Limit shapes of random Young tableaux and a discontinuity phenomenon

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joint work with J. Borga, C. Boutillier, P.-L. Méliot

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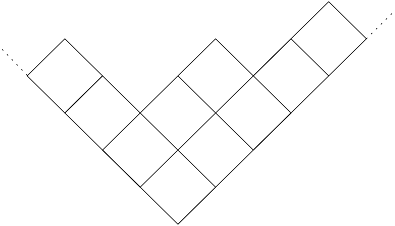
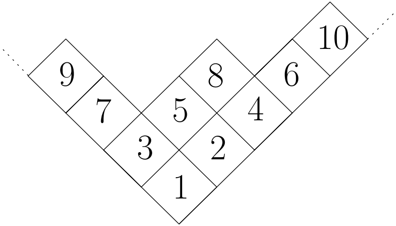
36th International Conference on
Formal Power Series and Algebraic Combinatorics
Bochum, Germany, July 2024

Young diagrams and tableaux



Young diagram

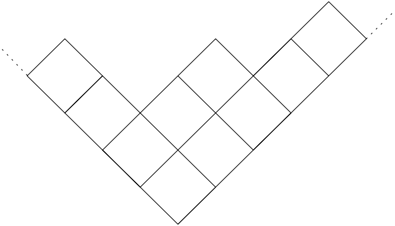


(Standard) Young tableau

Young diagram: stack of boxes in the upper quarter-plane.

(Standard) Young tableau: filling of a Young diagram with integers from 1 to n, increasing upwards.

Young diagrams and tableaux



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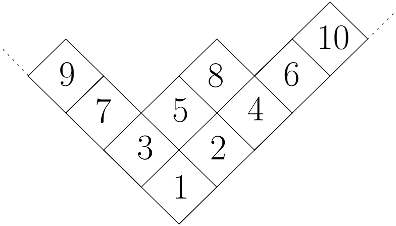

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Young diagram: stack of boxes in the upper quarter-plane.

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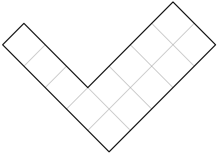
Our model: fix a Young diagram λ, and take a uniform random Young tableau T of shape λ (Biane, Pittel, Romik, Angel, Holroyd, Virag, Gorin, Rahman, Sun, Banderier, Marchal, Wallner, Śniady, Maślanka, Chan, Pak, Panova, Gordenko, Xu, ...).

Motivations

- Bijection with other models: constrained random permutations (RSK bijection), random sorting networks (Edelman–Greene bijection).
- Asymptotic representation theory: random tableaux encode some asymptotic information on restrictions of representations of large symmetric groups.
- Link with the well-studied lozenge tiling models (Young tableaux are in some sense a limit case of lozenge tilings);
- Tractable model of random linear extensions of 2-dimensional posets.

Simulation (first example)

We consider the n-th dilatation $n \cdot \lambda^0$ of the following diagram



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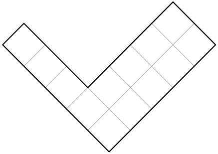

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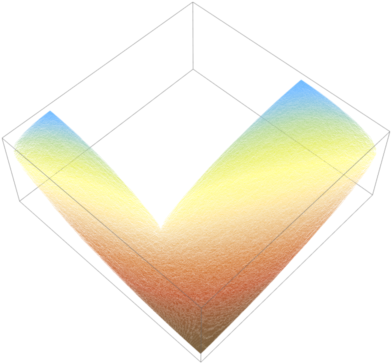
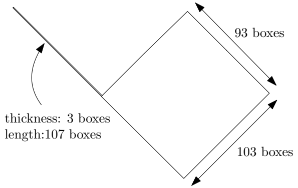
A uniform tableau T_N of shape $n \cdot \lambda^0$:



Here, $n = 100$ so the tableau T_N has $N = 130000$ cells. There seems to be a smooth limit surface.

Simulation (second example)

This time, take λ^0 to be



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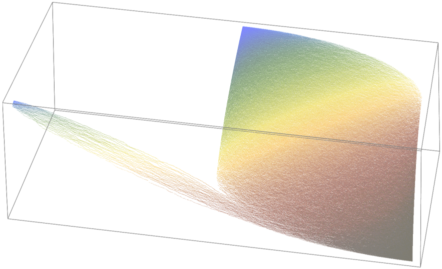
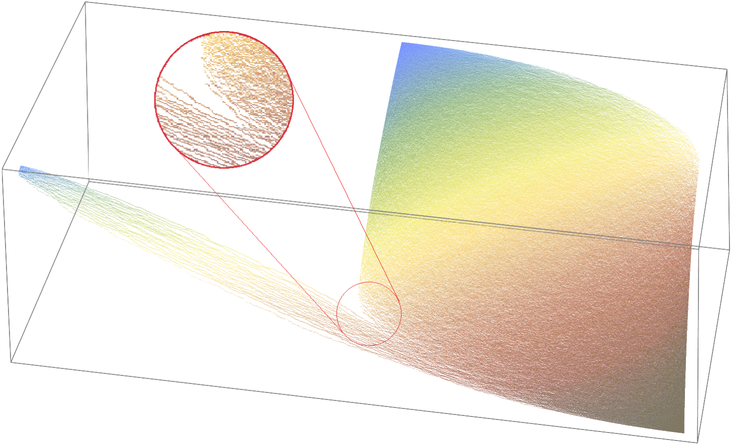


Here, $n = 6$ so the diagram/tableau has $N = 356400$ cells.

Simulation (second example, with a zoom)



There still seems to be a limiting surface, but this time it is discontinuous!

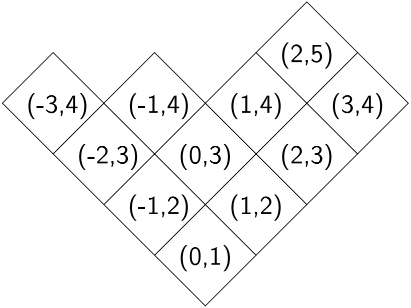
Results (informally)

- Previous contributions (Biane '03, Sun '18): convergence to a limiting surface with some implicit description (via Markov–Krein correspondence and free compression or via a variational principle).
- Our results:
  - a more explicit description of the limit surface in the multirectangular case (dilatation of a fixed diagram λ^0);
  - characterization of the diagrams λ^0 leading to discontinuous limit surfaces;
  - a local limit result (not in this talk).

Height function

Notation: if T is a tableau of size N, we let

- $T(x, y)$: content of the cell with coordinates (x, y) in T;



Cell coordinates

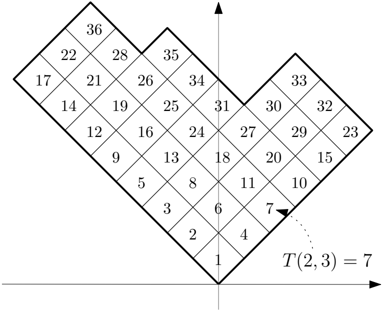


Tableau function

Height function

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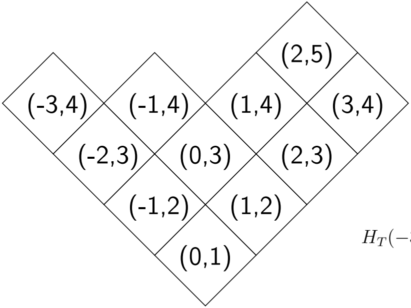
- $T(x,y)$: content of the cell with coordinates (x,y) in T;
- $H_T(x,t) = \#\{y : T(x,y) \le Nt\}$: number of entries on the vertical line x smaller than Nt.



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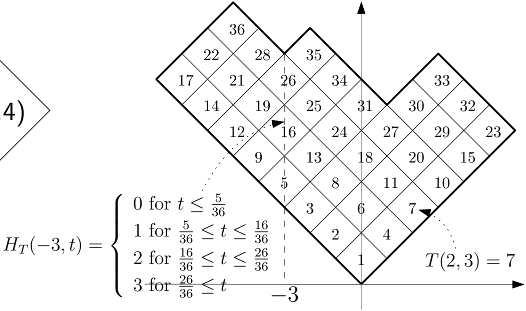

Tableau and height functions

($y \mapsto T(x,y)$ and $t \mapsto H_T(x,t)$ are roughly inverses of each other.)

Existence of the limiting height function

Theorem (Biane '03, Sun '18)

Let λ^0 be a fixed Young diagram. For $n \geq 1$, we let T_N be a uniform random Young tableau of shape $\lambda_N := n \cdot \lambda^0$. Then there exists a deterministic function H^∞ such that

$$\frac{1}{\sqrt{N}} H_{T_N}\left(\lfloor x\sqrt{N} \rfloor, t\right) \xrightarrow[N \to +\infty]{} H^\infty(x,t),$$

in probability, uniformly on (x,t).

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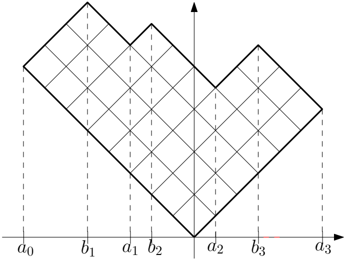
Question: How to compute H^∞?

The critical equation

We encode λ^0 by its interlacing coordinates $a_0 < b_1 < a_1 < \cdots < b_m < a_m$:



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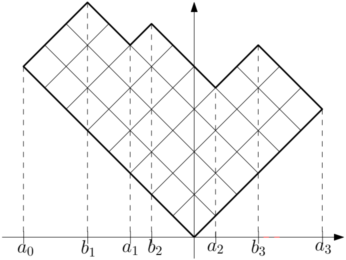

Definition: the critical equation

For parameters (x, t), we consider the polynomial equation

$$U \prod_{i=1}^{m} (x - \eta\, b_i + U) = (1-t) \prod_{i=0}^{m} (x - \eta\, a_i + U),$$

where $\eta = 1/\sqrt{|\lambda^0|}$.

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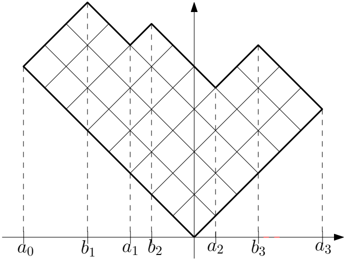

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Lemma

The critical equation has at least $m-1$ real roots.

We denote $U_c(x,t)$ its complex root with positive imaginary part, if it exists (in this case, we say that (x,t) is in the "liquid region").

Formula for the limiting height function

Theorem (Borga, Boutillier, F., Méliot, '23)

$$H^{\infty}(x,t) = \frac{1}{\pi} \int_0^t \frac{\operatorname{Im} U_c(x,s)}{1-s} \, \mathrm{d}s.$$

Convention: $\operatorname{Im} U_c(x,s) = 0$ if the critical equation has only real root ("frozen region").

Proof: uses a determinantal point process description of random tableaux (Gorin–Rahman, '19), and saddle point analysis (U_c is the saddle point).

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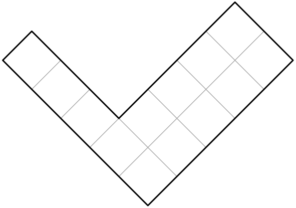
Example: square shape tableaux (Romik–Pittel, '07), $a_0 = -1, b_1 = 0, a_1 = 1$

The critical equation $U(x+U) = (1-t)(x+1+U)(x-1+U)$ is a second degree polynomial equation, and we get

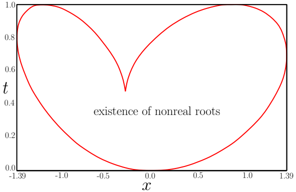
$$H^{\infty}_{\diamond}(x,t) = \frac{1}{\pi}\int_0^t \frac{\sqrt{4s-4s^2-x^2}}{2s-2s^2}\,\mathrm{d}s,$$

with the convention that $\sqrt{y} = 0$ if $y \le 0$.

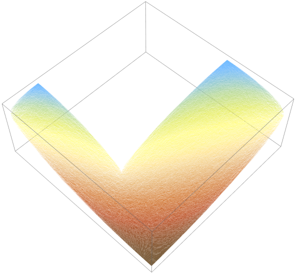
The heart example



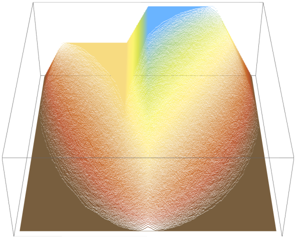
the Young diagram λ^0



boundary of the liquid region

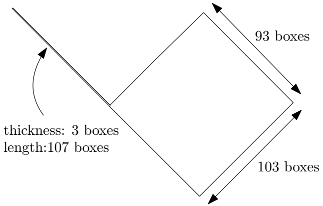


a realization of T_N



its height function H_{T_N}

The pipe example



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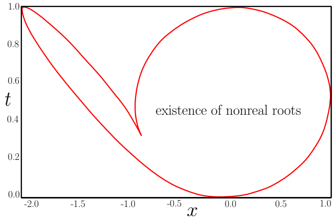

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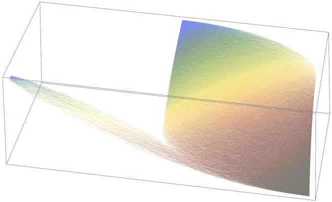

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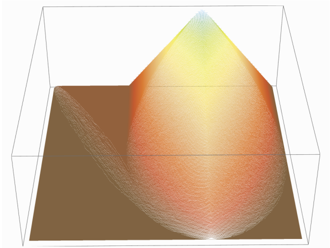
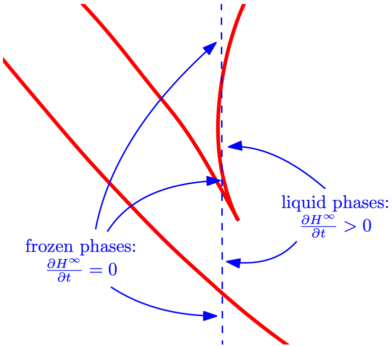


its height function H_{T_N}

Why is there a discontinuity in the pipe example?



Zoom on the boundary of the liquid region (blue line $x = x_0 \approx -0.9$)

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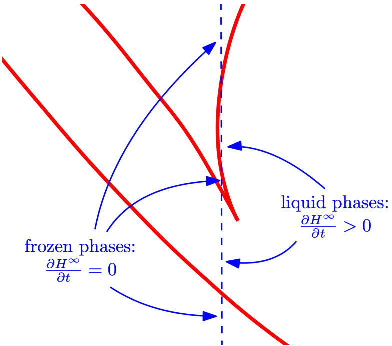

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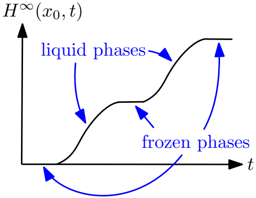

Schematic representation of the function $t \mapsto H^\infty(x_0, t)$

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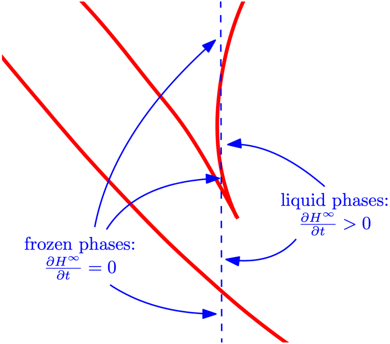

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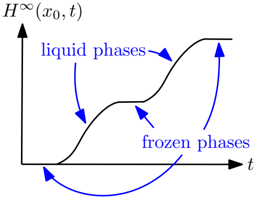

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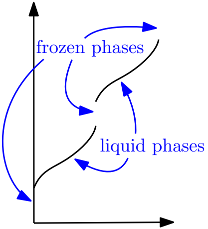
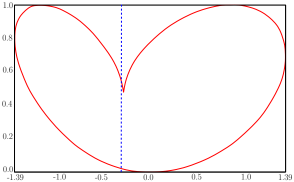
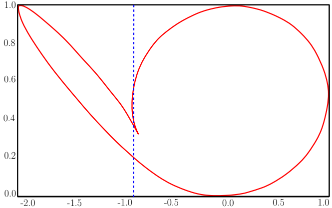


Schematic representation of the function $y \mapsto T^\infty(x_0, y)$

When is there a discontinuity?

There is a discontinuity as soon as the tangent at one of cusp is not vertical (both curves leaving a cusp have the same tangent; think at $x^2 = y^3$).





(In general, there are $m-1$ cusps, where m is the number of distinct parts in λ_0.)

When is there a discontinuity?

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With some computation, we get

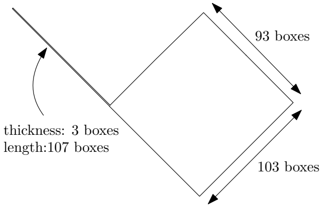
Theorem (Borga, Boutillier, F., Méliot, '23)

The limiting surface $T^{\infty}_{\lambda^0}$ is continuous if and only if the interlacing coordinates $a_0 < b_1 < a_1 < \cdots < b_m < a_m$ of λ^0 satisfy

$$\sum_{\substack{i=0 \\ i \neq i_0}}^{m} \frac{1}{a_{i_0} - a_i} = \sum_{i=1}^{m} \frac{1}{a_{i_0} - b_i}, \qquad \text{for all } i_0 = 1, \dots, m-1.$$

In particular, for $m > 1$, the limit surfaces are generically discontinuous!

Thanks for your attention!



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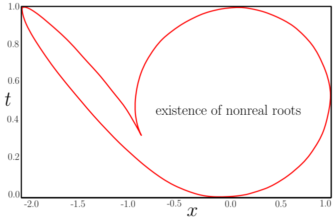

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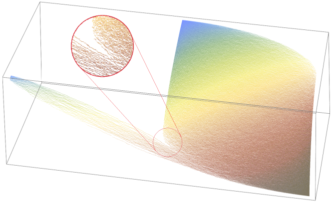

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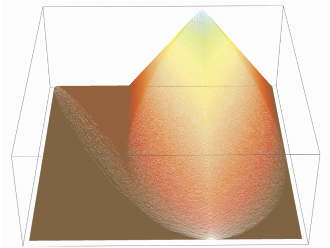

its height function H_{T_N}

Proof strategy 1 – determinantal point processes

Notation:
E: locally compact Polish space
μ: reference measure on E
K: measurable function $E^2 \to \mathbb{C}$.
X: simple point process on E

Definition (determinantal point process)

X is a determinantal point process on E with kernel K if it has a joint intensity with respect to μ given by

$$\rho_n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{1 \le i,j \le n},$$

for every $n \ge 1$ and distinct $x_1, \dots, x_n \in E$.

Used a lot in integrable probability theory/statistical physics since 90's, but also in random matrix theory, statistics, ...

Proof strategy 2 – tableaux and bead configurations

Definition (Poissonized tableaux)

A Poissonized tableau of shape λ is an upward increasing filling of λ with real numbers in $[0,1]$.

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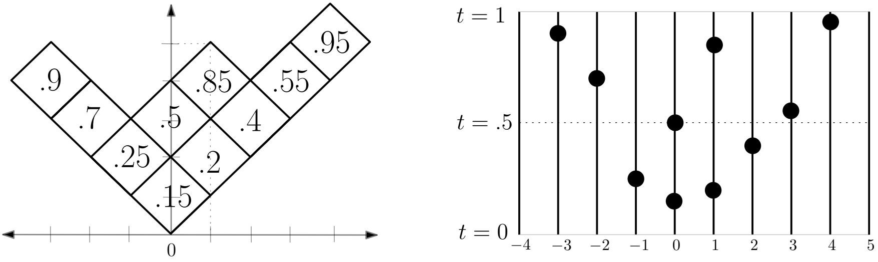
With a Poissonized tableau T, we associate a bead configuration

$$M_T := \Big\{(x, T(x,y)), (x,y) \in \lambda\Big\} \subseteq \mathbb{Z} \times [0,1].$$



Note: $H_T(x,t)$ is the number of beads in $\{x\} \times [0,t]$.

Proof strategy 3 – Gorin–Rahman theorem

Theorem (Gorin, Rahman, '19)

Let T be a uniform random Poissonized tableau of fixed shape λ. Then its associated bead process M_T is a determinantal point process on $\mathbb{Z} \times [0,1]$ with correlation kernel

Proof strategy 3 – Gorin–Rahman theorem

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$$K_\lambda((x_1,t_1),(x_2,t_2)) = -\frac{1}{(2\mathrm{i}\pi)^2}\cdot \oint_{\gamma_z}\oint_{\gamma_w} \frac{F_\lambda(z)}{F_\lambda(w)}\,\frac{\Gamma(w-x_1+1)}{\Gamma(z-x_2+1)}\,\frac{(1-t_2)^{z-x_2}(1-t_1)^{-w+x_1-1}}{z-w}\,\mathrm{d}w\,\mathrm{d}z,$$

where $F_\lambda(u)=\Gamma(u+1)\prod_{i=1}^{\infty}\frac{u+i}{u-\lambda_i+i}$ and the double contour integral runs over counterclockwise paths γ_w and γ_z such that

- *γ_w is inside (resp. outside) γ_z if $t_1 \geq t_2$ (resp. $t_1 < t_2$);*
- *γ_w and γ_z contain all the integers in $[-\ell(\lambda), x_1-1]$ and in $[x_2, \lambda_1-1]$ respectively;*
- *the ratio $\frac{1}{z-w}$ remains uniformly bounded.*

Proof strategy 4 – Rewriting the kernel

Consequence of Gorin–Rahman's formula:

$$\mathbb{E}[H_T(x,t)] = \int_0^t K((x,s),(x,s))ds$$

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To compute $\lim_{N\to+\infty} \frac{1}{\sqrt{N}} H_{T_N}\left(\lfloor x\sqrt{N}\rfloor, t\right)$, we look for a limit of

$$\frac{1}{\sqrt{N}} K((\lfloor x\sqrt{N}\rfloor, s),(\lfloor x\sqrt{N}\rfloor, s)).$$

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Via Stirling approximation and standard calculus, we get

$$\frac{1}{\sqrt{N}} K\big((\lfloor x\sqrt{N}\rfloor, s),(\lfloor x\sqrt{N}\rfloor, s)\big) \approx -\frac{1}{(2\mathrm{i}\pi)^2} \cdot \oint_{\gamma_Z}\oint_{\gamma_W} e^{\sqrt{N}(S(W)-S(Z))} \frac{h(W,Z)}{W-Z} \, \mathrm{d}W \, \mathrm{d}Z,$$

where

$S(U) = g(U) - U\log(1-t_0) - \sum_{i=0}^m g(x_0 - \eta\, a_i + U) + \sum_{i=1}^m g(x_0 - \eta\, b_i + U)$

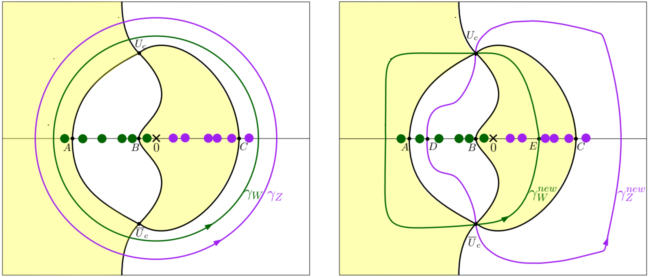
with $g(U) = U\log(U)$ and some function h.

Proof strategy 5 – Steepest descent analysis

Reminder: we are interested in

$$\oint_{\gamma_Z}\oint_{\gamma_W} e^{\sqrt{N}(S(W)-S(Z))}\,\frac{h(W,Z)}{W-Z}\,\mathrm{d}W\,\mathrm{d}Z\,.$$

Idea: deform γ_Z and γ_W such that $S(W) < S(Z)$ on the new contours.



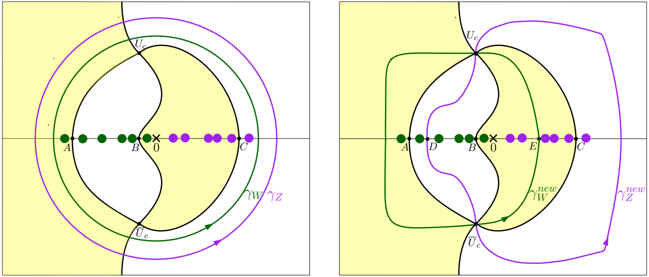
Schematic representation of the integration contours before and after transformation: in the white (resp. yellow) regions, we have $S(Z) > S(W)$ (resp. $S(Z) < S(W)$).

Proof strategy 5 – Steepest descent analysis

Reminder: we are interested in

$$\oint_{\gamma_Z}\oint_{\gamma_W} e^{\sqrt{N}(S(W)-S(Z))}\,\frac{h(W,Z)}{W-Z}\,\mathrm{d}W\,\mathrm{d}Z\,.$$

Idea: deform γ_Z and γ_W such that $S(W) < S(Z)$ on the new contours.



The point U_c on the above picture should satisfy $S'(U_c) = 0$, which is exactly the critical equation! (So the above picture is valid in the liquid region only.)

Proof strategy 5 – Steepest descent analysis

Reminder: we are interested in

$$\oint_{\gamma_Z}\oint_{\gamma_W} e^{\sqrt{N}(S(W)-S(Z))}\,\frac{h(W,Z)}{W-Z}\,\mathrm{d}W\,\mathrm{d}Z\,.$$

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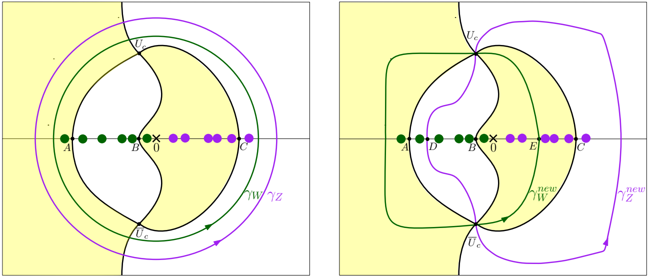

After change of contour, the integral tends to 0. The dominant term asymptotically is the residue term for the pole $W-Z$, which is an integral from $\overline{U_c}$ to U_c.