Bijections for Graphs and Lawrence Polytopes

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Introduction I

Let G be a finite undirected connected graph. By the matrix-tree theorem, we have

$$|\mathrm{Jac}(G)| = \#\text{spanning trees of } G$$

- $\mathrm{Jac}(G)$ is the Jacobian group/sandpile group/Picard group/critical group of G.
- A vague question: under certain conditions, is there a canonical simply transitive action of $\mathrm{Jac}(G)$ on the set of spanning trees? [Jordan Ellenberg, Mathoverflow 2011]
- For plane graphs (not just planar), the answer is yes. [Chan-Church-Grochow '18], [Baker-Wang '18], [Ganguly-McDonough '23], etc.
- For a general graph, we probably need some extra data to define the action.

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Introduction II

[Backman-Baker-Yuen '19] considered the following way to define the action:

$$\mathrm{Jac}(G) \curvearrowright \{\text{cycle-cocycle reversal classes of } G\} \longleftrightarrow \{\text{spanning trees of } G\}$$

- The group action is canonical and simply transitive [Backman '17].
- The bijection can be
  - the Backman-Baker-Yuen bijection, which relies on a cycle signature and a cocycle signature;
  - the Bernardi bijection, which relies on combinatorial embedding and initial data;
  - Gioan and Las Vergnas' active bijection, which relies on a total order on the edges of G.
- We introduce a family of bijections including BBY and Bernardi's bijections. The extra data needed to define the bijections are related to Lawrence polytopes.

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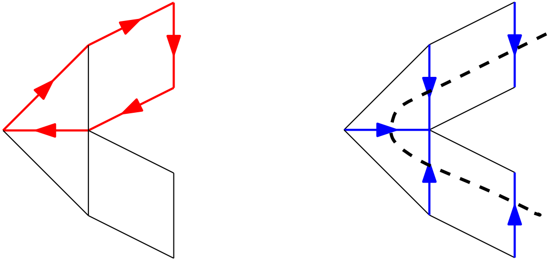
Section 1

The Action

$\mathrm{Jac}(G) \curvearrowright \{\text{cycle-cocycle reversal classes of } G\} \longleftrightarrow \{\text{spanning trees of } G\}$

Directed Cycles and Cocycles (Minimal Cuts)

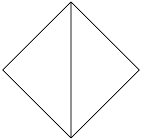


a directed cycle

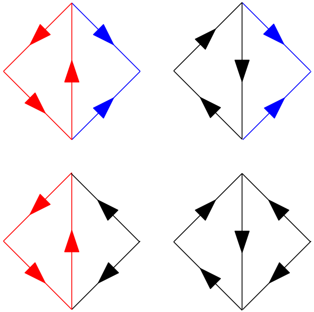
a directed cocycle

Cycle-Cocycle Reversals Classes of Orientations

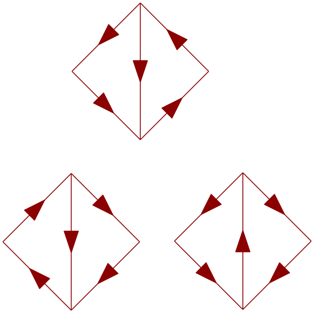
Gioan introduced the cycle-cocycle reversal classes of orientations of G. To obtain one class from a given orientation, we reverse the directed cycles and cocycles in all the possible ways.



G

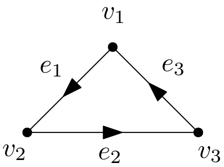


a cycle-cocycle reversal class



another cycle-cocycle reversal class

The Jacobian Group



Reference orientation

$$\begin{array}{cc} \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \end{array} & \begin{array}{c} \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \\ \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \end{array} \end{array}$$

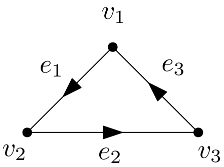
$$\begin{array}{cc} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} & \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \end{array}$$

directed cycle

- Let M be the incidence matrix of G w.r.t. the reference orientation.
- $\ker_{\mathbb{Z}}(M) := \ker(M) \cap \mathbb{Z}^E = \langle \text{directed cycles} \rangle$.
- $\mathrm{Im}_{\mathbb{Z}}(M^T) := \mathrm{Im}(M^T) \cap \mathbb{Z}^E = \langle \text{directed cocycles} \rangle$.
- The Jacobian group

$$\mathrm{Jac}(G) := \mathbb{Z}^E / (\ker_{\mathbb{Z}}(M) \oplus \mathrm{Im}_{\mathbb{Z}}(M^T)).$$

The Jacobian Group



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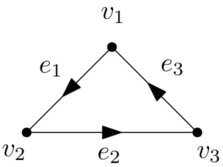

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The Canonical Action

$$\mathrm{Jac}(G) \curvearrowright \{\text{cycle-cocycle reversal classes of } G\} \longleftrightarrow \{\text{spanning trees of } G\}$$

- Cycle-cocycle reversal classes = orientations/(cycle reversals, cocycle reversals).
- $\mathrm{Jac}(G) = \mathbb{Z}^E/(\text{directed cycles, directed cocycles})$.

Theorem(Backman '17, Backman-Baker-Yuen '19)

The Jacobian group $\mathrm{Jac}(G)$ admits a simply transitive group action on the set of cycle-cocycle reversal classes of G.

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Section 2

Lawrence Polytopes and Bijections

$$\mathrm{Jac}(G) \curvearrowright \{\text{cycle-cocycle reversal classes of } G\} \longleftrightarrow \{\text{spanning trees of } G\}$$

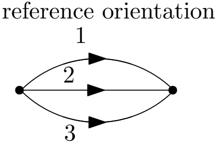
Lawrence Polytopes

- Let M be a totally unimodular matrix that represents the graphic matroid associated to a graph G. E.g., let M be the incidence matrix of G (and remove a row).
- Construct the Lawrence matrix ($n = |E|$, $r = |T|$):

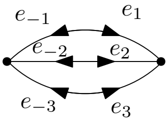
$$\begin{pmatrix} M_{r \times n} & 0 \\ I_{n \times n} & I_{n \times n} \end{pmatrix}.$$

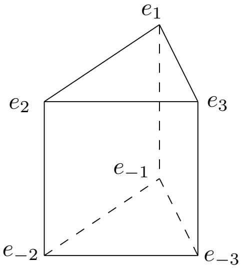
- Let $e_1, e_2, \cdots, e_n, e_{-1}, e_{-2}, \cdots, e_{-n}$ be the column vectors.
- The <u>Lawrence polytope</u> P is defined to be the convex hull of $e_1, \cdots, e_n, e_{-1}, \cdots, e_{-n}$.
- Take the dual matroid and define the <u>Lawrence polytope</u> P^*.

Example and Correspondence 1



e_1	e_2	e_3	e_{-1}	e_{-2}	e_{-3}
1	1	1	0	0	0
1	0	0	1	0	0
0	1	0	0	1	0
0	0	1	0	0	1

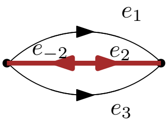




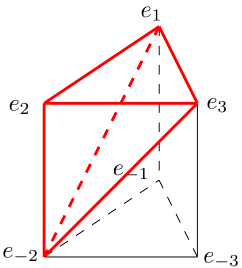
Correspondence 1:

$\{\text{vertices of } P\} \longleftrightarrow \{\text{arcs of } G\}$

Example and Correspondence 2



an externally oriented tree $\overrightarrow{T}$



Correspondence 2:

$\{\text{max simplices of } P\} \longleftrightarrow$

$\{\text{externally oriented trees of } G\}$

Definition

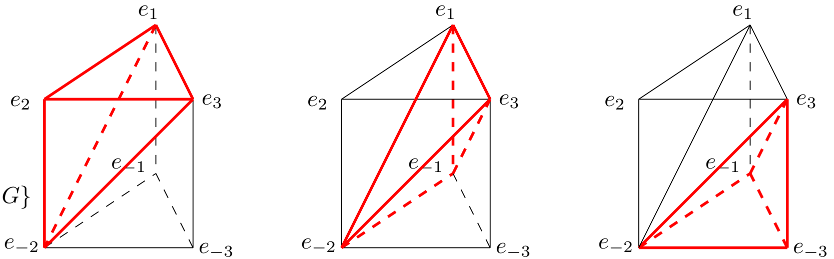
- A <u>fourientation</u> is a choice for each edge of G whether to make it one-way oriented, leave it unoriented, or biorient it.
- For a tree T, an <u>externally oriented tree</u> $\overrightarrow{T}$ is a fourientation where all the internal edges are bioriented and all the external edges are one-way oriented.

Example and Correspondence 3

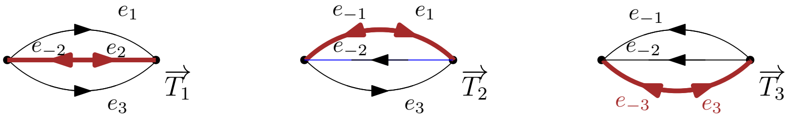
Correspondence 3:

$\{\text{triangulations of } P\} \longleftrightarrow$

$\{\text{triangulating external atlases of } G\}$



An external atlases $\mathcal{A}$:

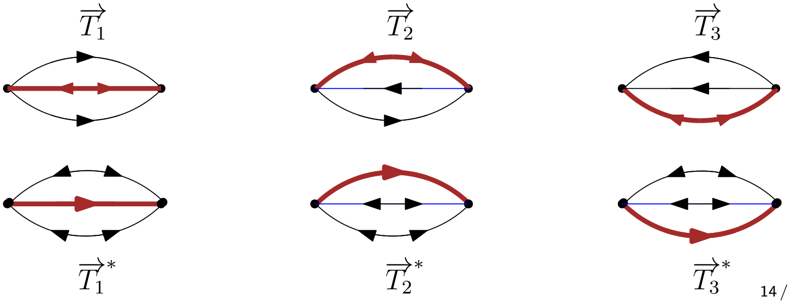


Definition

An <u>external atlas</u> $\mathcal{A}$ of G is a collection of externally oriented trees $\overrightarrow{T}$ such that each tree of G appears exactly once. It is called <u>triangulating</u> if the fourientation $\overrightarrow{T_i} \cap (-\overrightarrow{T_j})$ contains no directed cycle for any $\overrightarrow{T_i}, \overrightarrow{T_j} \in \mathcal{A}$.

The Lawrence Polytope P^*

- An external atlas $\mathcal{A}$ tells us how to orient external edges for every tree. Triangulations of P orient the external edges in an "interesting" way.
- Dually, triangulations of the Lawrence polytope P^* tell us how to orient internal edges for every tree in an "interesting" way.
- We define internally oriented trees $\overrightarrow{T}^*$ and internal atlases $\mathcal{A}^*$.



$\overrightarrow{T_1}$ $\overrightarrow{T_2}$ $\overrightarrow{T_3}$

An external atlases $\mathcal{A}$:

An internal atlases $\mathcal{A}^*$:

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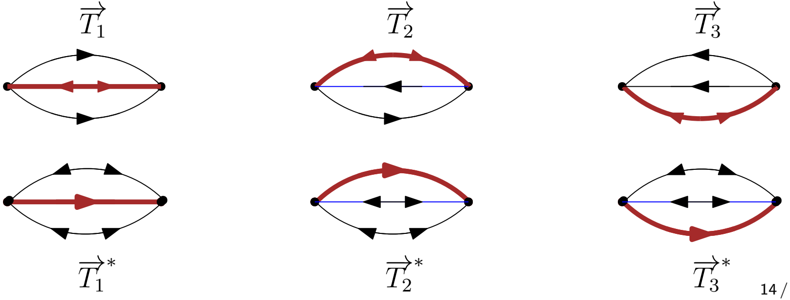

Triangulating Internal Atlases

Definition

1. An external atlas $\mathcal{A}$ is triangulating

   if the fourientation $\overrightarrow{T}_i \cap (-\overrightarrow{T}_j)$ contains no directed cycle for any $\overrightarrow{T}_i, \overrightarrow{T}_j \in \mathcal{A}$.

2. An internal atlas $\mathcal{A}^*$ is triangulating

   if the fourientation $\overrightarrow{T}_i{}^* \cap (-\overrightarrow{T}_j{}^*)$ contains no directed cocycle for any $\overrightarrow{T}_i{}^*, \overrightarrow{T}_j{}^* \in \mathcal{A}^*$.

Framework: From Atlases to Tree-Orientation Correspondences

From a pair of atlases $(\mathcal{A}, \mathcal{A}^*)$, we obtain a map

$$\begin{array}{rccc} f_{\mathcal{A},\mathcal{A}^*}: \{\text{trees}\} & \longrightarrow & \{\text{orientations}\} \\ T & \mapsto & \overrightarrow{T} \cap \overrightarrow{T}^* \end{array}$$

An external atlases $\mathcal{A}$:

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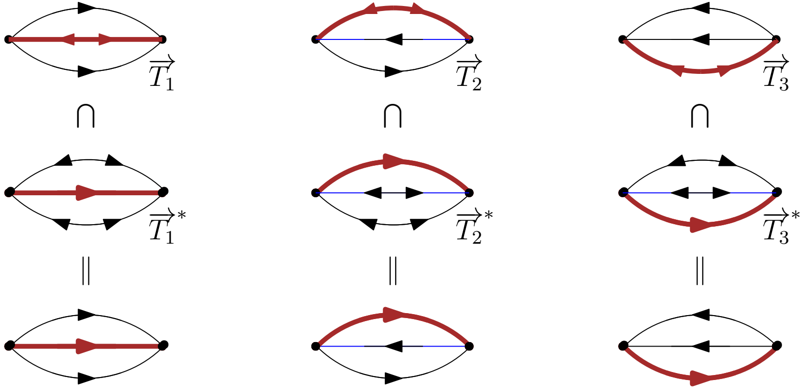
$f_{\mathcal{A},\mathcal{A}^*}$

trees $\longrightarrow$ orientations



Main Result

Theorem(D. '23+)

Given a pair of triangulating atlases $(\mathcal{A}, \mathcal{A}^*)$, the map

$$\begin{aligned} \overline{f_{\mathcal{A},\mathcal{A}^*}} : \{\text{trees}\} &\longrightarrow \{\text{cycle-cocycle reversal classes}\} \\ T &\mapsto [\overrightarrow{T} \cap \overrightarrow{T^*}] \end{aligned}$$

is bijective.

Remark 1: our proof uses the combinatorial definition of triangulating atlases rather than the geometric property of triangulations.

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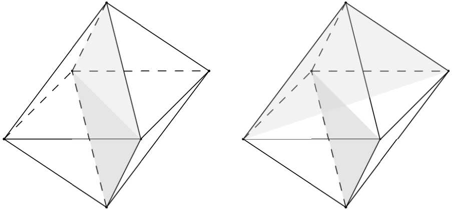
Remark 2: the result still holds if one of the triangulations is a dissection.



triangulation

dissection

Section 3

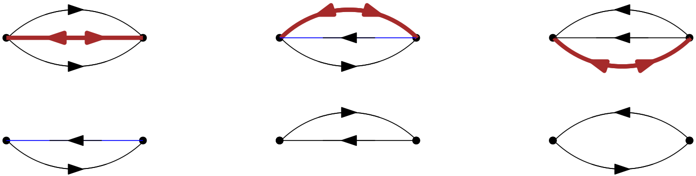
Specializations and Extensions

- Our bijections include the BBY bijections: they correspond to regular triangulations.
- Our bijections include the Bernardi bijections: they correspond to a subclass of dissections.
- Our bijections can be canonically extended to a subgraph-orientation correspondence.

Cycle Signatures

For a triangulating external atlas $\mathcal{A}$, we collect all the directed cycles $\overrightarrow{C}$ from $\mathcal{A}$:

$$\{\text{directed cycle } \overrightarrow{C} : \overrightarrow{C} \subseteq \overrightarrow{T} \text{ for some } \overrightarrow{T} \in \mathcal{A}\}.$$



A triangulating external atlas:

A cycle signature:

This set always gives a cycle signature.

Definition

A cycle signature of G is the choice of a direction for each cycle of G.

$\{$triangulating atlases$\}$ $\longleftrightarrow$ $\{$triangulating cycle signatures$\}$.

Dissections do not induce signatures in general.

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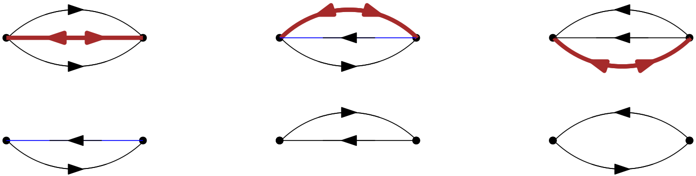

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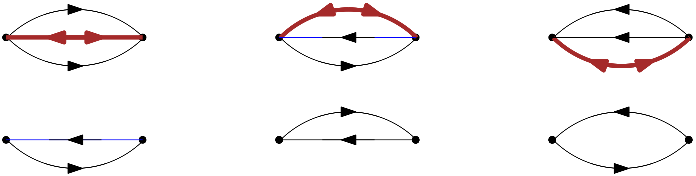

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Regular Triangulations and the BBY Bijections

Theorem(D. ’23+)

$$\begin{array}{ccc} \{\text{triangulations of } P\} & \longleftrightarrow & \{\underline{\text{triangulating cycle signatures}}\} \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ \{\underline{\text{regular triangulations}} \text{ of } P\} & \longleftrightarrow & \{\underline{\text{acyclic cycle signatures}}\} \end{array}$$

- Similar results hold for P^* and cocycle signatures.
- Backman, Baker, and Yuen use an acyclic cycle signature and an acyclic cocycle signature of G to define the BBY bijection.
- In our language, their BBY bijection is defined using a pair of special triangulating atlases/triangulations.

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- Similar results hold for P^* and cocycle signatures.
- Backman, Baker, and Yuen use an acyclic cycle signature and an acyclic cocycle signature of G to define the BBY bijection.
- In our language, their BBY bijection is defined using a pair of special triangulating atlases/triangulations.

Other Interesting Results on Signatures

- Gleb Nenashev helped me prove that

  a cycle signature of a graph is triangulating

  ⇔ for any three directed cycles in the signature, the sum is non-zero.

  We don't know whether similar results hold for regular matroids.

- (Backman-Santos-Yuen '23+) For oriented matroids M (which contain regular matroids),

  a circuit signature is triangulating

  ⇔ it is induced by a generic single-element lifting

  ⇔ it is induced by a triangulation of the Lawrence polytope of M.

  The dual result also holds.

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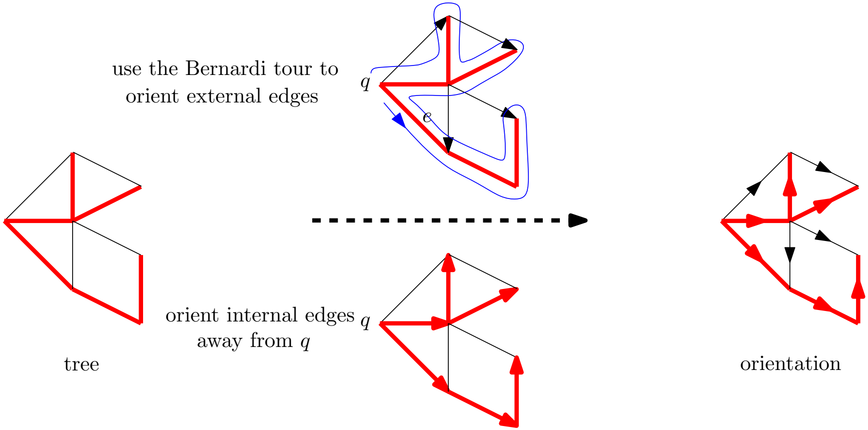
The Bernardi Bijections

The Bernardi bijection is defined for a connected graph embedded into an oriented surface with initial data (q, e).



The Bernardi Bijections and Dissections

- The Bernardi bijections cannot be defined using signatures in general.
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Extensions

type of subgraph	orientation class	cardinality
trees	cycle-cocycle reversal classes	$T_G(1,1)$
forests	cycle reversal classes	$T_G(2,1)$
connected subgraphs	cocycle reversal classes	$T_G(1,2)$
subgraphs	orientations	$T_G(2,2)$

- Recall: given one triangulations and one dissection, the map

  $$\begin{aligned} \overline{f_{\mathcal{A},\mathcal{A}^*}} : \{\text{trees}\} &\longrightarrow \{\text{cycle-cocycle reversal classes}\} \\ T &\mapsto [\overrightarrow{T} \cap \overrightarrow{T^*}] \end{aligned}$$

  is bijective.

- We extend the map $f_{\mathcal{A},\mathcal{A}^*}$ to a subgraph-orientation correspondence that establishes bijections for all the four rows in the table. The extension does not require extra data.

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The End

Thank you!