Tamari intervals and blossoming trees

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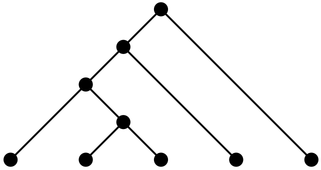
24 July 2024, Ruhr-Universität Bochum, FPSAC 2024

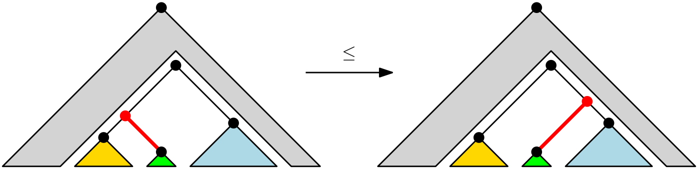
Binary trees

Binary trees : n binary internal nodes and $n+1$ leaves



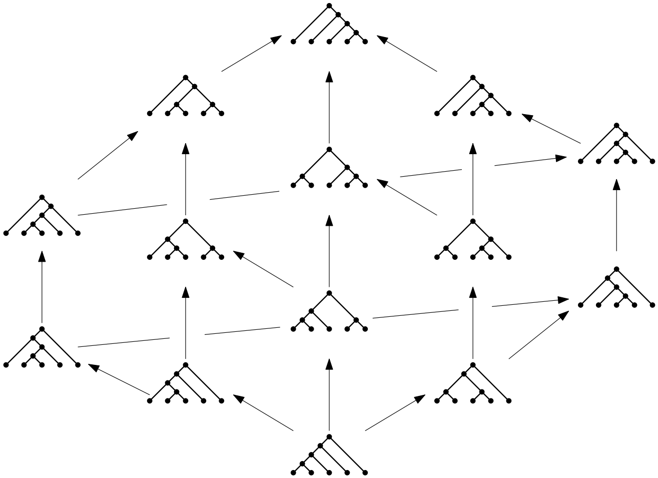
Counted by Catalan numbers: $\mathrm{Cat}_n = \frac{1}{2n+1}\binom{2n+1}{n}$

Rotation (from left to right) :



Tamari lattice

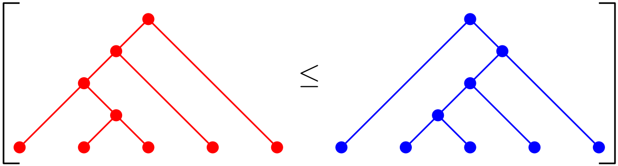
Left-to-right rotation defines a self-dual lattice (Tamari 1962)



Deep links with subjects in combinatorics, and many generalizations!

The next level: intervals

A Tamari interval: $[S, T]$ of binary trees with $S \leq T$



Motivation: conjecturally related to trivariate diagonal coinvariant spaces, also with operads… and nice numbers!

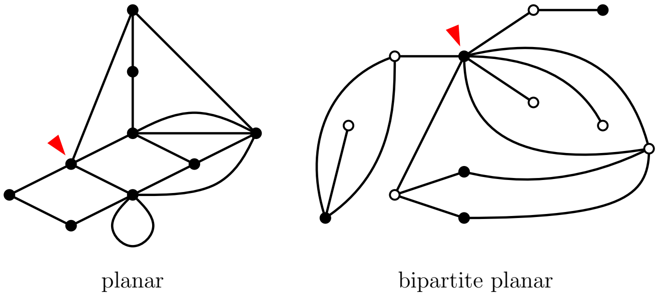
Many interesting families!

- Synchronized intervals: ν-Tamari
- New/modern intervals: enumeration, algebraic link
- Infinitely modern intervals: further restriction
- Kreweras intervals: algebraic link

They are often in bijection with families of planar maps!

What is a planar map?

Planar map: drawing of graphs on a plane without extra crossing



They are rooted, *i.e.*, with a marked corner.

Also many interesting families: triangulation, bipartite, ...

Tamari intervals and planar maps

Intervals	Formula	Planar maps
General	$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$	bridgeless 3-connected triangulation
Synchronized	$\frac{2}{n(n+1)}\binom{3n}{n-1}$	non-separable
New/modern	$\frac{3 \cdot 2^{n-2}}{n(n+1)}\binom{2n-2}{n-1}$	bipartite
Kreweras	$\frac{1}{2n+1}\binom{3n}{n}$	stacked triangulation

Also in bijection with other objects: interval posets, closed flow in forest, fighting fish, λ-term, ...

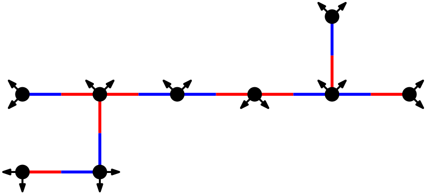
Many have worked on them: Bernardi, Bonichon, Bousquet-Mélou, Ceballos, Chapoton, Châtel, Chenevière, Combe, Duchi, F., Fusy, Henriet, Humbert, Préville-Ratelle, Pons, Rognerud, Viennot, Zeilberger, ...

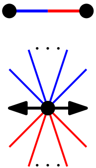
But a different equation / bijection for each family...

Our results

(Bicolored) Blossoming tree: an unrooted plane tree such that

- Each edge is half red and halef blue.
- Each node has two **buds**, splitting reds and blues.





Many variants, used a lot in enumeration of maps (Poulalhon–Schaeffer 2006)!

Theorem (F.–Fusy–Nadeau 2024+)

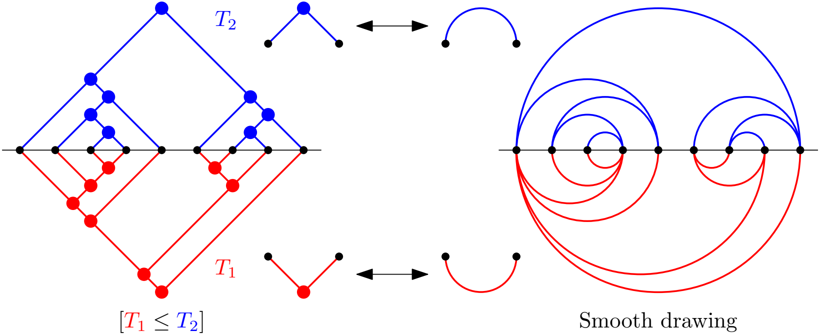
Tamari intervals of size n are in bijection with bicolored blossoming trees with n edges (thus $n+1$ nodes).

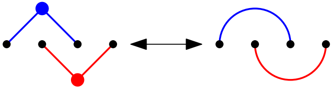
Inspired by interval-posets (Châtel–Pons 2015), giving **uniform** enumeration.

Many enumerative and structural consequences.

Canonical drawing and smooth drawing

Canonical drawing: larger tree on top, smaller tree flipped on bottom



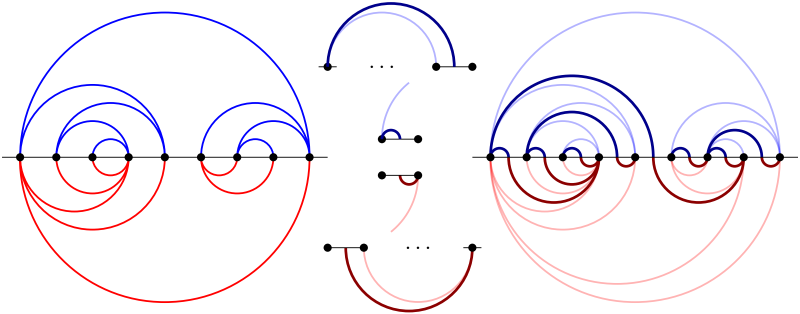
Forbidden  in Tamari interval

Smooth drawing: replace wedges by semi-circles

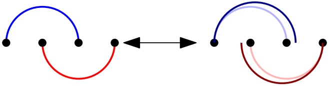
Each leaf has arcs of each color in **one direction**, due to **type**.

Smooth drawing and blossoming tree

To blossoming tree: each segment draws two half-edges

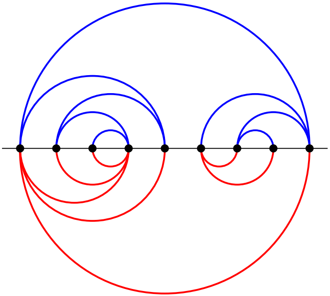


Smooth drawing

Forbidden  in Tamari interval

Smooth drawing and blossoming tree

Break the middle line into buds, conditions satisfied!



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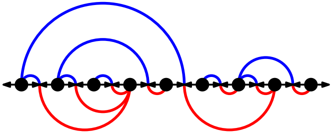
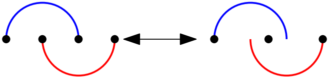
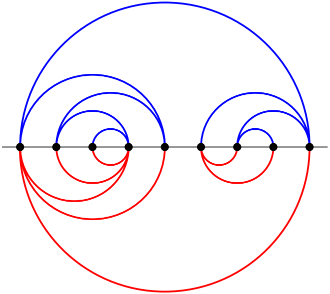


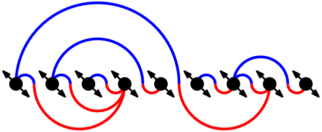
Forbidden  in Tamari interval

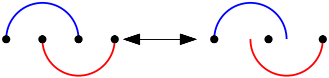
Smooth drawing and blossoming tree

Just wiggle a bit...



Smooth drawing



Forbidden  in Tamari interval

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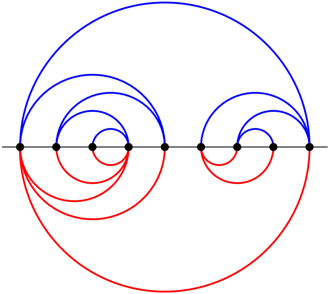

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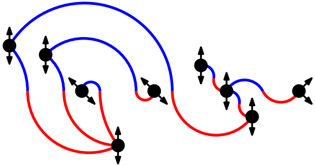
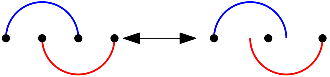


Forbidden  in Tamari interval

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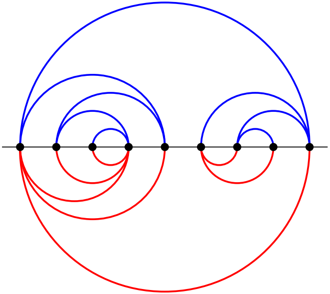

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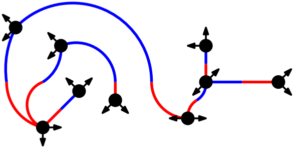
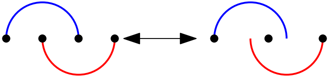


Forbidden  in Tamari interval

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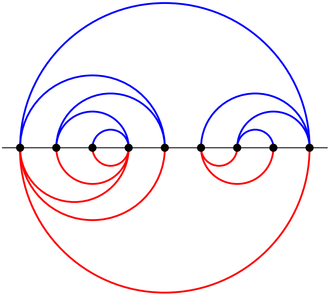
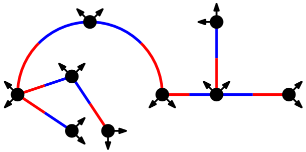


Smooth drawing



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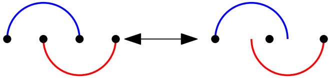
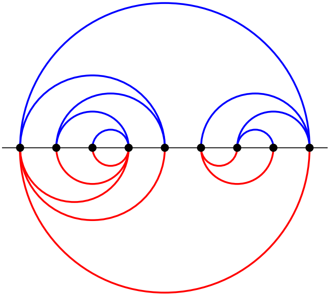


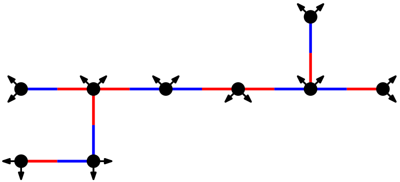
in Tamari interval

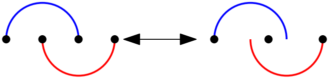
Smooth drawing and blossoming tree

… and we get a nice blossoming tree



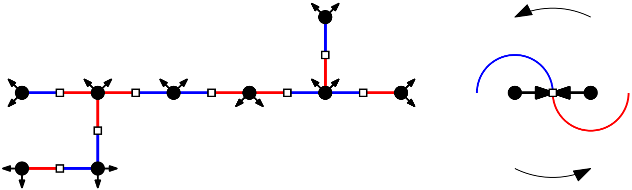
Smooth drawing



Forbidden  in Tamari interval

The reverse direction

To find the order of nodes, we do closure of buds to edges.



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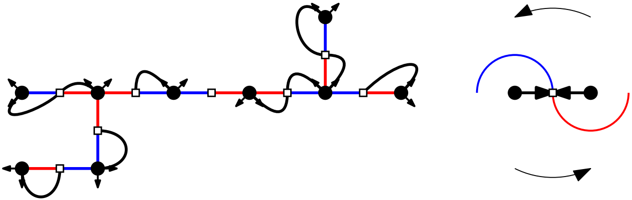

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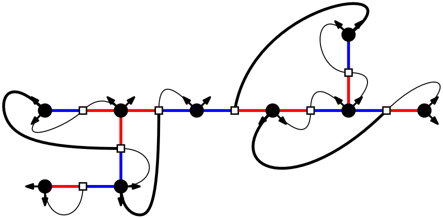

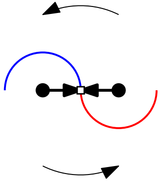

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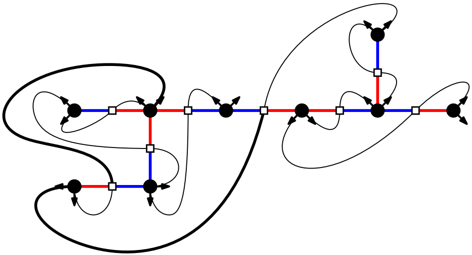
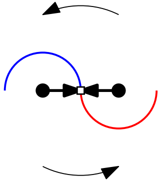
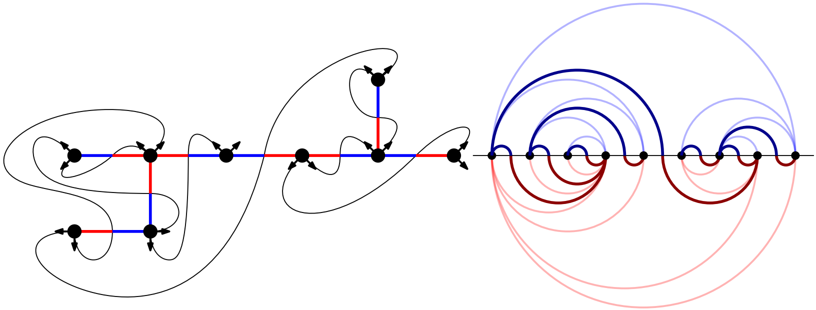




The reverse direction

Stretch the thread, and we get the trees.

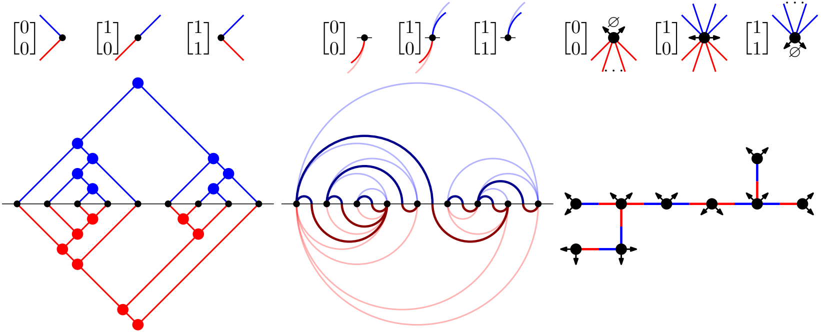


Refined statistics

Type of a leaf: 0 for right child, 1 for left child

Types of a node (pair of leaves): $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$

Statistics considered by Chapoton for new intervals.



Types in blossoming tree: presence of blue/red half-edges

First enumeration result

Theorem (Bostan–Chyzak–Pilaud 2023+)

The number of Tamari intervals of size n with k pairs of leaves of type $\begin{bmatrix}0\\0\end{bmatrix}$ or $\begin{bmatrix}1\\1\end{bmatrix}$ is

$$\frac{2}{n(n+1)}\binom{n+1}{k}\binom{3n}{k-2}.$$

Gives the f-vector of canonical complex of the Tamari lattice!

Synchronized intervals: special case $k = n + 1$

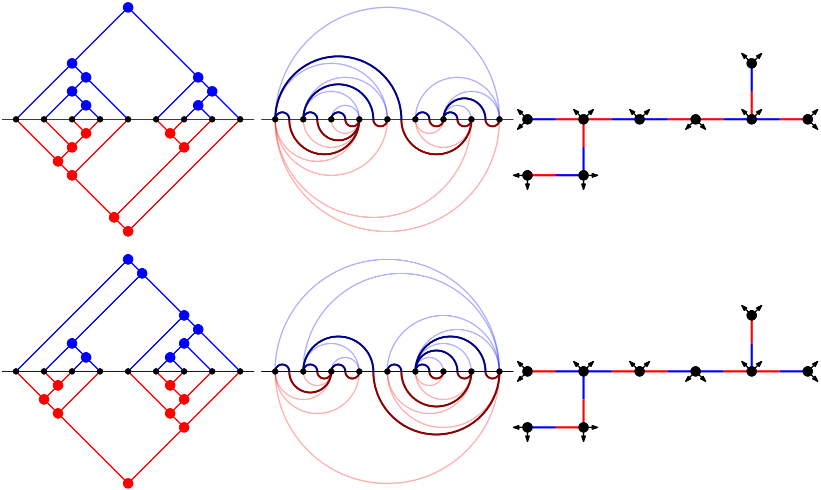
Obtained by solving functional equations.

Blossoming trees: k nodes with adjacent buds among $n + 1$ nodes.

Cyclic lemma suffices!

Duality

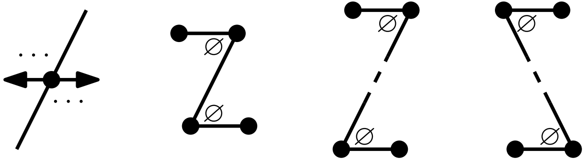
Duality on Tamari intervals: just a half-turn.



Duality on blossoming trees: just exchanging colors.

Unified enumeration

Interesting families can be described by forbidden patterns!



Synchronized Modern Infinitely modern Kreweras

Another proof of bijection in the spirit of (Poulalhon–Schaeffer 2006) for

- General intervals ↔ triangulations (Bernardi–Bonichon 2009)
- Synchronized ↔ non-separable maps (F.–Préville-Ratelle 2017)
- Kreweras ↔ ternary trees (Bernardi–Bonichon 2009)

Leads to different tree specifications, thus unified enumeration.

Even with refined by node types and intersection of families!

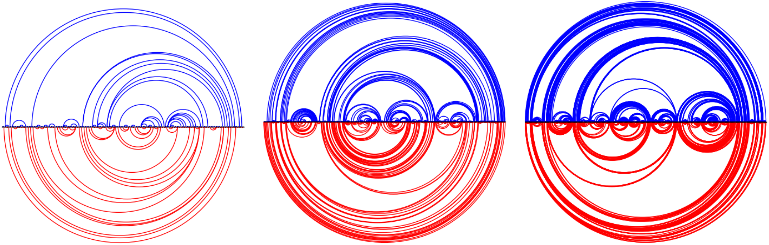
Self-dual sub-family: those stable by exchanging colors. Doable!

Enumeration results

Types	General size n	Self-dual size $2k$	Self-dual size $2k+1$
General	$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$	$\frac{1}{3k+1}\binom{4k}{k}$	$\frac{1}{k+1}\binom{4k+2}{k}$
Synchronized	$\frac{2}{n(n+1)}\binom{3n}{n-1}$	0	$\frac{1}{k+1}\binom{3k+1}{k}$
Modern / new for size-1	$\frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}$	$\frac{2^{k-1}}{k+1}\binom{2k}{k}$	$\frac{2^{k}}{k+1}\binom{2k}{k}$
Modern and synchronized	$\frac{1}{n+1}\binom{2n}{n}$	0	$\frac{1}{k+1}\binom{2k}{k}$
Inf. modern / Kreweras	$\frac{1}{2n+1}\binom{3n}{n}$	$\frac{1}{2k+1}\binom{3k}{k}$	$\frac{1}{k+1}\binom{3k+1}{k}$

Discussion

- Mysterious involution: reflection on blossoming trees
  - Exchanges infinitely modern and Kreweras
  - What are the images of modern intervals?
- How to explain Reiner's observation:
  self-dual intervals = q-analogue of # general intervals with $q = -1$?
- Breaks the order in canopy, so hard to get m-Tamari?
- Large scale structure?



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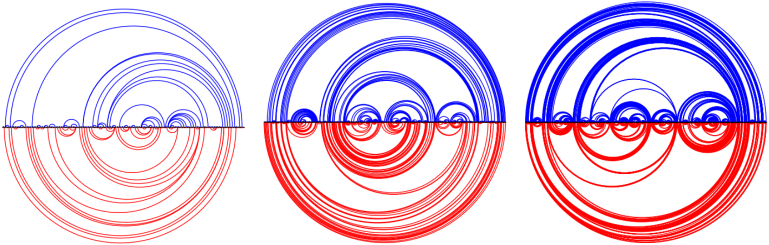

Thank you for listening!