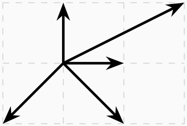
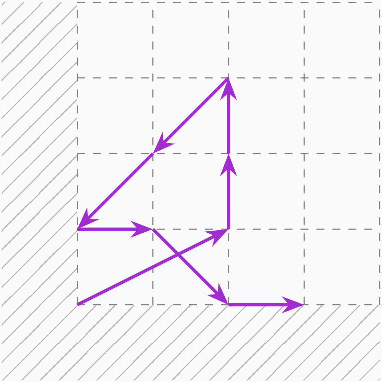
A Galois structure on the orbit of walks in the quadrant

Pierre Bonnet (LaBRI, Bordeaux)
joint work with Charlotte Hardouin (IMT, Toulouse)

Thursday July 25th, FPSAC '24

Walks in the quadrant

$\mathcal{S} =$  $\subset \mathbb{Z}^2$



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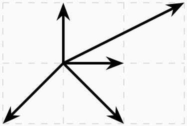
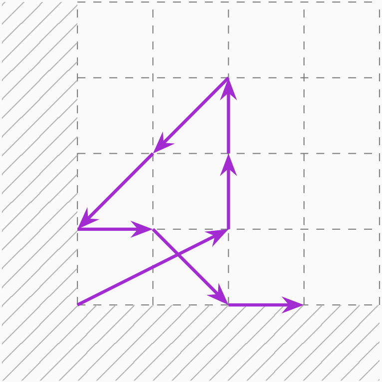
$\mathcal{S} =$  $\subset \mathbb{Z}^2$



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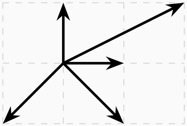
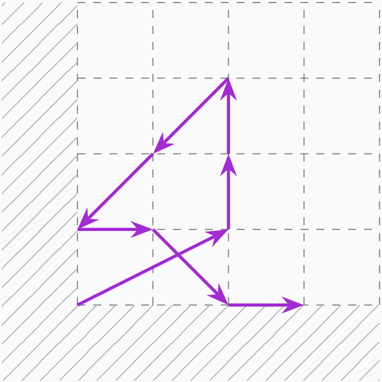
$\mathcal{S} =$  $\subset \mathbb{Z}^2$



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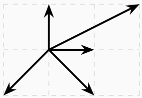
Problem

For a given model $\mathcal{S}$, determine $q_{i,j,n}$ the number of walks in the quadrant using n steps and terminating at (i, j).

The generating function and functional equation

$\mathcal{S} =$ 

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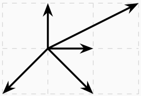
$\mathcal{S} =$ 

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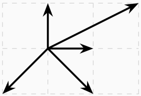
$\mathcal{S} =$ 

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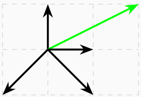
$\mathcal{S} =$ 

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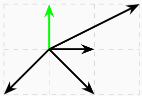
$\mathcal{S} =$ 

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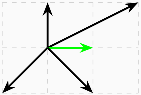
$\mathcal{S} =$ 

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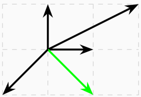
$\mathcal{S} =$ 

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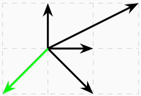
$\mathcal{S} =$ 

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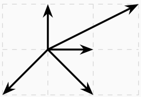
$\mathcal{S} =$ 

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$$\boxed{\underbrace{xy\left(1 - tS(x,y)\right)}_{K(x,y)\ \textit{(the kernel)}} Q(x,y) = xy - \left(t(x^2+1)Q(x,0) - tQ(0,0)\right) - tQ(0,y)}$$

Catalytic variables equation and classification

Polynomial equation in two catalytic variables

$$K(x,y)Q(x,y) = \Phi(t, x, y, Q(x,0), Q(0,y), Q(0,0))$$

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⇒ **algebraic case**

An algebraicity strategy

From two variables to one

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$\rightarrow$ derive equations in one catalytic variable for $Q(x,0)$ and $Q(0,y)$.

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[Bousquet-Mélou, Bernardi, Raschel, 21]: for small steps

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Invariants Lemma

The following assertions are equivalent:

- the coefficients in the t expansion of $(I(x) - J(y))/K(x, y)$ have no pole at $x = 0$ nor $y = 0$.
- $I(x) = J(y) = C(t)$ for some $C(t)$ independent from x and y

Obtain equations between invariants

Two pairs of invariants

$$(I_1(x), J_1(y)) = (\tfrac{1}{x} + c + O(x), O(1))$$

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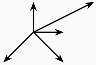
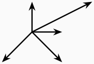
$$(I_1(x) - c)^2 - I_2(x) = C(t) \text{ and } (J_1(y) - c)^2 - J_2(y) = C(t)$$

Proof of algebraicity of $\mathcal{G}$



$$K(x,y)Q(x,y) = xy - \underbrace{\left(t(x^2+1)Q(x,0) - tQ(0,0)\right)}_{A(x)} - \underbrace{tQ(0,y)}_{B(y)}$$

Proof of algebraicity of $\mathcal{G}$



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Decoupling

$xy = F(x) + G(y) \mod K(x,y)$ for rational $F(x)$ and $G(y)$

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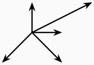

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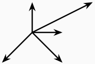
$xy = F(x) + G(y) \mod K(x,y)$ for rational $F(x)$ and $G(y)$

$\underbrace{A(x) - F(x)}_{I_1(x)} = \underbrace{G(y) - B(y)}_{J_1(y)} \mod K(x,y) \rightarrow$ **Invariants** $(I_1(x), J_1(y))$

Proof of algebraicity of $\mathcal{G}$



$$K(x,y)Q(x,y) = xy - \underbrace{\big(t(x^2+1)Q(x,0) - tQ(0,0)\big)}_{A(x)} - \underbrace{tQ(0,y)}_{B(y)}$$

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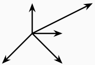
$$xy = -\frac{3tx^2 - t - 4x}{4t(x^2+1)} + \frac{-y-4}{4y} - \frac{K(x,y)}{ty(x^2+1)}$$

Proof of algebraicity of $\mathcal{G}$



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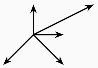

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Rational invariants

$$(I_2(x), J_2(y)) = \left(\frac{t^2x^6 + (t^2+1)x^4 + (t^2-1)x^3 - (t^2+1)x^2 - t^2}{t^2x(x^2+1)^2}, \frac{ty^4 - y^3 - ty - t}{ty^2}\right)$$

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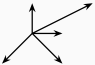

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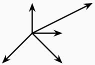

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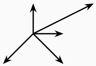
Pole elimination + Invariant Lemma

Pair of invariants $(I_3(x), J_3(y))$ without pole at $x = 0$ and $y = 0$.

Proof of algebraicity of $\mathcal{G}$



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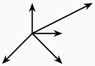
Proof of algebraicity of $\mathcal{G}$



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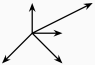
each equation (1) and (2) is of **one** catalytic variable

Proof of algebraicity of $\mathcal{G}$



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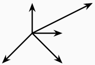
each equation (1) and (2) is of **one** catalytic variable

$\Rightarrow$ $Q(x,0)$ and $Q(0,y)$ are algebraic by [BMJ06]

Proof of algebraicity of $\mathcal{G}$



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Decoupling

$$(I_1(x), J_1(y)) = \left(\frac{3tx^2 - t - 4x}{4t(x^2+1)} + A(x), \frac{-y-4}{4y} - B(y)\right)$$

Rational invariants

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$\Rightarrow$ $Q(x,0)$ and $Q(0,y)$ are algebraic by [BMJ06] $\Rightarrow$ $Q(x,y)$ is algebraic $\square$

Another algebraic model

$$K(x,y)Q(x,y) = x^{7+1}y^{1+1} - \underbrace{(tQ(x,0) - tQ(0,0))}_{A(x)} - \underbrace{t(y+1)Q(0,y)}_{B(y)}$$

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$$K(x,y)Q(x,y) = x^{7+1}y^{1+1} - \underbrace{(tQ(x,0) - tQ(0,0))}_{A(x)} - \underbrace{t(y+1)Q(0,y)}_{B(y)}$$

Decoupling

$$x^8y^2 = \frac{x^8t^2 - 2x^5t + x^2 - 2xt}{t^2} + \frac{y^2-1}{y^2} + K(x,y)\frac{x^4y^2t - x^4yt + xy - yt + t}{y^2t^2}$$

$$(I_1(x), J_1(y)) = \left(-\frac{x^8t^2 - 2x^5t + x^2 - 2xt}{t^2} + A(x), \frac{y^2-1}{y^2} - B(y)\right)$$

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Rational invariants

$$(I_2(x), J_2(y)) =$$

$$\left(\frac{-x^{16}t^3 + 3x^{13}t^2 + 4x^{12}t^3 - 3x^{10}t - 5x^9t^2 - 6x^8t^3 + x^7 + x^6t + x^5t^2 + xt^2 - t^3}{x^4t^3}, \right.$$

$$\left. \frac{y^8t^4 + 4y^7t^4 + 12y^6t^4 + 32y^5t^4 + 54y^4t^4 + 52y^3t^4 + y^6 + 28y^2t^4 + 8yt^4 + t^4}{y^3t^4(y^4 + 4y^3 + 6y^2 + 4y + 1)} \right)$$

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$$(I_1(x), J_1(y)) = \left(-\frac{x^8t^2 - 2x^5t + x^2 - 2xt}{t^2} + A(x), \frac{y^2-1}{y^2} - B(y)\right)$$

Rational invariants

$$(I_2(x), J_2(y)) = \left(\frac{-x^{16}t^3 + 3x^{13}t^2 + 4x^{12}t^3 - 3x^{10}t - 5x^9t^2 - 6x^8t^3 + x^7 + x^6t + x^5t^2 + xt^2 - t^3}{x^4t^3}, \frac{y^8t^4 + 4y^7t^4 + 12y^6t^4 + 32y^5t^4 + 54y^4t^4 + 52y^3t^4 + y^6 + 28y^2t^4 + 8yt^4 + t^4}{y^3t^4(y^4 + 4y^3 + 6y^2 + 4y + 1)}\right)$$

Another algebraic model

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Decoupling

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Pole elimination + Invariants Lemma

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Another algebraic model

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Rational invariants

$$(I_2(x), J_2(y)) = \left(\frac{-x^{16}t^3 + 3x^{13}t^2 + 4x^{12}t^3 - 3x^{10}t - 5x^9t^2 - 6x^8t^3 + x^7 + x^6t + x^5t^2 + xt^2 - t^3}{x^4t^3}, \frac{y^8t^4 + 4y^7t^4 + 12y^6t^4 + 32y^5t^4 + 54y^4t^4 + 52y^3t^4 + y^6 + 28y^2t^4 + 8yt^4 + t^4}{y^3t^4(y^4 + 4y^3 + 6y^2 + 4y + 1)} \right)$$

Pole elimination + Invariants Lemma

Pair of invariants $(I_3(x), J_3(y))$ without pole at $x = 0$ and $y = 0$.

$$I_3(x) = U(x, t, Q(x,0), Q(0,0), (\partial_y Q)(0,0), \ldots, (\partial_y^5 Q)(0,0)) = C(t) \quad (1)$$

$$J_3(y) = V(y, t, Q(0,y), Q(0,0), (\partial_y Q)(0,0), \ldots, (\partial_y^5 Q)(0,0)) = C(t) \quad (2)$$

each equation (1) and (2) is of **one** catalytic variable

$\Rightarrow$ $Q(x,0)$ and $Q(0,y)$ are algebraic by [BMJ06] $\Rightarrow$ $Q(x,y)$ is algebraic $\square$

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□

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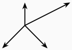
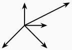
-   first models with large steps to have been proved algebraic

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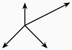
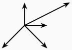
-   first models with large steps to have been proved algebraic
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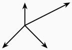
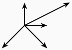
-   first models with large steps to have been proved algebraic
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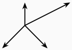
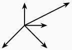
-   first models with large steps to have been proved algebraic
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- New algebraic models with large steps extending Gessel's:

   ...

  (given $n \geq 1$, algebraic for s.p. $(n-1, 0)$ and $((n+1)k-1, k-1)$ for all k)

Construction of rational invariants and decoupling

The orbit

[Bostan, Bousquet-Mélou, Melczer 2018]

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Idea: starting from (x, y), construct pairs of variables (u, v) such that $S(u, v) = S(x, y)$, changing one coordinate at a time.

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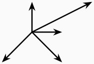
Example:  $S(x, y) = \frac{1}{xy} + \frac{x}{y} + x + x^2y + y$

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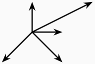
Example:  $S(x, y) = \frac{1}{xy} + \frac{x}{y} + x + x^2y + y$

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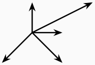
Example: 

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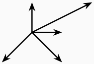
Example: 

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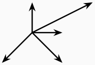
Example: 

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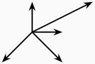
Example: 

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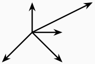
Example:  $$S(x, y) = \frac{1}{xy} + \frac{x}{y} + x + x^2y + y$$

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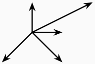
Example:  $S(x, y) = \frac{1}{xy} + \frac{x}{y} + x + x^2y + y$

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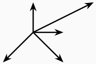
Example:  $S(x, y) = \frac{1}{xy} + \frac{x}{y} + x + x^2y + y$

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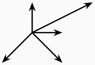
Example: 

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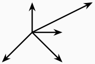
Example: 

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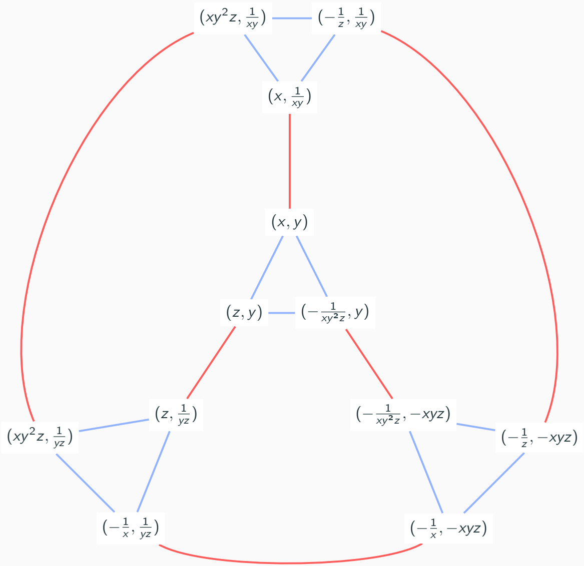
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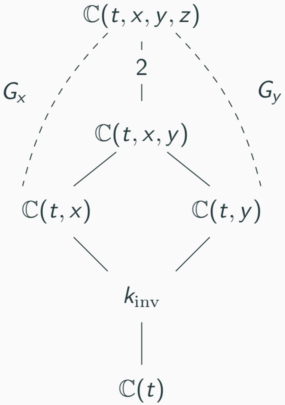
Key point: Galois symmetries = orbit symmetries



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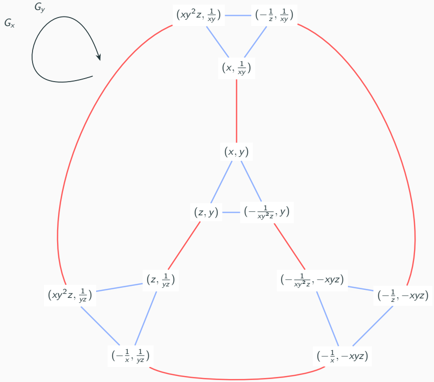

$$G_x = \mathrm{Gal}(\mathbb{C}(t,x,y,z)/\mathbb{C}(t,x)) = \langle \psi, \tau \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

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Theorem ((Fried 1978), (B., Hardouin 2023+))

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Construction in case of a finite orbit

Rational invariants

Theorem ((Fried 1978), (B., Hardouin 2023+))

The following are equivalent:

- *The orbit is finite*
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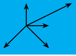
Any nonconstant coefficient of $\mu_x(Z)$ generates the field of rational invariants.

Rational invariants for 

$$\mu_x(Z) = Z^6 - \frac{\left(x^3 + x^6 + x^4 - x^2 - 1\right) t^2 + x^2 \left(x^2 - 1\right) t - x^3}{t^2 x \left(x^2+1\right)^2} Z^5 + \frac{t+1}{t} Z^4$$
$$- 2\frac{x^6 t^2 + \left(-\frac{t^2}{2} + \frac{1}{2}\right) x^5 + t\left(t+1\right) x^4 + \left(-t^2 - t\right) x^2 - \frac{\left(t^2-1\right)x}{2} - t^2}{t^2 x \left(x^2+1\right)^2} Z^3$$
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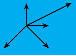
Rational invariants for 

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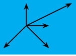
Rational invariants for 

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Linear combination of pairs of the orbit $c = \sum_{u,v} c_{u,v}(u, v)$.

For $H(x, y)$ a fraction, define $H_c = \sum_{u,v} c_{u,v} H(u, v)$.

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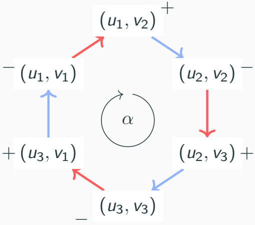
Example: killing decoupled fractions $F(x) + G(y)$



$$\alpha = (u_1, v_2) - (u_1, v_1) + (u_2, v_3) - (u_2, v_2) + (u_3, v_1) - (u_3, v_3)$$

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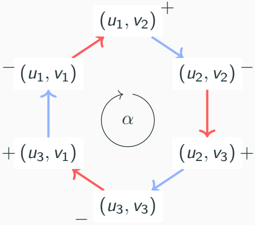

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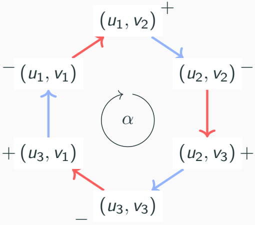

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$\Rightarrow H_c = H_c(x)$ by Galois

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Theorem (Bonnet, Hardouin 2023+)

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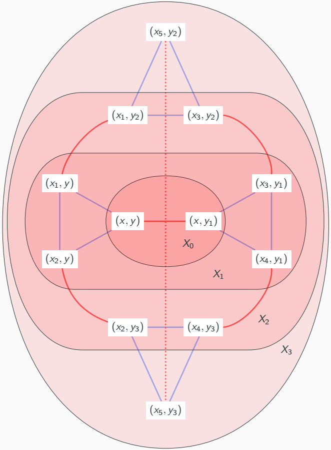
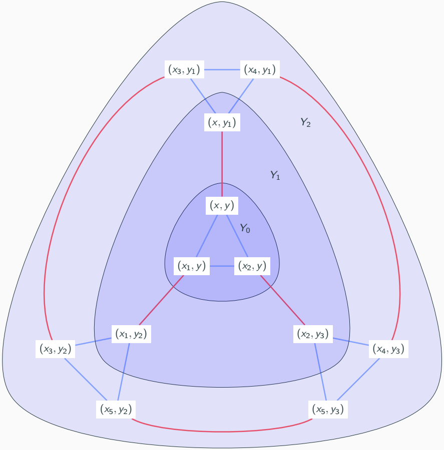

(x5, y2)
(x1, y2)
(x3, y2)
(x1, y)
(x3, y1)
(x, y)
(x, y1)
X0
(x2, y)
(x4, y1)
X1
(x2, y3)
(x4, y3)
X2
X3
(x5, y3)


(x3, y1)
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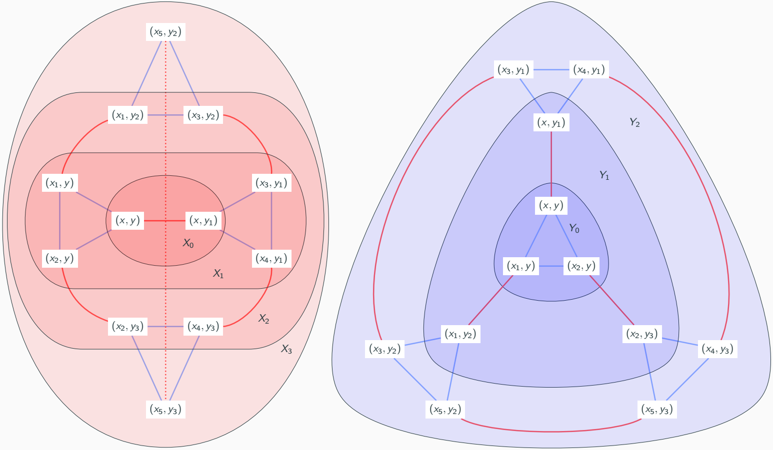
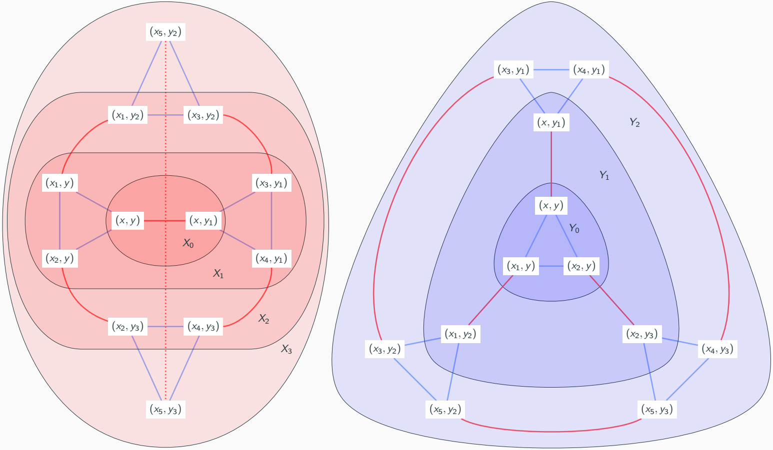


$$(x, y) = \left(\frac{X_0}{2} - \frac{X_1}{8} + \frac{X_2}{8}\right) + \left(\frac{Y_0}{4} - \frac{Y_1}{4}\right) + \alpha,$$



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$$(xy)_{(x,y)} = xy = -\frac{3tx^2 - t - 4x}{4t(x^2+1)} + \frac{-y-4}{4y} + 0 \mod K(x, y).$$

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Thank you for your attention!