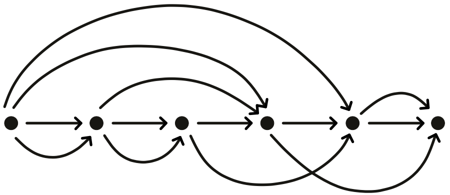
Wilting Theory of Flow Polytopes

Jonah Berggren and Khrystyna Serhiyenko

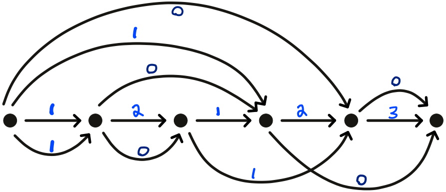
DAGs and Flow Polytopes

A **DAG** (Directed Acyclic Graph) G is a collection of vertices and edges between those vertices with no oriented cycles.

- We additionally require there to be a unique **source vertex** with no incoming edges and a unique **sink vertex** with no outgoing edges.



DAGs and Flow Polytopes



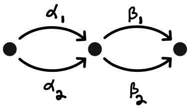
A **flow** on G is a labelling $F : E \to \mathbb{R}^{\geq 0}$ of the edges with the property of **conservation of flow** at each internal edge α:

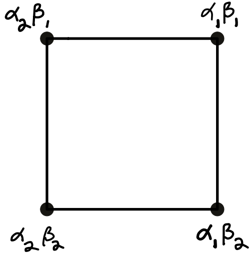
$$\sum_{\alpha \text{ ends at } v} F(\alpha) = \sum_{\beta \text{ starts at } v} F(\beta)$$

- The flow is **unit** if the sum of $F(\alpha)$ over all source edges is 1.

The **(unit) flow polytope** $\mathcal{F}_1(G)$ is the space of unit flows.

DAGs and Flow Polytopes





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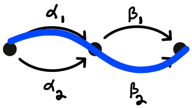
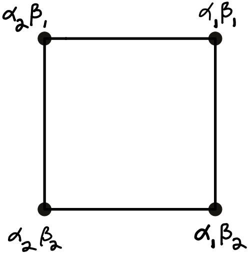
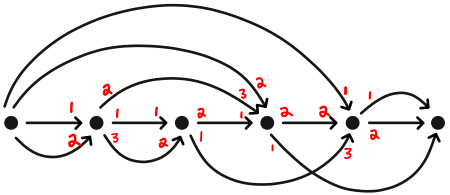


$$\begin{matrix} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ (1 & 0 & 0 & 1) \end{matrix}$$



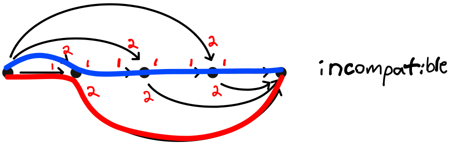
Framed DAGs

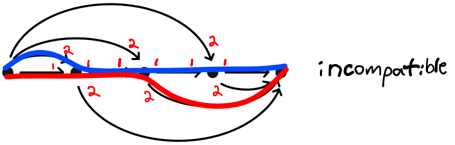
A **framing** on a DAG is the data of, for every internal vertex v, a total order on the incoming edges to v and a total order on the outgoing arrows from v. We think of framings as being induced by labellings of the (internal) half-edges.

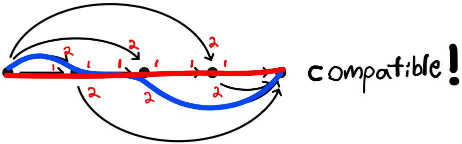


Framed DAGs

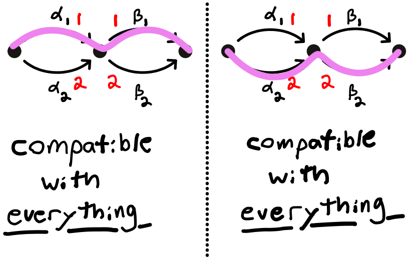
- A route p is **incompatible** to q if they share a segment σ that p enters high and leaves low compared with q, WLOG. Otherwise, they are **compatible**.
- We are interested in the **clique complex** of sets of pairwise compatible routes.



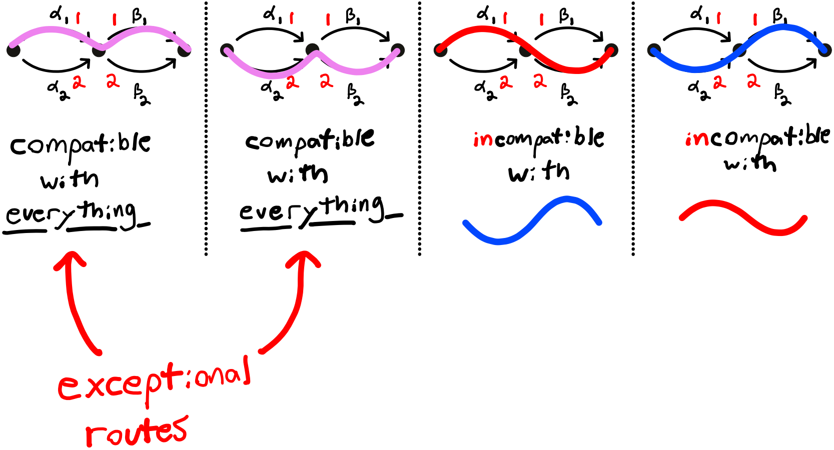




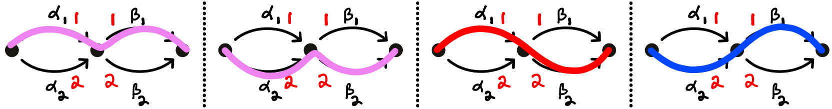
Framed DAGs


α1 1
1 β1
α2 2
2 β2
compatible with everything
compatible with everything
incompatible with
incompatible with

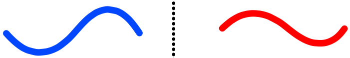
Framed DAGs


α_1 1
1 β_1
α_2 2
2 β_2
compatible with everything
compatible with everything
incompatible with
incompatible with
exceptional routes

Framed DAGs

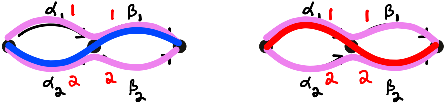


compatible with everything | compatible with everything | incompatible with | incompatible with



exceptional routes

Maximal Cliques:



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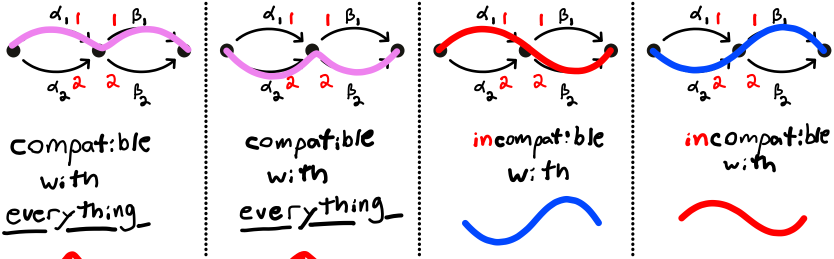
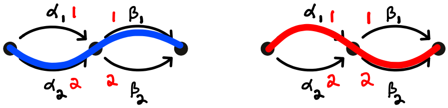


exceptional routes

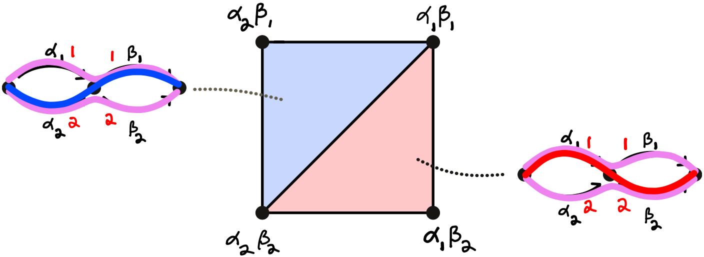
(Reduced) Maximal Cliques:



DKK Triangulations

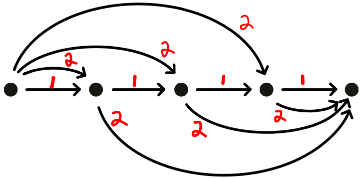
Theorem (Danilov-Karzanov-Koshevoy)

The clique complex on G induces a regular unimodular triangulation on $\mathcal{F}_1(G)$.



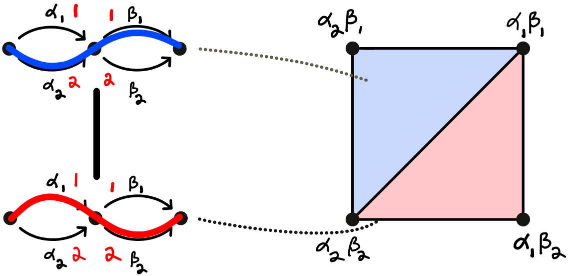
We call this the **DKK triangulation**.

Amply Framed DAGs



- A DAG is **full** if every internal vertex is incident to precisely two incoming and two outgoing edges.
- An **amply framed DAG** is a full DAG with framing induced by a labelling of the edges in $\{1, 2\}$.
  - In other words, we can't have any "steep" arrows starting with a 1 and ending with a 2, or vice versa.

Lattice Structure

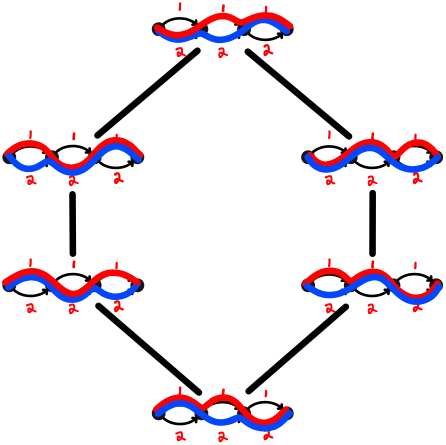


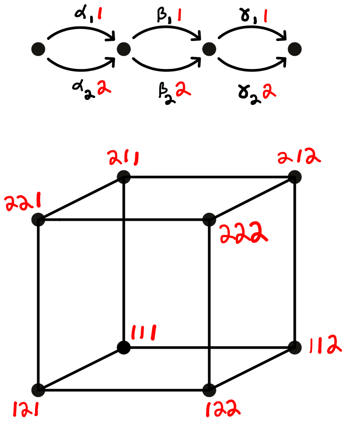
When G is amply framed, we may define a lattice structure on the dual graph of its DKK triangulation:

Theorem (von Bell-Braun-Bruegge-Hanely-Peterson-Serhiyenko-Yip)

The dual graph of the DKK lattice of an amply framed DAG Γ arises as the Hasse diagram of a lattice structure on the maximal cliques, known as the **framing lattice**.

Larger Example



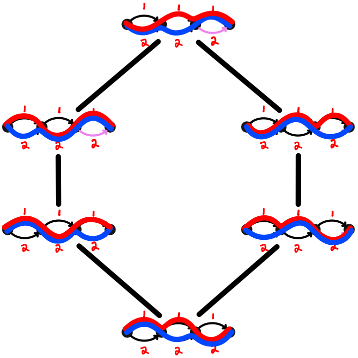


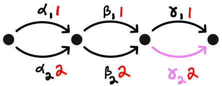
New Stuff

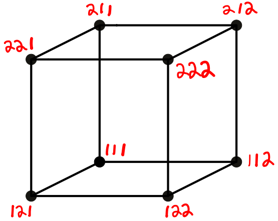
Part 1: Let's take an amply framed DAG, but try to avoid some set W of "wilted" arrows.

Wilted Framed DAGs

A **wilted framed DAG** (Γ, W) is a framed DAG Γ along with a choice W of **wilted** arrows. All other arrows are **lush**.







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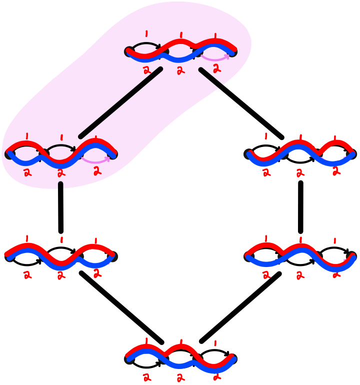
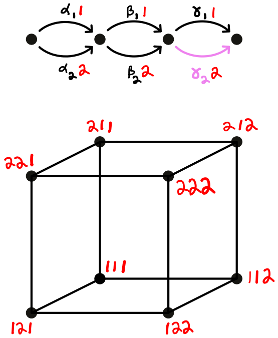




- A maximal clique whose nonexceptional routes avoid all arrows of W is **lush**.

Wilted Framed DAGs

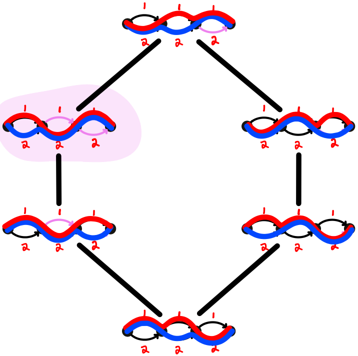
Here is an important initial result, proven using the representation theory of algebras:

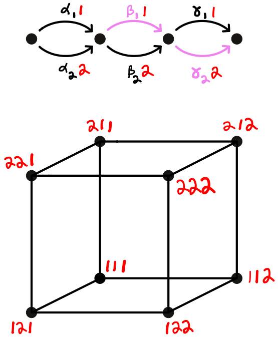
Theorem (BS)

When Γ is amply framed and $W \subseteq E$, **the lush maximal cliques of** (Γ, W) **form an interval in the framing lattice of** Γ*.*

We call this the **lush interval** $\mathcal{L}_{(\Gamma,W)}$.

Wilted Framed DAGs





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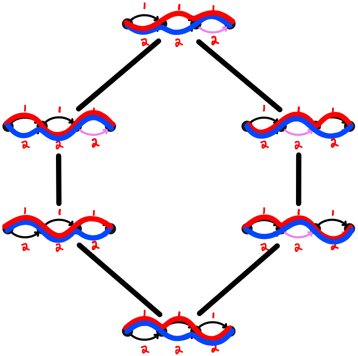
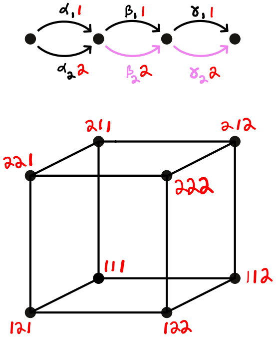


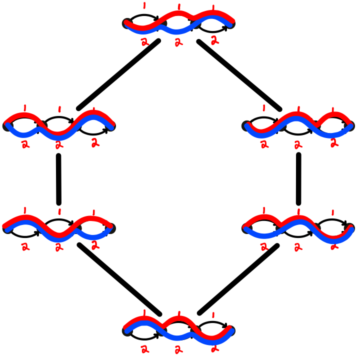


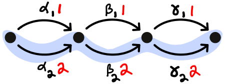
Sometimes the interval is empty!

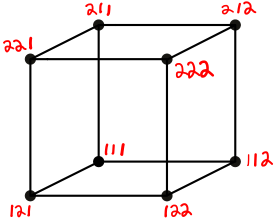
Decomposing

Lemma

Let p be an exceptional route (i.e., consisting only of 1's or only of 2's). Then any maximal clique misses precisely one arrow of p.







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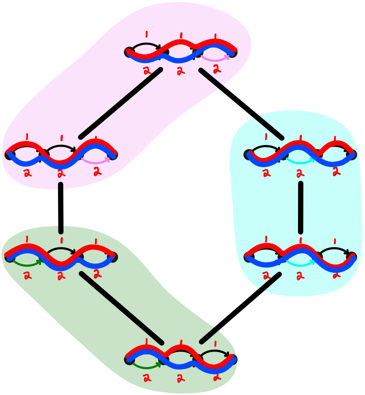
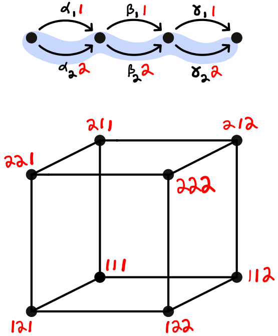
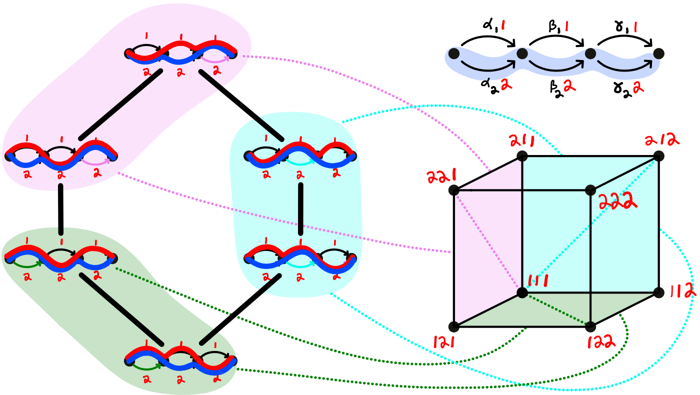




Polyhedral Interpretation: take all facets avoiding the vertex 222; each of these is a lush interval, triangulated according to its induced DKK triangulation.



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This motivates a more general construction! Start with a framed DAG Γ.

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   - (since we know we can't wilt more than one arrow from a given exceptional route)

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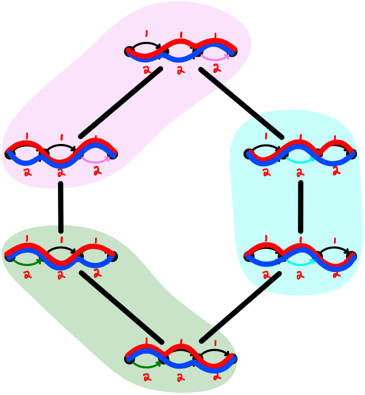
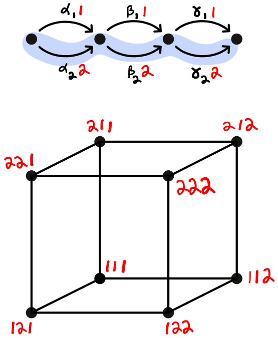
Theorem (BS)

This partitions the framing lattice into lush intervals.

Wilting Decompositions





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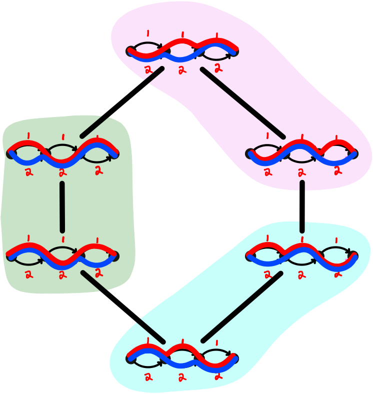

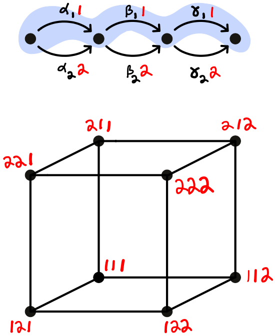

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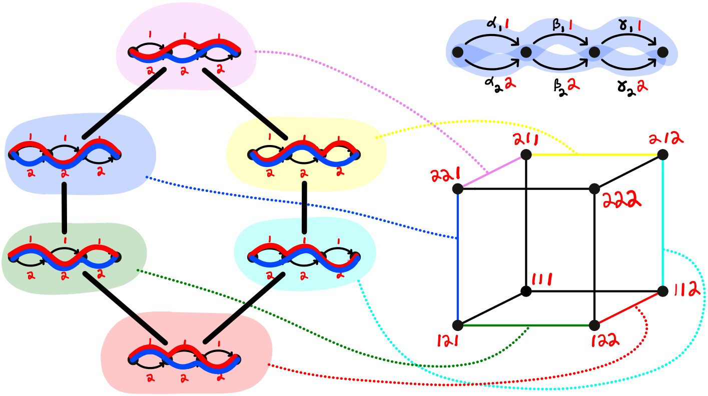
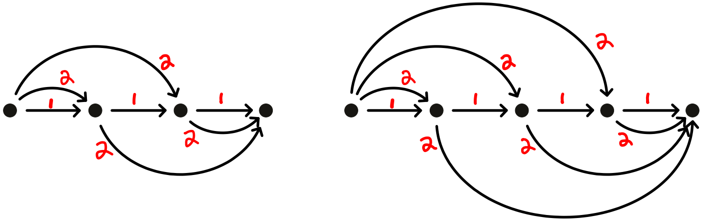


Motivating Example: Tamari

A motivating example for the preceding recovers various decompositions of the **Tamari lattice** into ν-**Tamari intervals**.

Motivating Example: Tamari

The following are two examples of **caracol DAGs** Car(n), for $n = 2$ and $n = 3$.



Theorem (von Bell, Gonzalez D'León, Mayorga Cetina, Yip)

The framing lattice of Car(n) is Tamari.

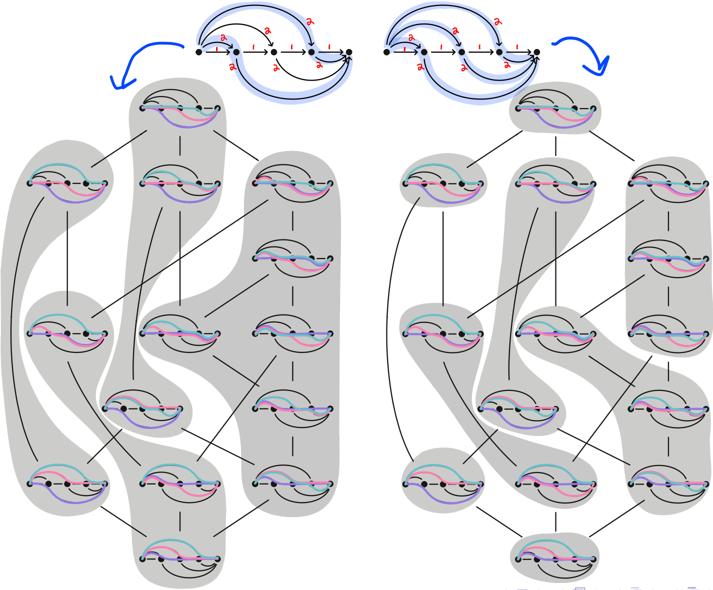
Motivating Example: Tamari

- ν-**Tamari lattices** are productive generalizations of the Tamari lattice introduced by Prèville-Ratelle and Viennot.
- Building on work of von Bell, Gonzalez D'León, Mayorga Cetina, and Yip connecting ν-Tamari lattices to DKK triangulations, we show the following:

Theorem (BS)

If S is any set of 2-routes of $Car(n)$, then the wilting decomposition with respect to S partitions the Tamari lattice into ν-Tamari intervals.

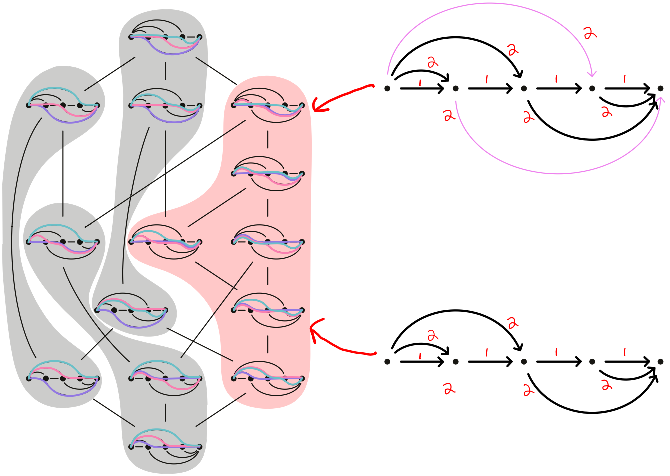
Motivating Example: Tamari



Part Two

Part Two: Lattice Structures for More Framed DAGs

More Lattice Structures



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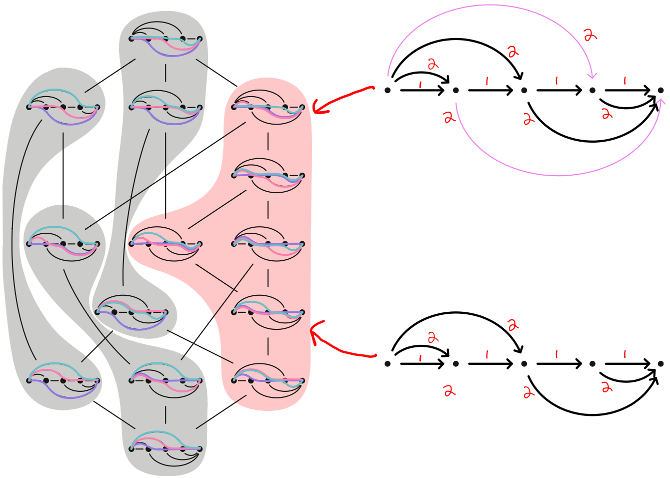
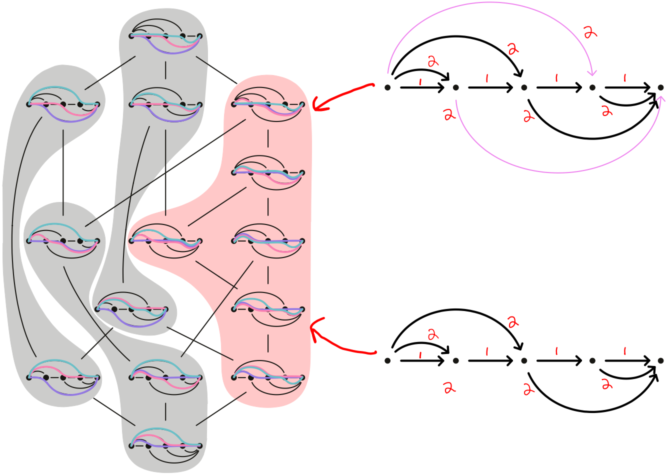


The literature does not give a lattice structure on the DKK triangulation of non-amply-framed DAGs.

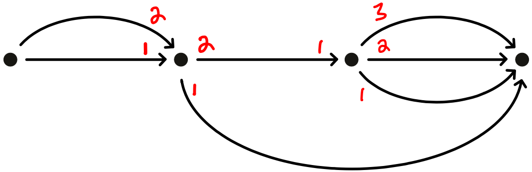
More Lattice Structures



On the other hand, we know that the DKK triangulation of the lower (non-amply-framed) DAG has a lattice structure, since it embeds as an interval into that of the upper (amply framed) DAG.

Ample Envelopes

New Goal: Start with a more general DAG and realize its triangulation as a lush interval in the framing lattice of an amply framed DAG.



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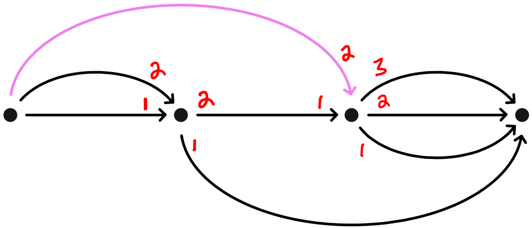

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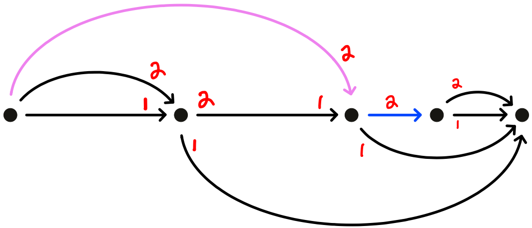

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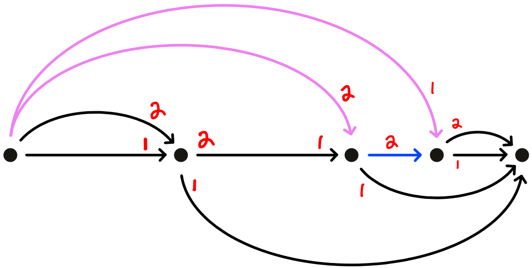

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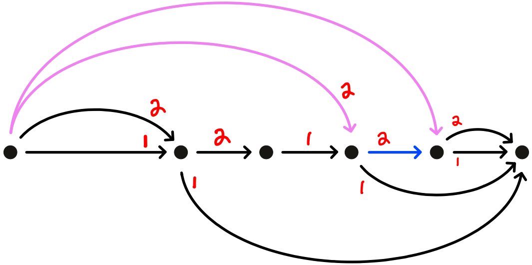

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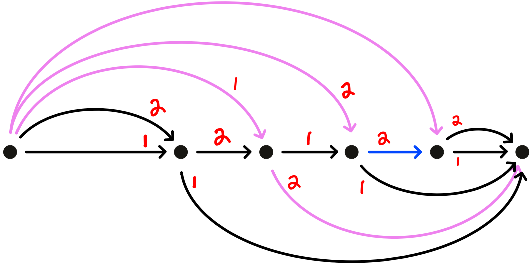

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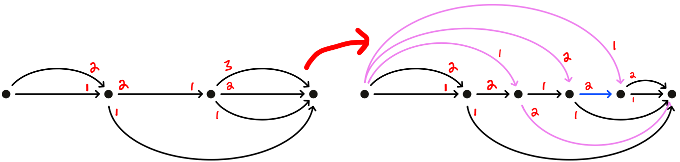
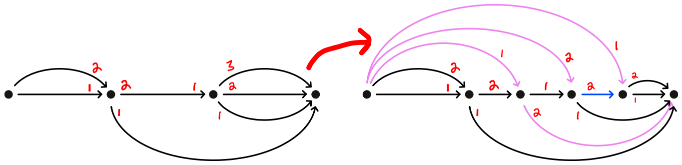


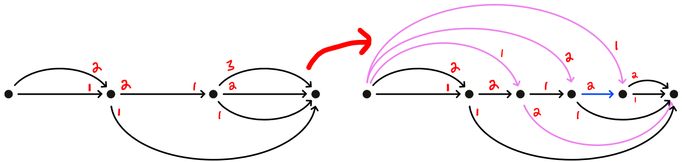
- This strategy always realizes the original flow polytope Γ as a face of a wilted amply framed flow polytope (Γ', W), called an **ample envelope**, given by zeroing out the wilted arrows.

Ample Envelopes



- This strategy always realizes the original flow polytope Γ as a face of a wilted amply framed flow polytope (Γ', W), called an **ample envelope**, given by zeroing out the wilted arrows.
- On the other hand, sometimes this goes wrong – in particular, (Γ', W) may have an empty lush interval (i.e., no maximal reduced clique avoids all of the wilted arrows).
  - When it works, we call Γ **rooted**.

Ample Envelopes



- This strategy always realizes the original flow polytope Γ as a face of a wilted amply framed flow polytope (Γ', W), called an **ample envelope**, given by zeroing out the wilted arrows.
- On the other hand, sometimes this goes wrong – in particular, (Γ', W) may have an empty lush interval (i.e., no maximal reduced clique avoids all of the wilted arrows).
  - When it works, we call Γ **rooted**.
  - We have a nice combinatorial characterization of rooted framed DAGs – see the abstract for a nice intuitive proof!
  - These include, for example, ν-caracol and s-oruga graphs.

Consequences

We can extend results about amply framed DAGs and their DKK triangulations to the rooted case! In particular:

Theorem (BS)

The dual graph of the DKK lattice of a rooted DAG Γ arises as the Hasse diagram of a **lattice structure on the maximal cliques** *(with the same covering relation description as the amply framed case), denoted $\mathcal{L}_\Gamma$.*

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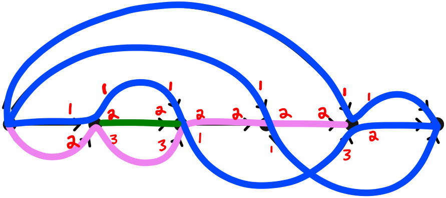
(And more!)

Thank You

We have a concrete combinatorial description for rooted framed DAGs.



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An **exceptional segment** of a framed DAG Γ is a maximal path which is compatible with every path.

Extra Innings: Rooted Framed DAGs

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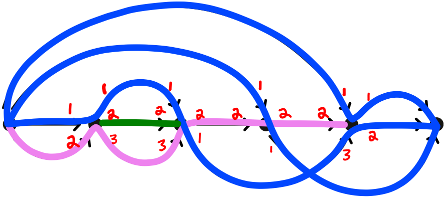

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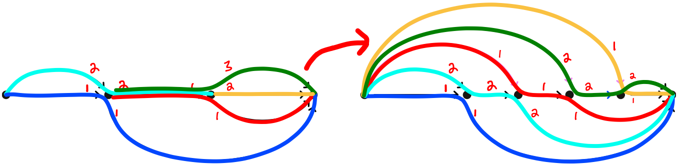

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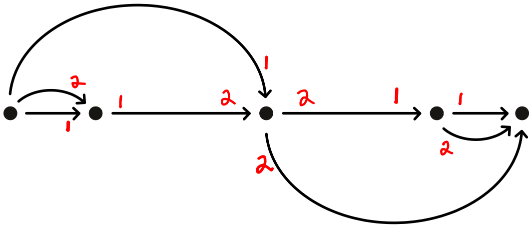

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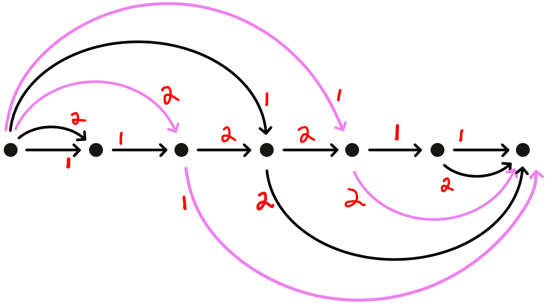

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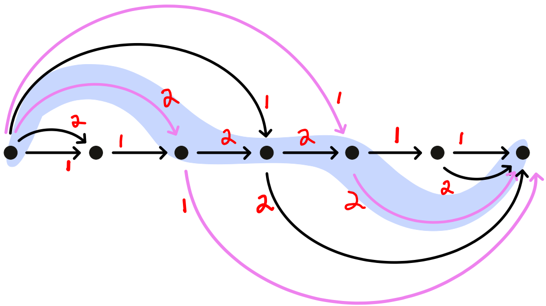

In the end, this exceptional route has two wilted arrows, so the lush interval will be empty (even though the simplicial complex of lush cliques still matches that of the original).