On the size of Bruhat intervals

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Joint work with F. Castillo, N. Libedinsky and D. Plaza

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Definition of Bruhat intervals

Let (W, S) be a Coxeter system, equipped with a length funcion ℓ and a partial order $\leq$ (the Bruhat order):

- For $w \in W$, write

$$w = s_1 s_2 \cdots s_k, \quad s_i \in S.$$

  If k is minimal, we define $\ell(w) = k$ and we say that $s_1 s_2 \cdots s_k$ is a reduced expression of w.
- We say that $u \leq w$ if each reduced expression of w has a subexpression which is a reduced expression of u.

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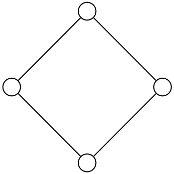
Main problem: Compute the cardinalities of Bruhat intervals

$$[u, w] := \{z \in W \mid u \leq z \leq w\},$$

in particular of lower Bruhat intervals $\leq w := [id, w]$.

Some known results

- If $\ell(w) - \ell(u) = 2$, then $[u, w] \cong$ 

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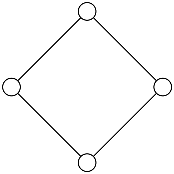
- If $\ell(w) - \ell(u) = 2$, then $[u, w] \cong$ 
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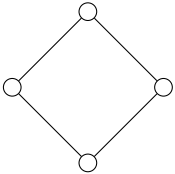
- If $\ell(w) - \ell(u) = 2$, then $[u, w] \cong$ 
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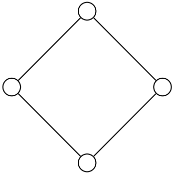
- If $\ell(w) - \ell(u) = 2$, then $[u, w] \cong$ 
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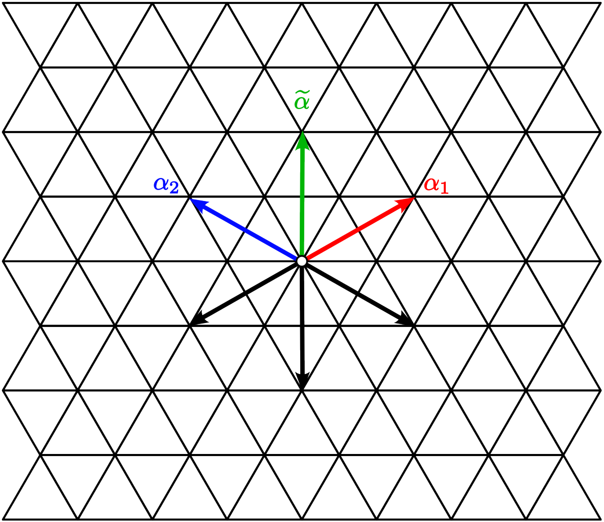
- (Libedinsky-Patimo, 2023) and (Batistelli-Bingham-Plaza, 2023): In affine type $\widetilde{A}_2$ ($\widetilde{B}_2$, respectively), explicit formula of all $| \leq w|$. Under some parametrization of the group, these are polynomial formulas.

The Lattice Formula (example in $\widetilde{A}_2$)



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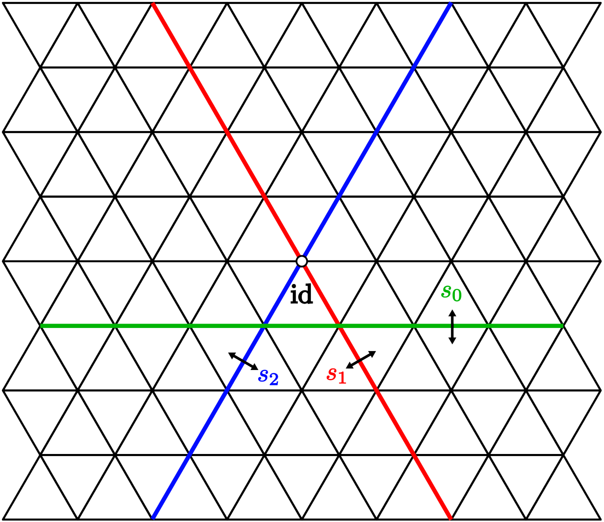

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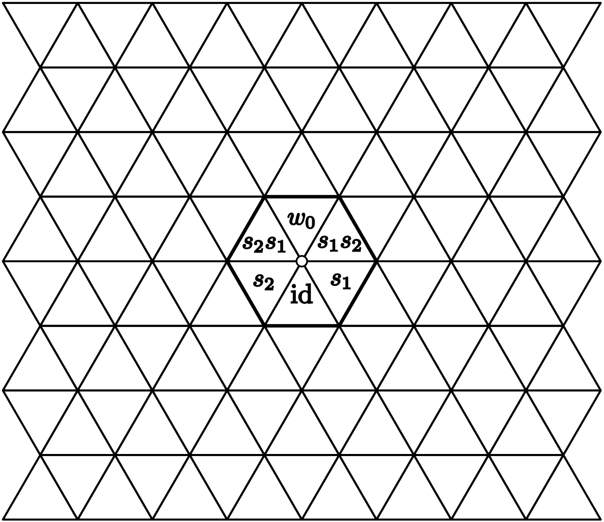

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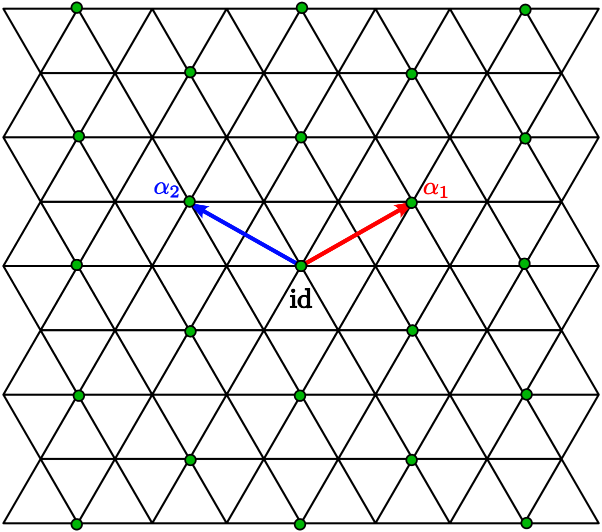

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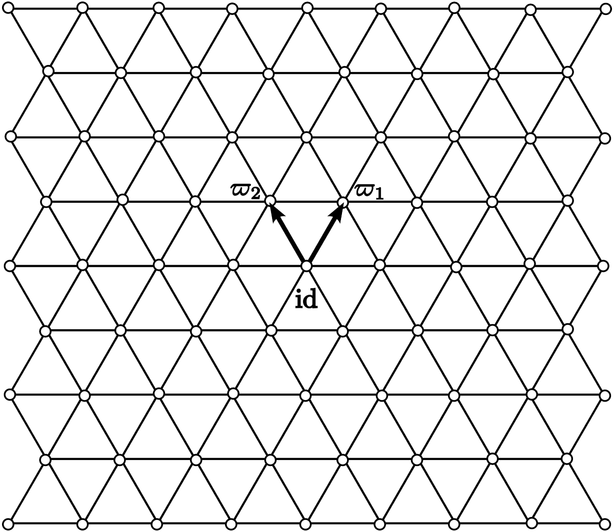

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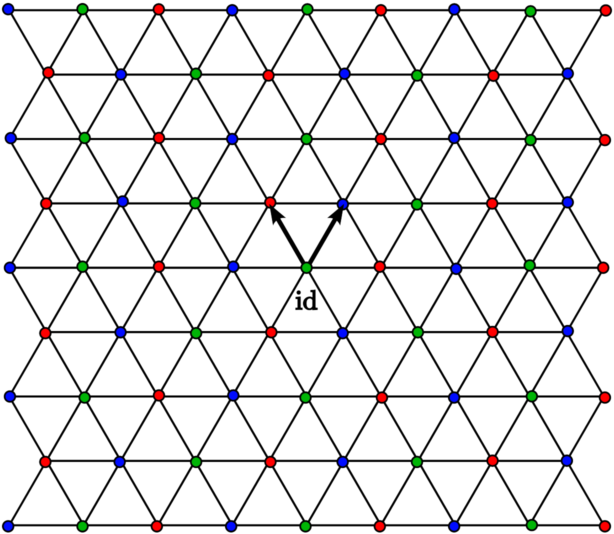

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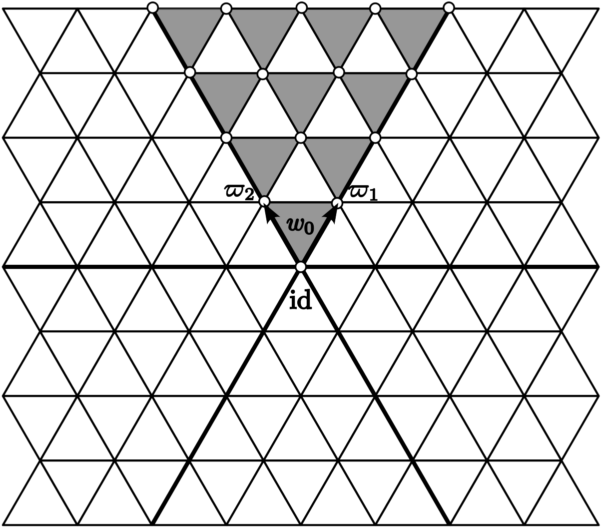

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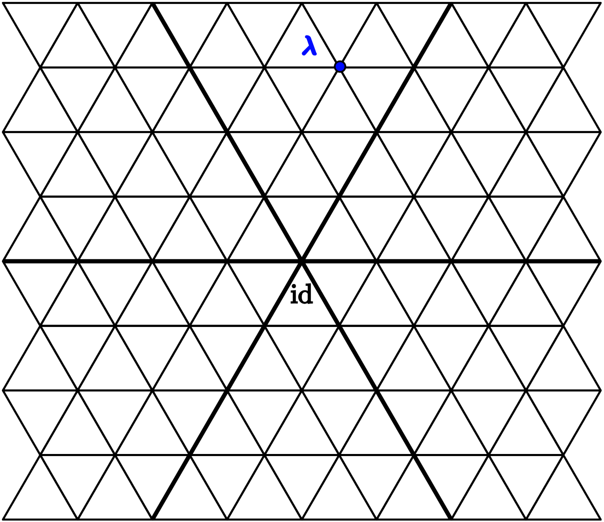

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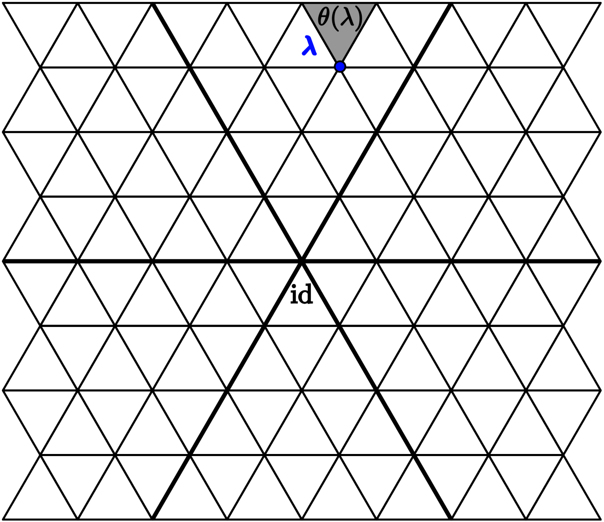

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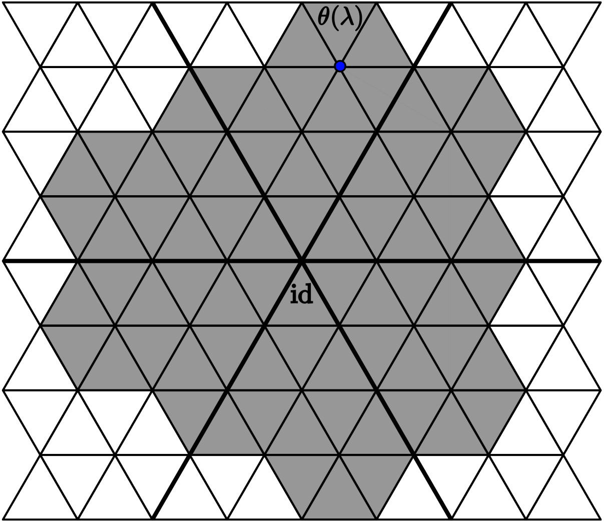

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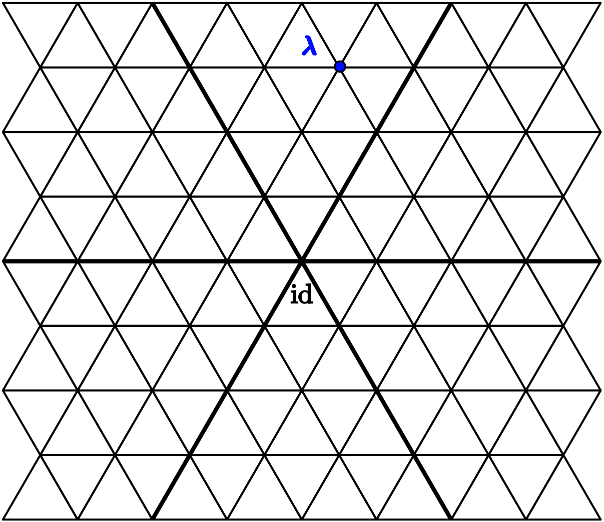

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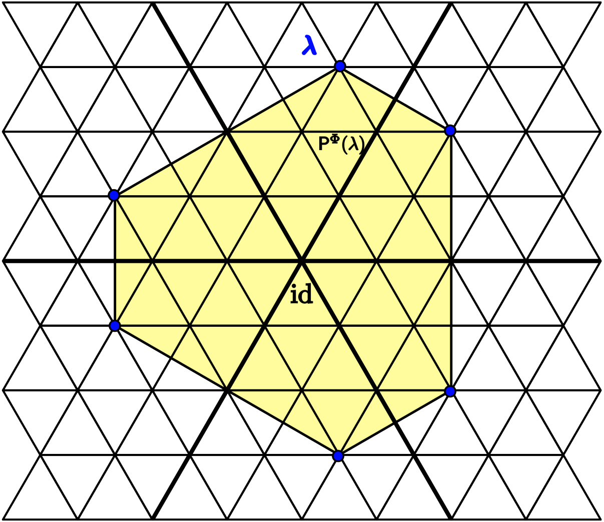

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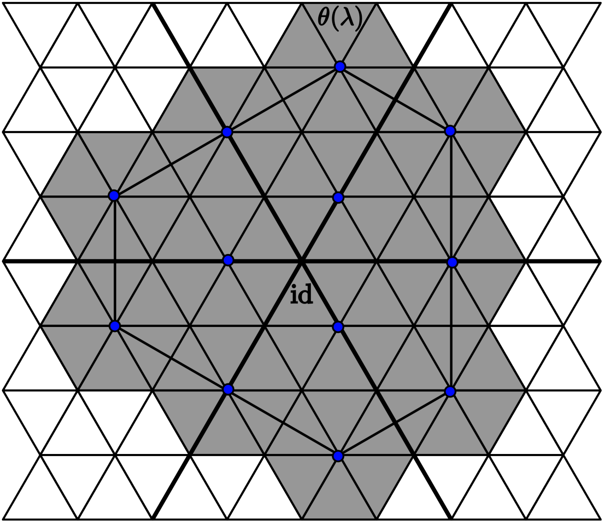

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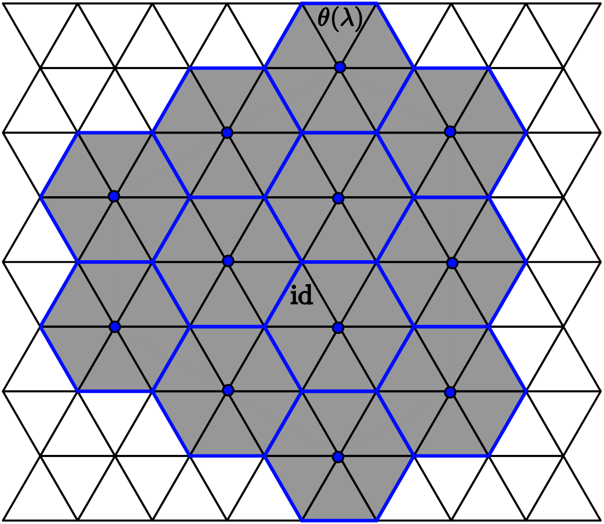

The Lattice Formula

Let Φ be an irreducible root system with finite Weyl group W_f and affine Weyl group W_a. For a dominant coweight λ, we define

$$\mathrm{P}^{\Phi}(\lambda) \coloneqq \mathrm{Conv}(W_f \cdot \lambda).$$

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Let Φ be an irreducible root system with finite Weyl group W_f and affine Weyl group W_a. For a dominant coweight λ, we define

$$\mathrm{P}^{\Phi}(\lambda) := \mathrm{Conv}(W_f \cdot \lambda).$$

Theorem: Lattice Formula

For every dominant coweight λ, we have

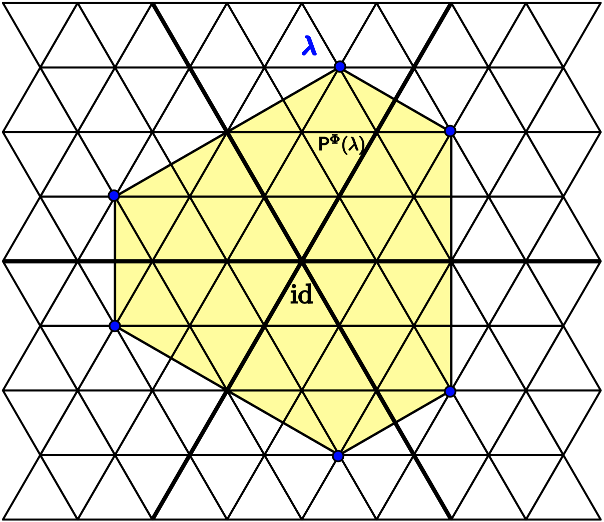
$$\mathrm{Alcoves}(\leq \theta(\lambda)) = \bigsqcup_{\mu} \mathrm{Alcoves}(W_f) + \mu,$$

where μ ranges over $\mathrm{P}^{\Phi}(\lambda) \cap (\lambda + \mathbb{Z}\Phi^{\vee})$.

In particular,

$$|\leq \theta(\lambda)| = |W_f||\mathrm{P}^{\Phi}(\lambda) \cap (\lambda + \mathbb{Z}\Phi^{\vee})|.$$

The Geometric Formula (example in $\widetilde{A}_2$)



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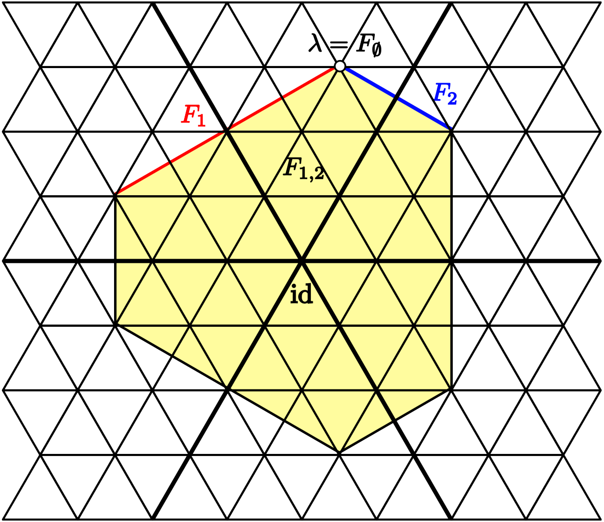

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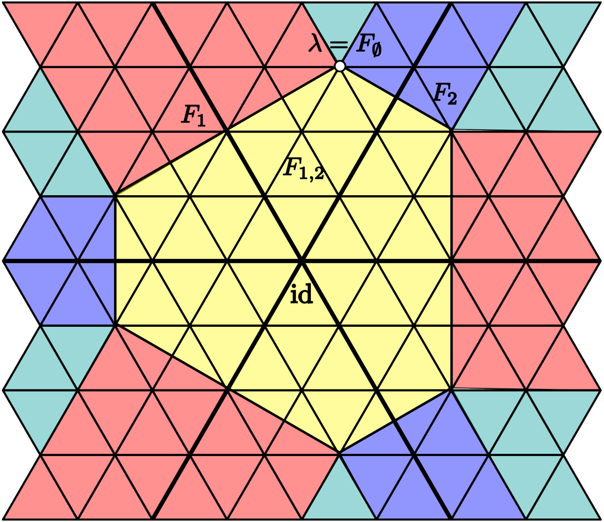

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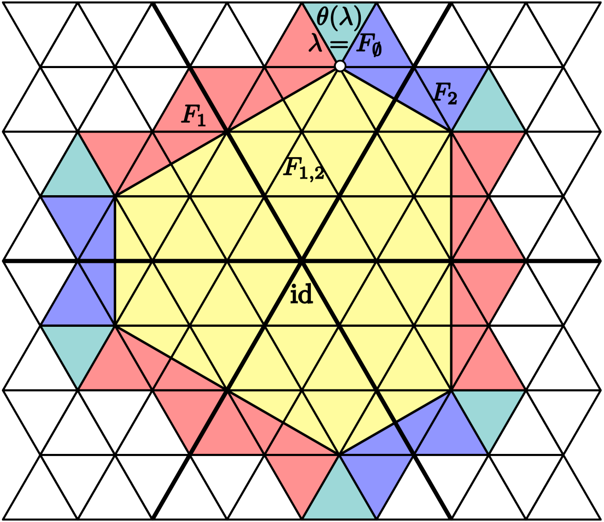
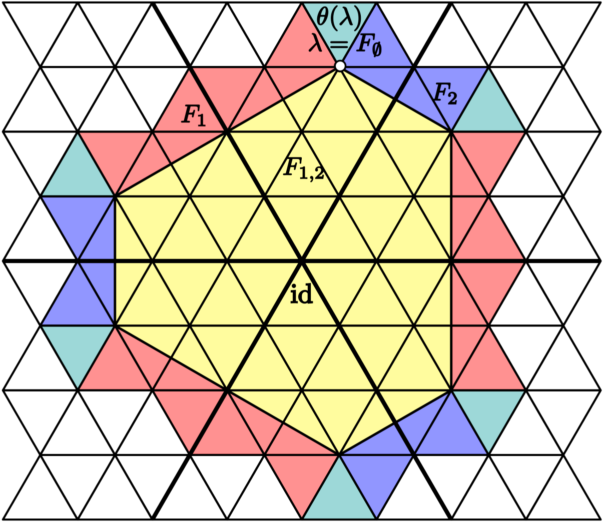


The Geometric Formula (example in $\widetilde{A}_2$)

$$|\leq \theta(\lambda)| = \mu_{1,2}\text{Area}(F_{1,2}) + \mu_1\text{Length}(F_1) + \mu_2\text{Length}(F_2) + \mu_\emptyset\text{Card}(F_\emptyset)$$



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\end{aligned}
$$

If $\lambda = (a, b)$ in the fundamental weight basis, then:

- $V_{1,2}(a, b) = \frac{\sqrt{3}}{2}(a^2 + 4ab + b^2)$, $\quad \mu_{1,2} = 2\sqrt{3}$
- $V_1(a, b) = a\sqrt{2}$, $\quad \mu_1 = \frac{9}{2}\sqrt{2}$
- $V_2(a, b) = b\sqrt{2}$, $\quad \mu_2 = \frac{9}{2}\sqrt{2}$
- $V_\emptyset(a, b) = 1$, $\quad \mu_\emptyset = 6$

Thus,

$$|\leq \theta(a, b)| = 3a^2 + 3b^2 + 12ab + 9a + 9b + 6$$

The Geometric Formula

Let Φ be any irreducible root system and let S_f be the set of simple reflections of the finite Weyl group W_f.

Theorem: Geometric Formula

There exist unique real numbers μ_J^Φ such that for any dominant coweight λ,

$$|\leq \theta(\lambda)| = \sum_{J \subset S_f} \mu_J^\Phi V_J^\Phi(\lambda).$$

Important: The coefficients μ_J^Φ **do not** depend on λ.

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Important: The coefficients μ_J^Φ **do not** depend on λ.
If $\lambda = (m_1, \dots, m_n)$ in the fundamental coweight basis, then the volumes $V_J^\Phi(\lambda)$ are polynomials in $m_1, \dots, m_n$. ($n =$rank of Φ)
Thus, the Geometric Formula implies that $|\leq \theta(\lambda)|$ is also a polynomial of degree n in $m_1, \dots, m_n$.

The Geometric Formula (example in $\widetilde{A}_2$)

$$|\leq \theta(\lambda)| = \mu_{1,2} V_{1,2}(\lambda) + \mu_1 V_1(\lambda) + \mu_2 V_2(\lambda) + \mu_\emptyset V_\emptyset(\lambda)$$