Invariant Theory for the Face Algebra of the Braid Arrangement

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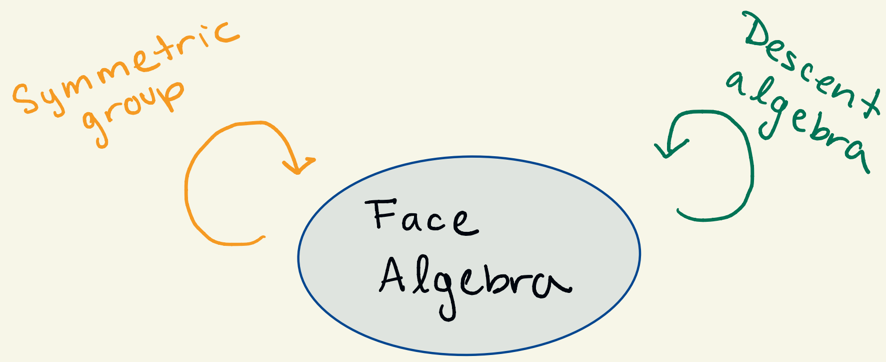
FPSAC 2024 BOCHUM, GERMANY

Big Ideas

① Two connected combinatorial algebras:

② Interplay between two actions:

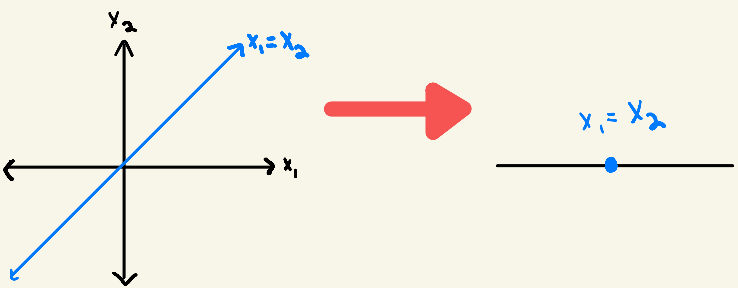


The Braid Arrangement

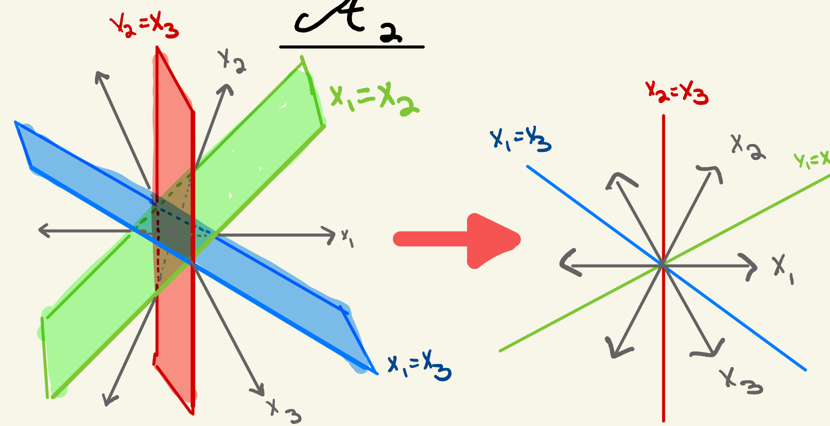
The Braid Arrangement $\mathcal{A}_{n-1}$ is the collection of hyperplanes

$$\{ x_i = x_j \mid 1 \leq i < j \leq n \} \subseteq \mathbb{R}^n.$$

$\mathcal{A}_1$

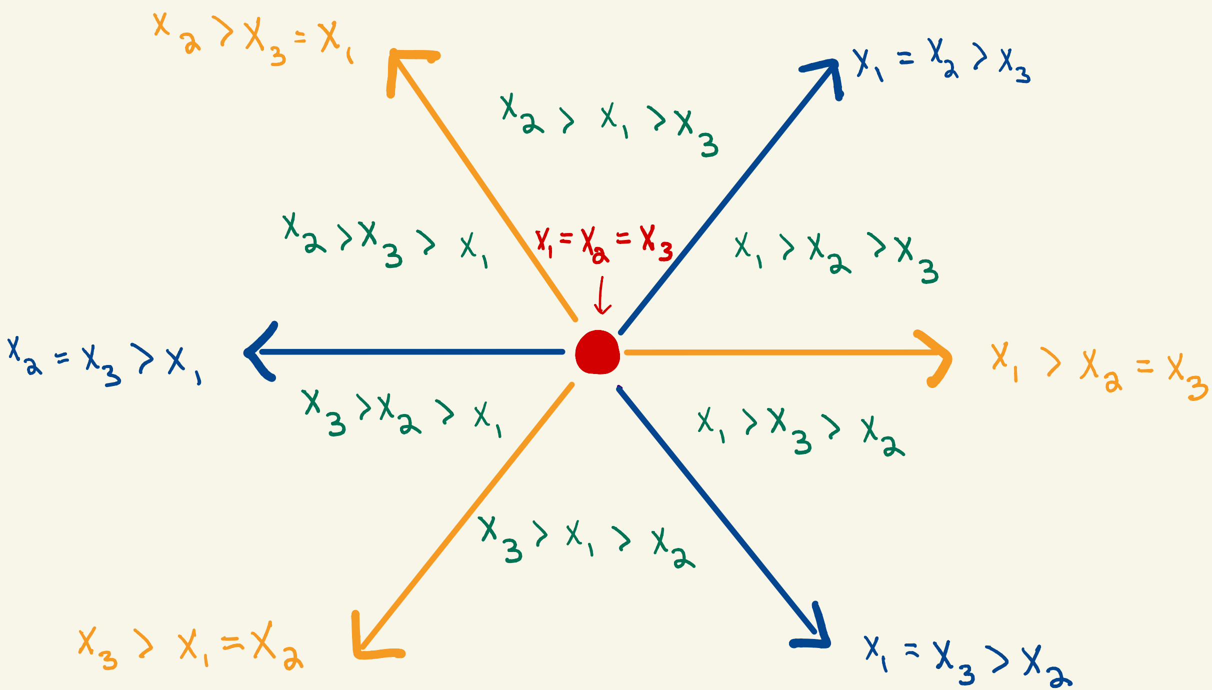


$\mathcal{A}_2$

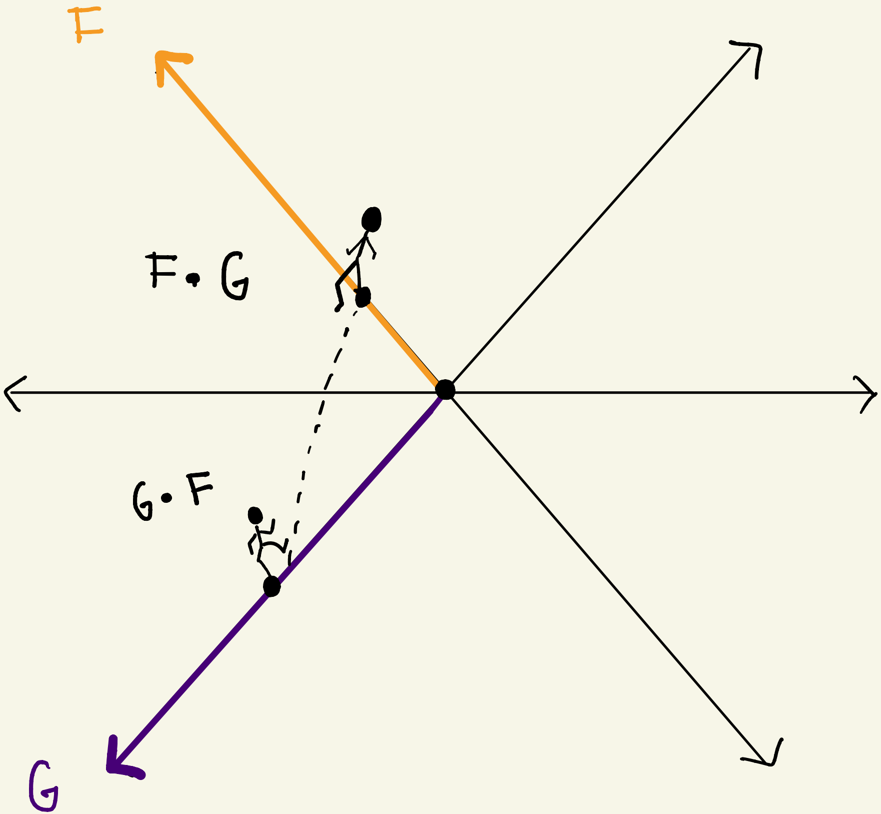


Faces of $\mathcal{A}_{n-1}$

The faces of $\mathcal{A}_{n-1}$ are indexed by ordered set partitions of $\{1, 2, \ldots, n\}$ and acted on by the symmetric group S_n.



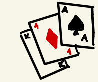
Face Multiplication



- The faces of $\mathcal{A}_{n-1}$ form a monoid $\widetilde{\mathcal{F}}_n$.
- S_n acts on $\widetilde{\mathcal{F}}_n$ by monoid automorphisms

The face algebra $\mathbb{C}\mathcal{F}_n$:

The face algebra $\mathbb{C}\mathcal{F}_n$ is studied:

①  In the context of Markov Chains / Card Shuffling 

Bidigare, Brown, Diaconis, Denham, Hanlon, Lafrenière, Reiner, Rockmore Saliola, Uyemura-Reyes, Welker, and more

② As an algebra with combinatorial representation theory:

Aguiar-Mahajan, Bastidas, Bidigare, Saliola, Schocker, and more

③ For its connections to Solomon's descent algebra

Aguiar-Mahajan, Saliola, Schocker, and more

Solomon's descent algebra

- The descent set $\mathrm{Des}(\sigma)$ of a permutation $\sigma \in S_n$ consists of the positions $1 \le i \le n-1$ for which $\sigma(i) > \sigma(i+1)$.

  ↳ $\mathrm{Des}(\underline{7}\,\underline{5}\,1\,\underline{3}\,2\,\underline{6}\,4\,8\,9) = \{1,2,4,6\}$

- For $J \subseteq \{1,2,\ldots,n-1\}$, define

$$X_J := \sum_{\substack{\sigma \in S_n : \\ \mathrm{Des}(\sigma) = J}} \sigma \in \mathbb{C} S_n.$$

- $\mathrm{span}_{\mathbb{C}} \{X_J \mid J \subseteq \{1,2,\ldots,n-1\}\}$ forms a subalgebra of $\mathbb{C}S_n$ called Solomon's descent algebra: Σ_n.

The key connection

Thm (Bidigare): The descent algebra is anti-isomorphic to the S_n-invariant subalgebra of the face algebra:

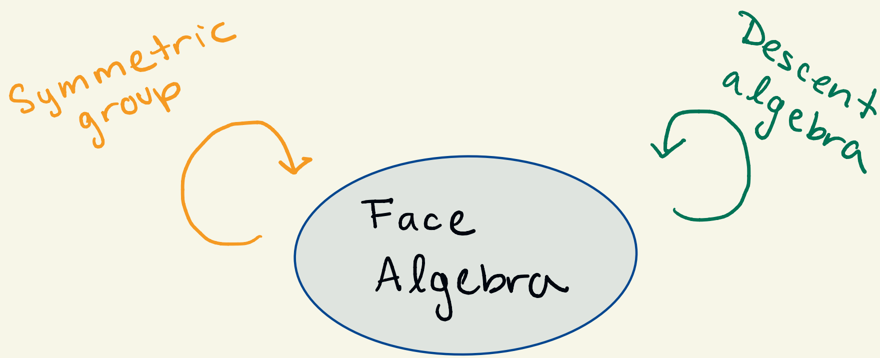
$$\Sigma_n^{opp} \cong (\mathbb{C}\mathcal{F}_n)^{S_n}.$$

↳ A useful way to view Σ_n: Saliola computed its quiver using this embedding

Big Ideas

① Two connected combinatorial algebras:

② Interplay between two actions:

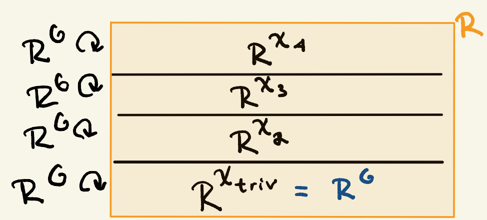


Isotypic components: Generalizations of invariant subalgebras

Finite group G acting on a fin. dim. $\mathbb{C}$-algebra R by algebra homomorphisms

Invariant Subalgebra   **G-isotypic components**

Each χ-isotypic subspace R^{χ} is an R^G-module.



S_n- isotypic component of $\mathbb{C}\mathcal{F}_n$

Bidigare studied $(\mathbb{C}\mathcal{F}_n)^{\chi^\nu}$ for χ^ν ← symmetric group character

- the trivial character χ^n
- the sign character χ^{1^n}

Each S_n- isotypic component is a $(\mathbb{C}\mathcal{F}_n)^{S_n}$- module ..... or equivalently, a (right) Σ_n- module.

Main Question

How does each S_n-isotypic component $(\mathbb{C}\mathcal{F}_n)^{\chi^\nu}$ look as a Σ_n- module?

$(\mathbb{C}\mathcal{F}_n)^{\chi_\mu}$? ?	$\circlearrowleft \Sigma_n$
$\vdots$	
$(\mathbb{C}\mathcal{F}_n)^{\chi_\lambda}$? ?	$\circlearrowleft \Sigma_n$
$(\mathbb{C}\mathcal{F}_n)^{\chi_\nu}$? ?	$\circlearrowleft \Sigma_n$
$(\mathbb{C}\mathcal{F}_n)^{S_n} \cong \Sigma_n^{opp}$	$\circlearrowleft \Sigma_n$

(Right) Representation Theory of Σ_n

- Studied in depth by Garsia–Reutenauer, generalized to other types by F. Bergeron, N. Bergeron, Howlett, Taylor

- Integer partitions of n $\longleftrightarrow$ simple modules $\longleftrightarrow$ Idempotents in a complete family of primitive, orthogonal idempotents

  $\lambda \qquad M_\lambda \qquad E_\lambda$

  one-dimensional!

- Σ_n is not semisimple $\longrightarrow$ can't (necessarily) decompose Σ_n-rep'ns into direct sums of simples

- Alternative way to understand a Σ_n-rep V:

  ↳ Count the composition multiplicity $[V : M_\lambda]$ for each λ

The isotypic components: A first approximation

Proposition As Σ_n-representations

$$(\mathbb{C}\mathcal{F}_n)^{\chi_\nu} = \bigoplus_{\mu \text{ dominates } \nu} (\mathbb{C}\mathcal{F}_n E_\mu)^{\chi_\nu}, \quad \text{with}$$

$$\dim_{\mathbb{C}} (\mathbb{C}\mathcal{F}_n E_\mu)^{\chi_\nu} = \#\{\text{compositions } \alpha \mid \alpha \text{ rearranges to } \mu\} \cdot \#SYT(\nu) \cdot \underbrace{K_{\nu,\mu}}_{\text{Kostka numbers}}$$

Abuse of notation*: viewing E_μ in $(\mathbb{C}\mathcal{F}_n)^{S_n}$ rather than Σ_n

A first approximation

$\mathbb{C}\mathcal{I}_4$

S_n-isotypic subspaces (ν)

	$\mathbb{C}\mathcal{I}_n E_4$	$\mathbb{C}\mathcal{I}_n E_{31}$	$\mathbb{C}\mathcal{I}_n E_{22}$	$\mathbb{C}\mathcal{I}_n E_{211}$	$\mathbb{C}\mathcal{I}_n E_{1111}$
χ_{1111}					$1\cdot 1\cdot 1$
χ_{211}				$3\cdot 3\cdot 1$	$1\cdot 3\cdot 3$
χ_{22}			$1\cdot 2\cdot 1$	$3\cdot 2\cdot 1$	$1\cdot 2\cdot 2$
χ_{31}		$2\cdot 3\cdot 1$	$1\cdot 3\cdot 1$	$3\cdot 3\cdot 2$	$1\cdot 3\cdot 3$
$\chi_4 = \chi_{triv}$	$1\cdot 1\cdot 1$	$2\cdot 1\cdot 1$	$1\cdot 1\cdot 1$	$3\cdot 1\cdot 1$	$1\cdot 1\cdot 1$

$\#\alpha \models n \sim \mu$

$\#SYT(\nu)$

$K_{\nu,\mu}$

μ

A Better Answer

Fill in submodules with Σ_n-composition factors M_λ

S_n-isotypic subspaces (ν)

	$\mathbb{C}\mathcal{J}_n E_4$	$\mathbb{C}\mathcal{J}_n E_{31}$	$\mathbb{C}\mathcal{J}_n E_{22}$	$\mathbb{C}\mathcal{J}_n E_{211}$	$\mathbb{C}\mathcal{J}_n E_{1111}$
χ_{1111}					M_{22}
χ_{211}				$3M_{22}$ $3M_{31}$ $3M_{4}$	$3M_{211}$ $3M_{4}$ $3M_{31}$
χ_{22}			$2M_{22}$	$2M_{211}$ $2M_{31}$ $2M_{4}$	$2M_{31}$ $2M_{22}$
χ_{31}		$3M_{4}$ $3M_{31}$	$3M_{4}$	$3M_{211}$ $6M_{31}$ $3M_{22}$ $6M_{4}$	$3M_{211}$ $3M_{4}$ $3M_{31}$
$\chi_4 = \chi_{triv}$	M_{4}	M_{4} M_{31}	M_{22}	M_{211} M_{31} M_{4}	M_{1111}

μ

Previously Known

S_n-isotypic subspaces	$\mathbb{C}\mathcal{J}_n E_4$	$\mathbb{C}\mathcal{J}_n E_{31}$	$\mathbb{C}\mathcal{J}_n E_{22}$	$\mathbb{C}\mathcal{J}_n E_{211}$	$\mathbb{C}\mathcal{J}_n E_{1111}$
χ_{1111}					$M_{(2,2)}$
χ_{211}				$3M_{(2,2)}$ $3M_{(3,1)}$ $3M_{(4)}$	$3M_{(2,1,1)}$ $3M_{(4)}$ $3M_{(3,1)}$
χ_{22}			$2M_{(2,2)}$	$2M_{(2,1,1)}$ $2M_{(3,1)}$ $2M_{(4)}$	$2M_{(3,1)}$ $2M_{(2,2)}$
χ_{31}		$3M_{(4)}$ $3M_{(3,1)}$	$3M_{(4)}$	$3M_{(2,1,1)}$ $6M_{(3,1)}$ $3M_{(2,2)}$ $6M_{(4)}$	$3M_{(2,1,1)}$ $3M_{(4)}$ $3M_{(3,1)}$
χ_4	$M_{(4)}$	$M_{(4)}$ $M_{(3,1)}$	$M_{(2,2)}$	$M_{(2,1,1)}$ $M_{(3,1)}$ $M_{(4)}$	$M_{(1,1,1,1)}$

Rightmost column: Uyemura-Reyes's shuffling rep'ns

Bottom row: Garsia–Reutenauer's Cartan invariants of Σ_n

Previously Known: At a glance

χ_{1111}					M_{22}
χ_{211}				$3M_{22}$ $3M_{31}$ $3M_{4}$	$3M_{211}$ $3M_{4}$ $3M_{31}$
χ_{22}			$2M_{22}$	$2M_{211}$ $2M_{31}$ $2M_{4}$	$2M_{31}$ $2M_{22}$
χ_{31}		$3M_{4}$ $3M_{22}$	$3M_{4}$	$3M_{211}$ $6M_{31}$ $3M_{22}$ $6M_{4}$	$3M_{211}$ $3M_{4}$ $3M_{31}$
χ_{4}	M_{4}	M_{4} M_{31}	M_{22}	M_{211} M_{31} M_{4}	M_{1111}
	$\mathbb{C}\mathcal{F}_n E_4$	$\mathbb{C}\mathcal{F}_n E_{31}$	$\mathbb{C}\mathcal{F}_n E_{22}$	$\mathbb{C}\mathcal{F}_n E_{211}$	$\mathbb{C}\mathcal{F}_n E_{1111}$

- **Thm (Garsia–Reutenauer, 1989)**

$$\left[\left(\mathbb{C}\tilde{\mathcal{F}}_n\right)^{\chi_{(n)}} E_\mu : M_\lambda \right] = \#\left\{ \begin{array}{l} \text{compositions} \\ \alpha \text{ rearranging to } \mu \end{array} \;\middle|\; \text{LyndonType}(\alpha) = \lambda \right\}.$$

- **Thm (Uyemura-Reyes, 2002)**

$$\left[\left(\mathbb{C}\tilde{\mathcal{F}}_n\right)^{\chi_\nu} E_{1111} : M_\lambda \right] = \#\mathrm{SYT}(\nu) \cdot \langle s_\nu , \mathcal{L}_\lambda \rangle .$$

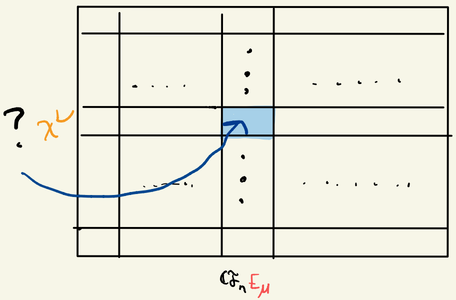
"Higher Lie" symmetric function

Schur functions

Full Answer

To fill out the whole table...

How many M_λ's to draw?



Thm: (C., 2024)

$$\left[(\mathbb{C}\mathcal{F}_n)^{\chi_\nu} E_\mu : M_\lambda\right] = \#SYT(\nu)\cdot\left\langle s_\nu \,,\, \text{coefficient of } \underline{x}_\lambda \underline{y}_\mu \text{ in } F\right\rangle,$$

where

$$F = \prod_{\substack{\text{Lyndon} \\ \text{words} \\ w}} \sum_{\substack{\text{Partitions} \\ \tau}} \underline{x}_{\tau\cdot|w|}\, \underline{y}_w^{|\tau|}\, \mathcal{L}_\tau[h_w].$$

Notation

- $\underline{x}_\lambda := x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_k}$
- $|\nu|, |w|$:= sum of the parts of ν and the letters of w.
- $\nu, |w|$:= scaling of the parts of ν by $|w|$

Proof Ingredients

Understanding a Σ_n-representation

$[(\mathbb{C}\mathcal{F}_n)^{\chi_V} E_\mu : M_\lambda]$?

$\xrightarrow{\text{Fin. dim. algebra theory}}$

Understanding an S_n-representation

$E_\lambda \mathbb{C}\mathcal{F}_n E_\mu$?

Saliola's work on quiver of $\mathbb{C}\mathcal{F}_n$

Poset Topology

$$\bigoplus_{\substack{[X\leq Y]\\ X\in\mu\\ Y\in\lambda}} \mathcal{H}_{top}\left(\Pi_n(X,Y)\right)\otimes \mathrm{Det}(Y)\otimes \mathrm{Det}(X)\Big\uparrow_{\mathrm{Stab}_{S_n}(X)\cap \mathrm{Stab}_{S_n}(Y)}^{S_n}\ ?$$

$\xrightarrow{\text{Sundaram's work on } \Pi_n}$

Symmetric Functions

$$\prod_{\substack{\text{Lyndon}\\ \text{words}\\ w}} \sum_{\substack{\text{Partitions}\\ \tau}} \frac{x}{(\tau\cdot|w|)} (y_w)^{|\tau|}\, \mathcal{L}_\tau[h_w].$$

Thank you!

For full preprint:

2404.00536