Supersolvable Posets & Fiber-type Arrangements

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Outline

1. Arrangements (of hyperplanes & toric analogue)
2. Locally geometric posets
3. Stanley's supersolvable (geometric) lattices
4. Supersolvable locally geometric posets
5. Strict supersolvability
6. The characteristic polynomial
7. Arrangement bundles

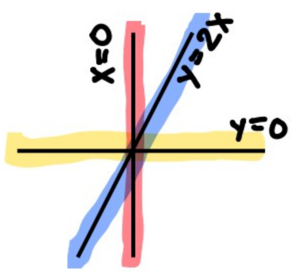
Arrangements

$\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}^n - 0$

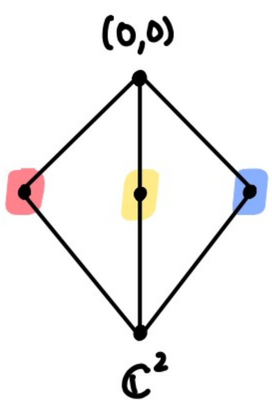
$$H_\alpha = \ker\begin{pmatrix} \mathbb{C}^n \longrightarrow \mathbb{C} \\ \vec{x} \longmapsto a_1x_1 + \ldots + a_nx_n \end{pmatrix}$$

$\{\alpha_1, \ldots, \alpha_\ell\} \subseteq \mathbb{Z}^n - 0$

hyperplane arrangement



geometric lattice
(flats of matroid)



Example:

$\alpha_1 = (1,0)$

$\alpha_2 = (0,1)$

$\alpha_3 = (2,-1)$

Poset $P(\mathcal{A})$:

Intersections $\bigcap_{H \in S} H \quad (S \subseteq \mathcal{A})$

ordered by reverse inclusion

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$$H_\alpha = \ker\begin{pmatrix} (\mathbb{C}^\times)^n \longrightarrow \mathbb{C}^\times \\ \vec{t} \mapsto t_1^{a_1} \cdots t_n^{a_n} \end{pmatrix}$$

$\{\alpha_1, \dots, \alpha_\ell\} \subseteq \mathbb{Z}^n - 0$

hyperplane arrangement

toric arrangement

Example:

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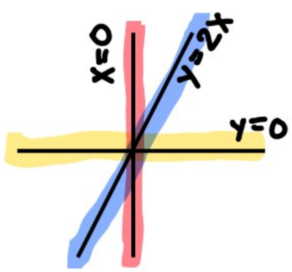
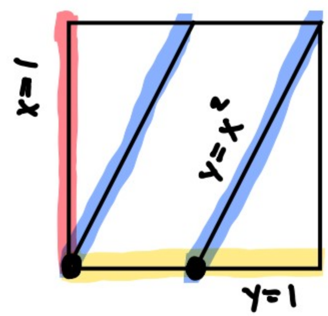


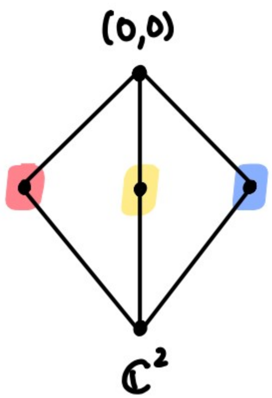


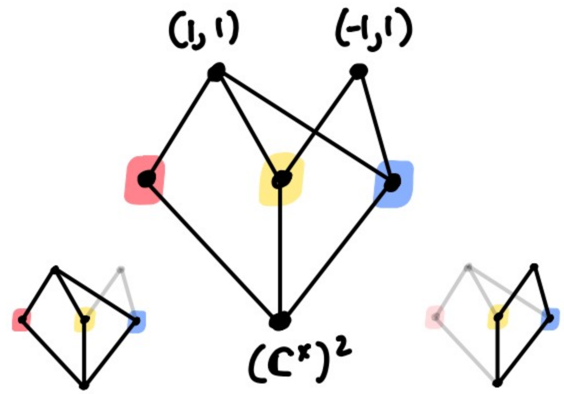
Poset $P(\mathcal{A})$:

connected components of intersections $\bigcap_{H \in S} H$ $(S \subseteq \mathcal{A})$

ordered by reverse inclusion

geometric lattice
(flats of matroid)



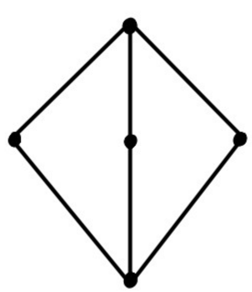


Locally geometric posets

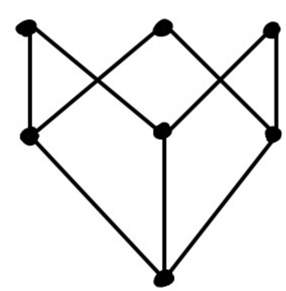
In a finite poset P let

$$x \vee y = \min\{u \in P : u \geq x \ \& \ u \geq y\} \qquad x \wedge y = \max\{\ell \in P : \ell \leq x \ \& \ \ell \leq y\}$$

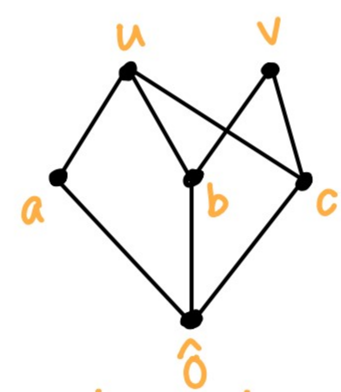
P is a **lattice** if $|x \vee y| = 1 = |x \wedge y|$ for all $x, y \in P$.



lattice

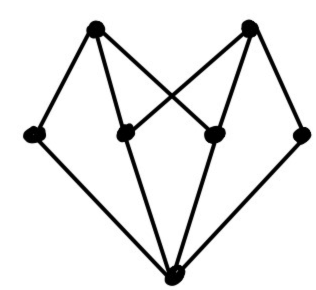


semilattice



$a \vee v = \phi$ $\qquad$ $b \vee c = \{u, v\}$

$a \wedge v = \hat{0}$ $\qquad$ $u \wedge v = \{b, c\}$



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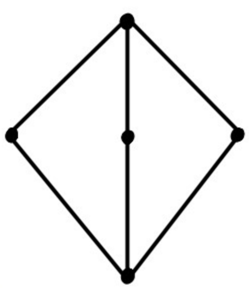
P is a **lattice** if $|x \vee y| = 1 = |x \wedge y|$ for all $x, y \in P$.

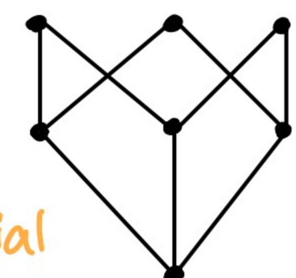
P is a **geometric lattice** if it is a lattice and

$$x \lessdot y \iff \exists \text{ atom } a \in P \text{ s.t. } a \not\leq x \;\&\; y = x \vee a.$$

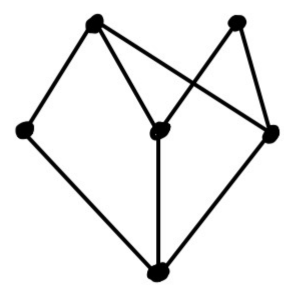
P is a **locally geometric poset** if it is bounded below (min $= \hat{0}$) and every closed interval is a geometric lattice

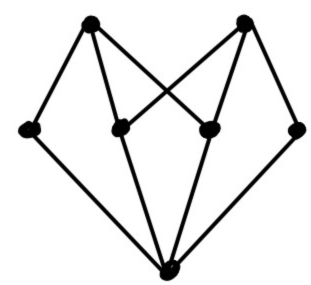


geometric lattice

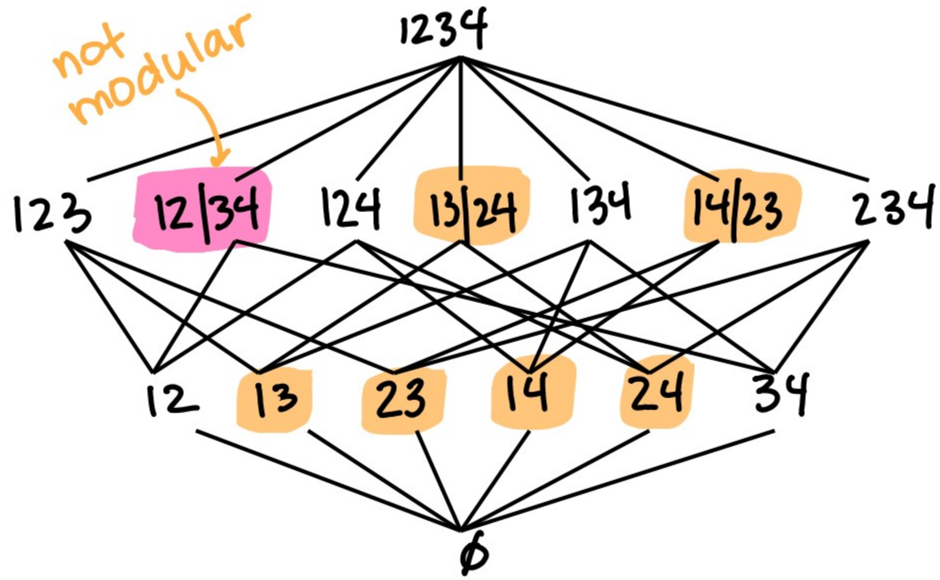


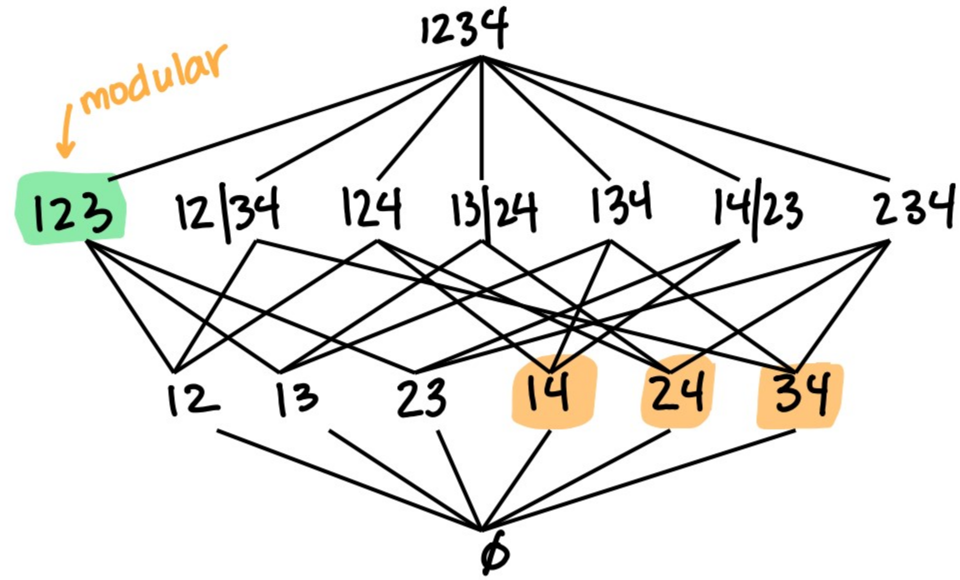
simplicial poset / geometric semilattice





Supersolvable geometric lattices

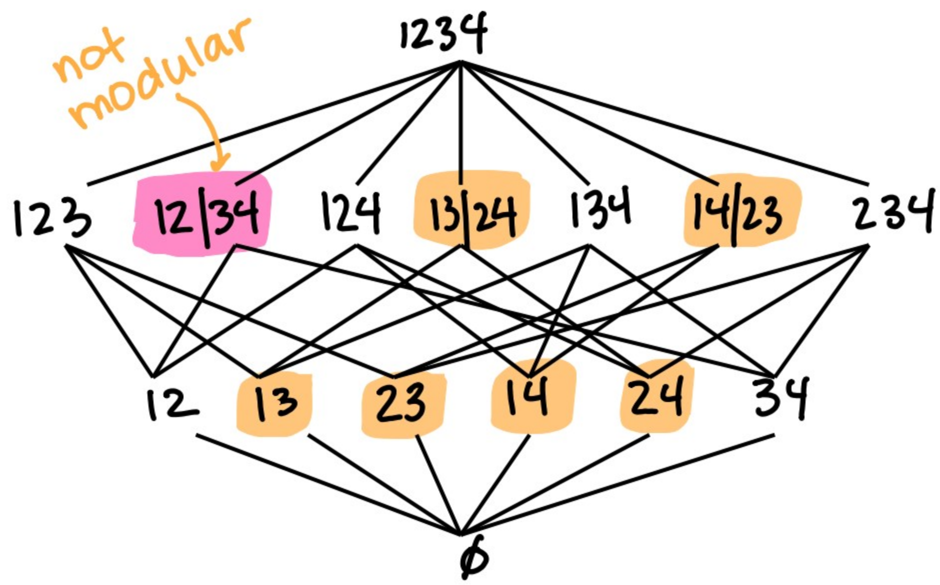


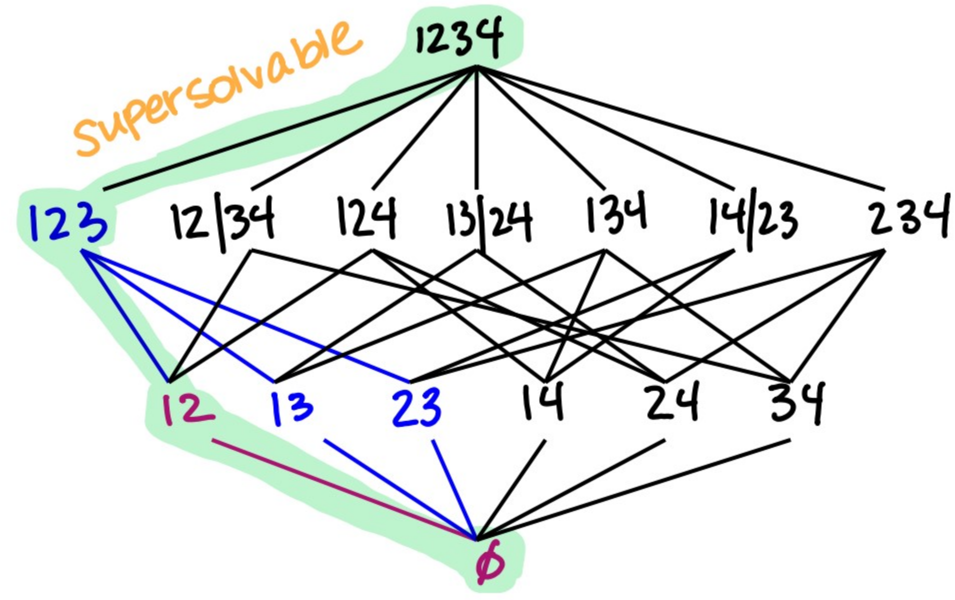


An element x of a geometric lattice $\mathcal{L}$ is **modular** if its complements form an antichain

$$\left(y \in \mathcal{L} \text{ s.t. } x \wedge y = \hat{0} \;\&\; x \vee y = \hat{1} \right)$$

Supersolvable geometric lattices





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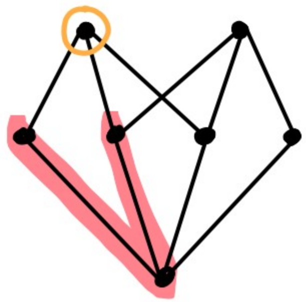
$$\left(y \in \mathcal{L} \text{ s.t. } x \wedge y = \hat{0} \;\&\; x \vee y = \hat{1}\right)$$

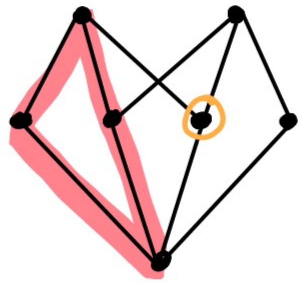
A geometric lattice is **supersolvable** if it has a chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_r = \hat{1}$ with each x_i modular

(Stanley '72)

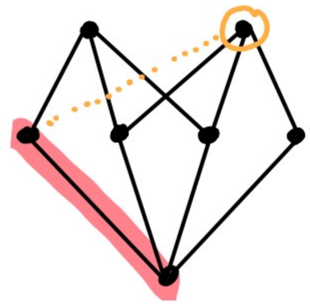
Supersolvable locally geometric posets



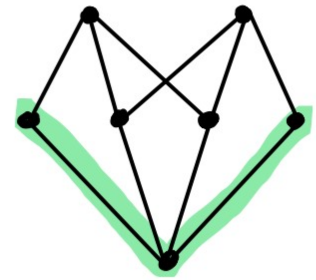
not join-closed



not an order ideal



not an M-ideal

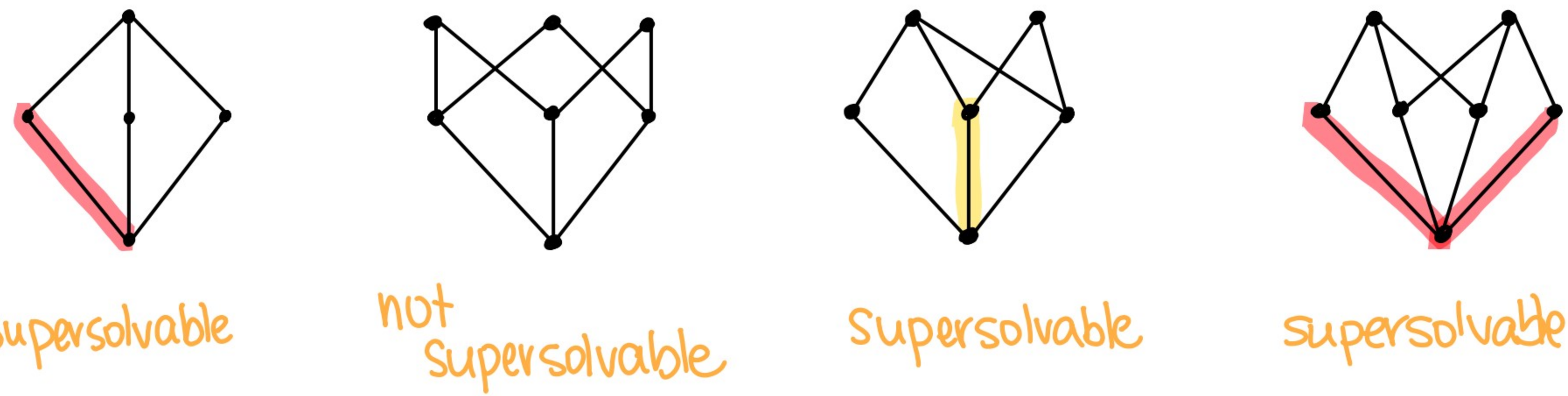


M-ideal!

An **M-ideal** in a locally geometric poset P is a pure, join-closed order ideal $Q \subseteq P$ s.t.

- if $x \in Q$, atom $a \notin Q$, then $a \vee x \neq \emptyset$
- for every $u \in \max P$, there is some $x \in \max Q$ s.t. x is a modular element of the geometric lattice $P_{\leq u}$

Supersolvable locally geometric posets



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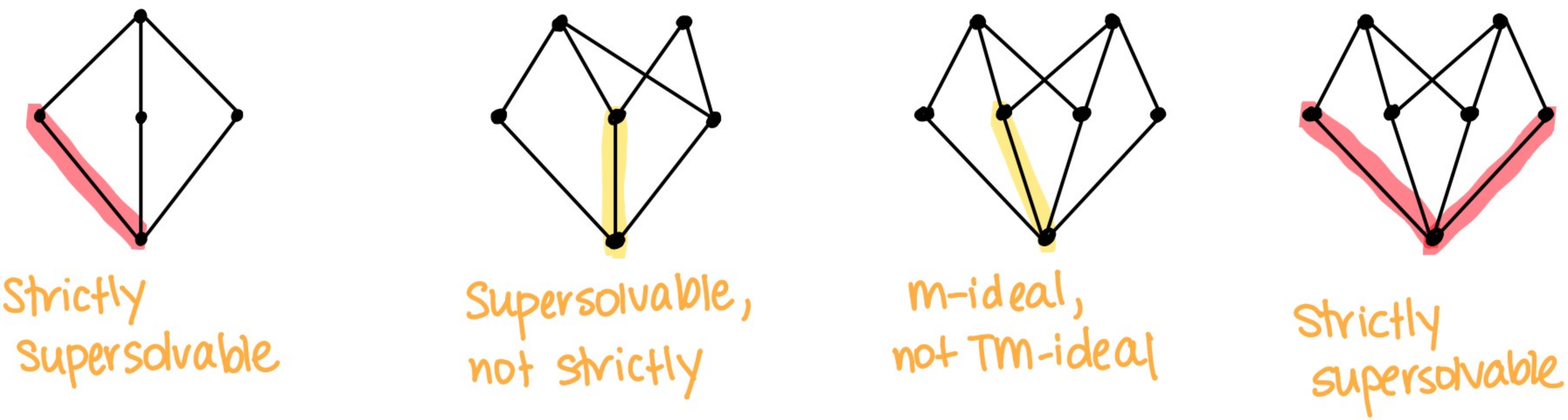
A locally geometric poset P is **supersolvable** if it has a chain $\{\hat{0}\} = Q_0 \subset Q_1 \subset \cdots \subset Q_r = P$ with each Q_i an M-ideal and $\operatorname{rank}(Q_i) = i$.

Strictly Supersolvable locally geometric posets



A **TM-ideal** in a locally geometric poset P is a pure, join-closed order ideal $Q \subseteq P$ s.t.

- if $x \in Q$, atom $a \notin Q$, then $|a \vee x| = 1$
- for every $u \in \max P$, there is some $x \in \max Q$ s.t. x is a modular element of the geometric lattice $P_{\leq u}$

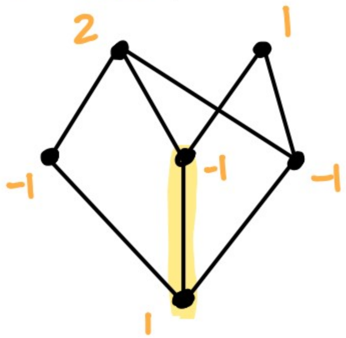
A locally geometric poset P is **strictly supersolvable** if it has a chain $\{\hat{0}\} = Q_0 \subset Q_1 \subset \cdots \subset Q_r = P$ with each Q_i a TM-ideal and $\operatorname{rank}(Q_i) = i$.

Characteristic Polynomial

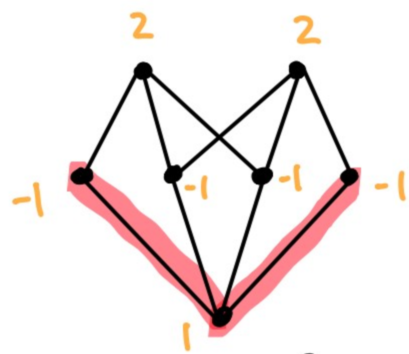
The **characteristic polynomial** of a locally geometric poset P is $\chi_P(t) = \sum_{x \in P} \mu(x)\, t^{rk(P)-rk(x)}$

möbius function: $\mu(\hat{0}) = 1$ & for $x > \hat{0}$, $\sum_{y \le x} \mu(y) = 0$

Example



$\chi_p(t) = t^2 - 3t + 3$



$\chi_p(t) = t^2 - 4t + 4$

Characteristic Polynomial

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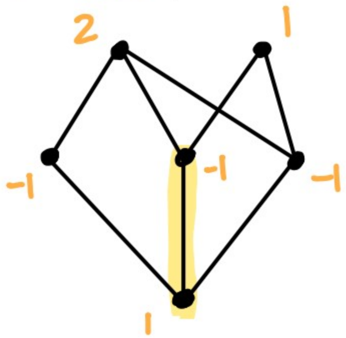

supersolvable, not strictly

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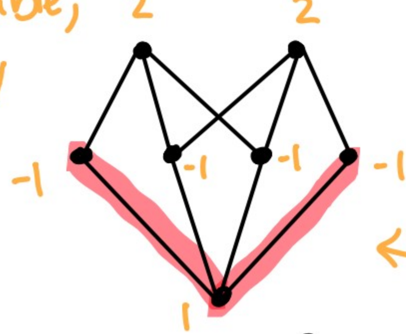


strictly supersolvable

$\chi_P(t) = t^2 - 4t + 4 = (t-2)(t-2)$

Theorem (B.-Delucchi '24, Stanley '72 for lattices)

If P is strictly supersolvable via $\{\hat{0}\} = Q_0 < Q_1 < \cdots < Q_r = P$

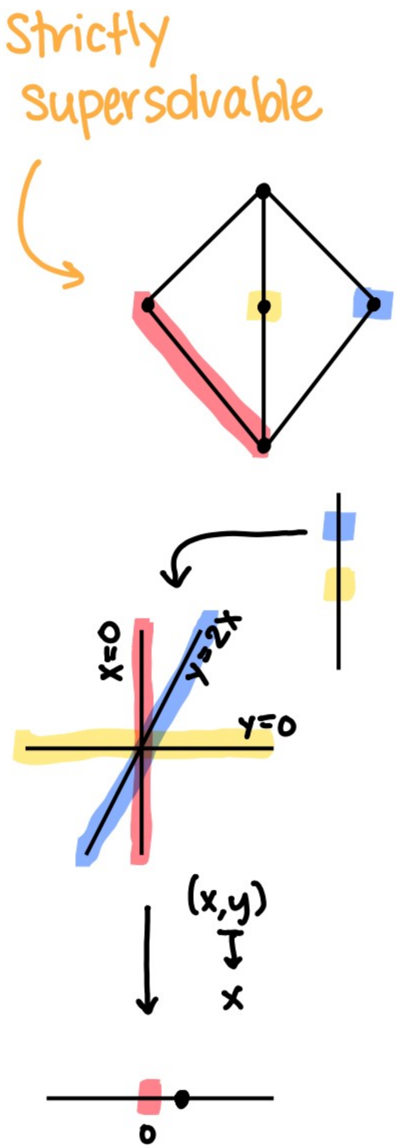
then $\chi_P(t) = \prod_{i=1}^{r} (t - a_i)$ with $a_i = \#$atoms in Q_i not in Q_{i-1}

Arrangement bundles

Theorem (B.-Delucchi '24, Terao '86 for hyperplane arrangements)

A toric arrangement $\mathcal{A}$ is supersolvable if and only if there is a choice of coordinates s.t. the projections $(\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^{n-1} \to \dots \to \mathbb{C}^\times$ restrict to fiber bundles on arrangement complements with fiber $\mathbb{C}^\times \setminus \{\text{some points}\}$

Arrangement bundles



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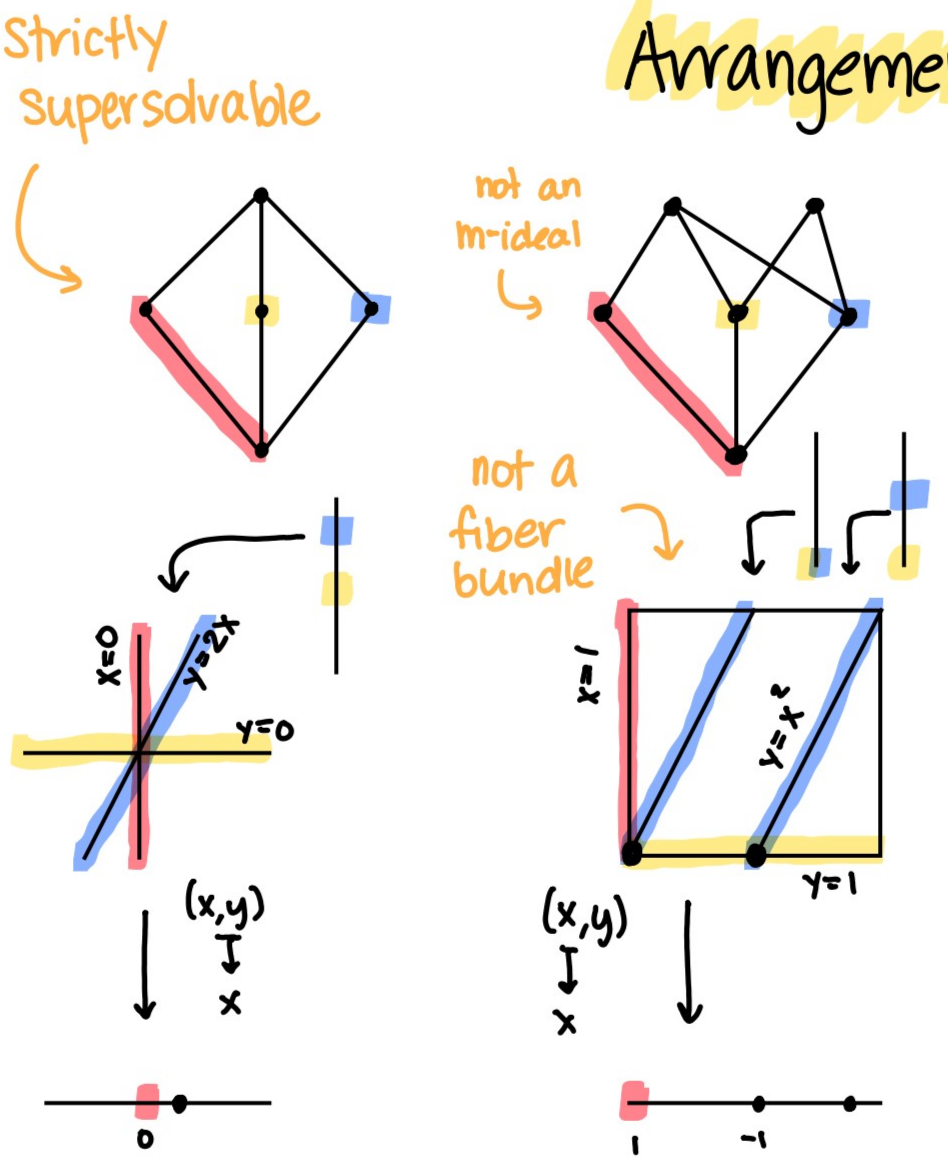

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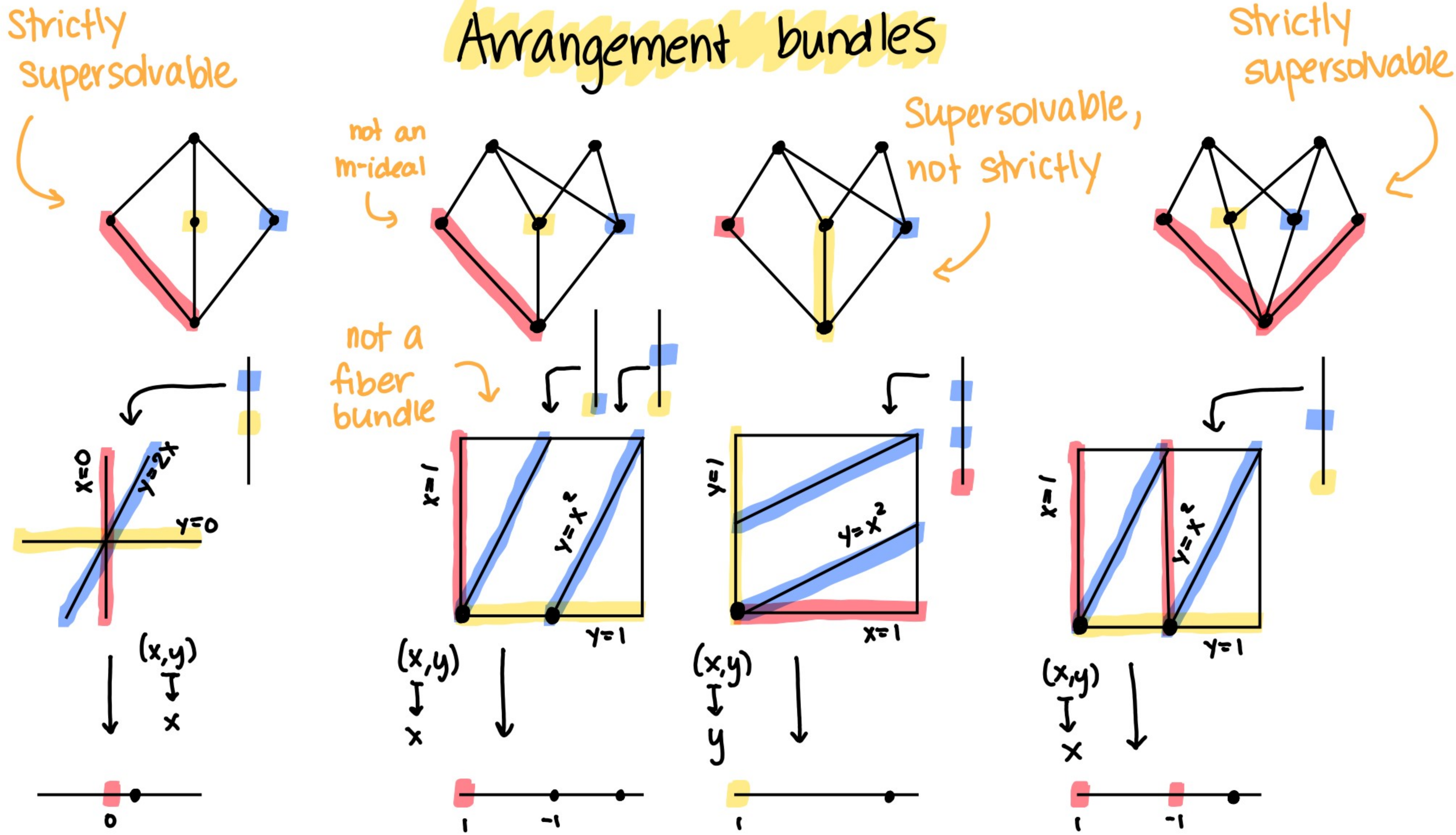

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Thank you