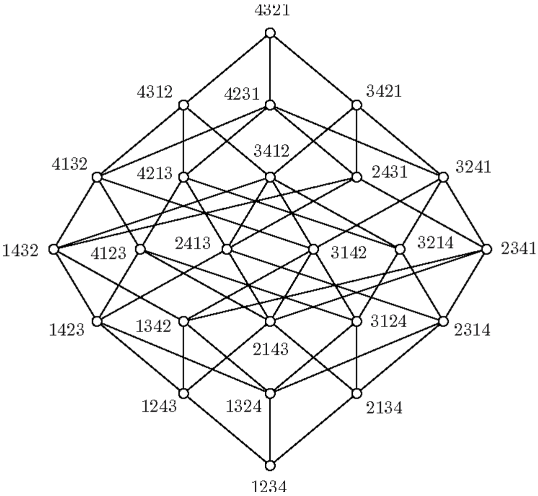
Kazhdan–Lusztig polynomials

The *Kazhdan–Lusztig polynomials* $P_{uv}(q) \in \mathbb{N}[q]$ for u, v in a Coxeter group W are of foundational importance in geometric representation theory.



Bruhat order on the symmetric group





Recursive definition of KL polynomials

The Kazhdan–Lusztig polynomials are the unique family of polynomials $P_{uv}(q) \in \mathbb{Z}[q]$ satisfying:

(1) $P_{uv}(q) = 0$ unless $u \leq v$.

(2) $P_{uu}(q) = 1$ for all u.

(3) $\deg P_{uv}(q) \leq \frac{1}{2}(\ell(v) - \ell(u) - 1)$ if $u < v$.

(4) And, if $u < v$ then

$$q^{\ell(v)-\ell(u)} P_{uv}\left(\frac{1}{q}\right) = \sum_{x \in [u,v]} R_{ux}(q) P_{xv}(q).$$

Definition of R polynomials

The R-polynomials are the unique family of polynomials $R_{uv}(q) \in \mathbb{Z}[q]$ satisfying:

(1) $R_{uv}(q) = 0$ unless $u \leq v$.

(2) $R_{uu}(q) = 1$ for all u.

(3) If s is a descent of both u and v:

$$R_{uv}(q) = R_{us,vs}(q).$$

(4) If s is a descent of v but not of u:

$$R_{uv}(q) = qR_{us,vs}(q) + (q-1)R_{u,vs}(q).$$

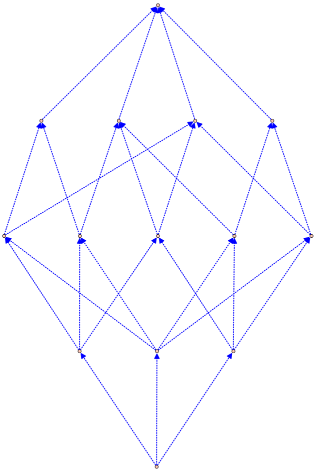
Smooth Schubert varieties and permutation patterns

Theorem (Kazhdan–Lusztig; Peterson; Lakshmibai–Sandhya)

TFAE for W simply laced:

- $P_{ew} = 1$,
- X_w *is smooth,*
- $[e, w]$ *is rank-symmetric,*
- w *avoids* 3412 *and* 4231 *(if $W = S_n$)*

$w = 3412$



For example, $P_{e,3412} = 1 + q$.

Positivity

Theorem (Kazhdan–Lusztig; Elias–Williamson)

For any $v \leq w \in W$ we have $P_{vw} \in \mathbb{N}[q]$.

It is a major open problem to give a positive combinatorial formula for P_{vw}.

For any $v \leq w \in W$ we have $P_{vw}(0) = 1$. We'll study the next term in the case $v = e$.

Definition

Let $w \in W$ be such that X_w is singular (so $P_{ew} \neq 1$). Define

$$h(w) = \min\{i > 0 \mid [q^i]P_{ew} \neq 0\}.$$

So we have $P_{ew} = 1 + cq^{h(w)} +$ higher order terms.

How large can $h(w)$ be?

Theorem (Björner–Ekedahl '09)

If X_w is singular we have:

$$h(w) = \min\{i \mid \#[e,w]_i < \#[e,w]_{\ell(w)-i}\} \\ = \min\{i \mid \dim H^{2i}(X_w) < \dim H^{2(\ell(w)-i)}(X_w)\}.$$

Conjecture (Billey–Postnikov '02)

Let W be of finite simply laced type (A, D, or E) with rank r. Suppose that X_w is singular. Then $h(w) \leq r$.

This conjecture is surprising, since P_{ew} is of degree $O(r^2)$.

Pattern heights

Definition (Cortez; Woo)

Suppose $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ *are the positions of a* 3412 *pattern in* w*. The* **height** *of this occurrence is* $w(i_1) - w(i_4)$*. The* **minimum height** $\mathrm{minHeight}(w)$ *is the smallest height among all occurrences of* 3412 *in* w*.*

Example

We have $\mathrm{minHeight}(3412) = 1$ *and* $\mathrm{minHeight}(45312) = 2$*.*

An exact formula for $h(w)$ in Type A

Theorem (G.–Gao)

Let $w \in S_n$ with X_w singular. Then

$$h(w) = \begin{cases} 1, & \text{if } w \text{ contains } 4231; \\ \mathrm{minHeight}(w), & \text{otherwise.} \end{cases}$$

Example

The permutation $w = 45312$ avoids 4231 and has $\mathrm{minHeight}(w) = 2$. We have $P_{ew} = 1 + q^2$, so $h(w) = 2$.

A tight bound

We resolve Billey and Postnikov's conjecture.

Theorem (G.–Gao)

Let W be of finite simply laced type (A, D, or E) with rank r. Suppose that X_w is singular. Then $h(w) \leq r - 2$, and this bound is tight.

Example

For $w \in S_n$, $\mathrm{minHeight}(w)$ is maximized when

$$w = (n-1)\, n\, (n-2)\, (n-3) \cdots 3\, 1\, 2,$$

with $\mathrm{minHeight}(w) = (n-1) - 2 = \mathrm{rank}(S_n) - 2$.

For types D and E we prove the bound but we don't have an exact formula for $h(w)$.

Techniques

- We use a hybrid between the Billey and Billey–Postnikov characterizations of smooth Schubert varieties in finite type by generalized permutation patterns.
- For certain of these generalized patterns, we show directly that containment implies $h(w) = 1$ or 2 using Björner–Ekedahl.
- If w avoids these patterns and contains others, we find J so that the projection $X_w \to X^J(w^J)$ induced by $G/B \to G/P_J$ is a fiber bundle. In this case we can understand $[e, w]$ in terms of $[e, w^J] \cap W^J$ and $[e, w_J]$.

Are there any questions?