# On lattice path matroids subdivisions of the permutahedron

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Universidad de los Andes

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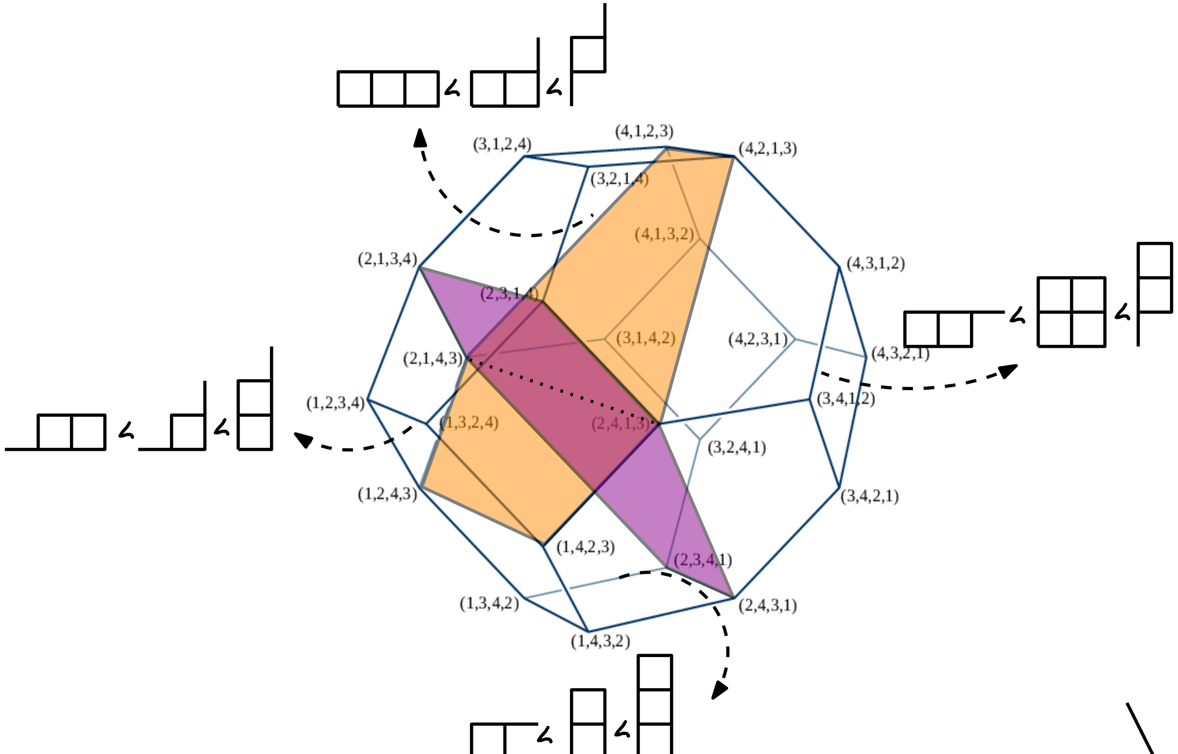
FPSAC Bochum

# Overview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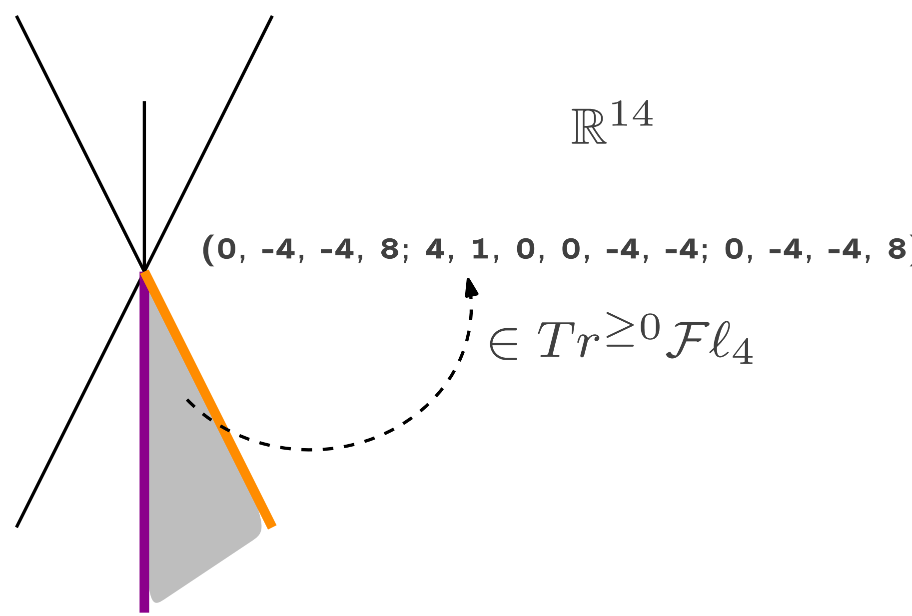
$\mathbb{R}^{14}$
(0, -4, -4, 8; 4, 1, 0, 0, -4, -4; 0, -4, -4, 8
$\in Tr^{\geq 0}\mathcal{F}\ell_4$

# Lattice path matroids

A matroid $M$ is a pair $([n], \mathcal{B})$ where $\emptyset \neq \mathcal{B} \subseteq 2^{[n]}$ satisfies:
given $A, B \in \mathcal{B}$, if $a \in A \setminus B$ there is $b \in B \setminus A$ s.t. $A - a + b \in \mathcal{B}$.

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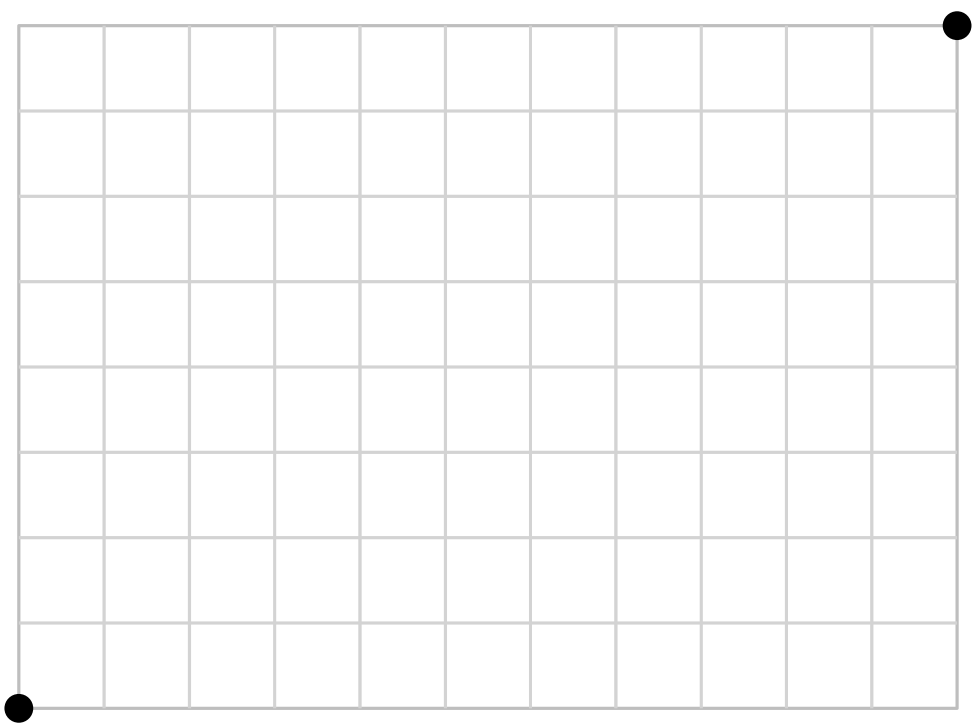

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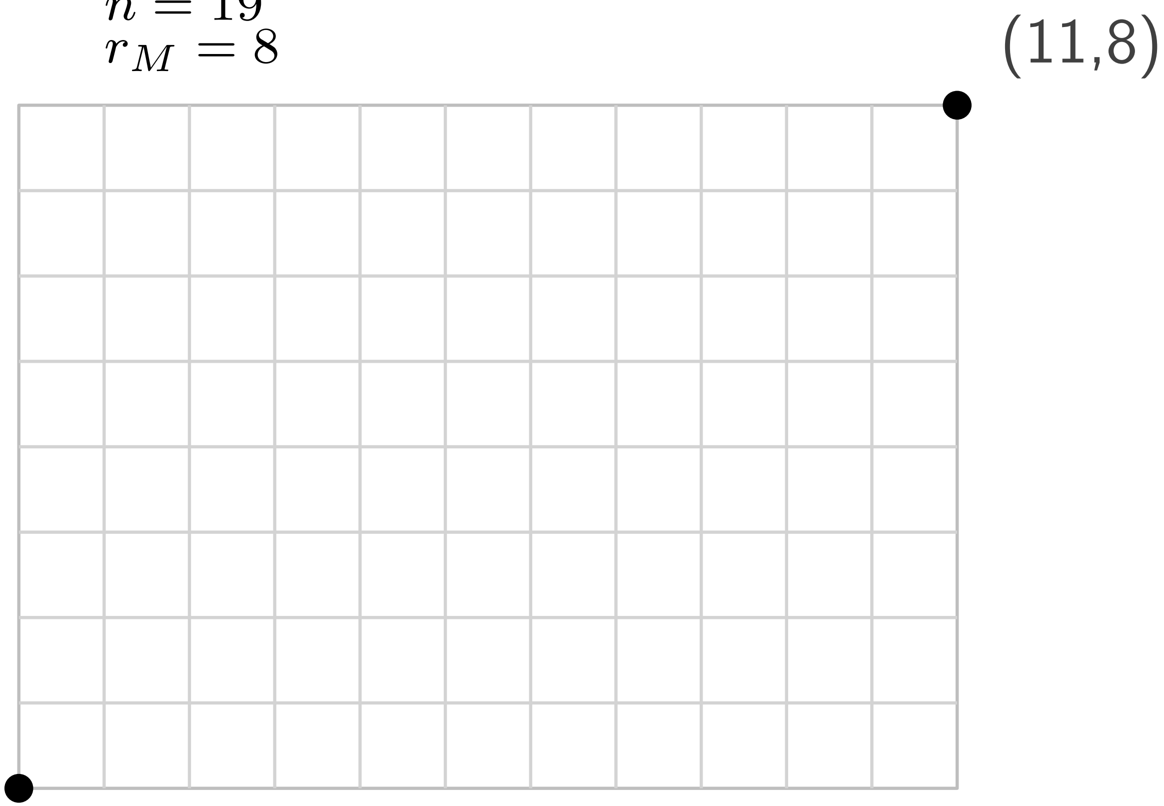

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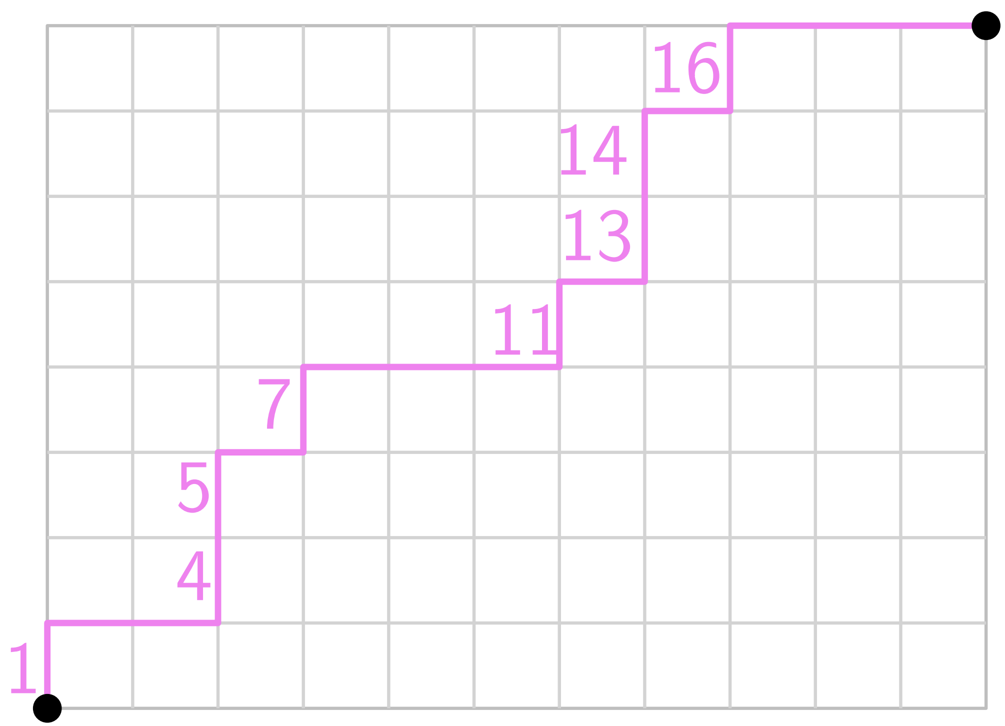

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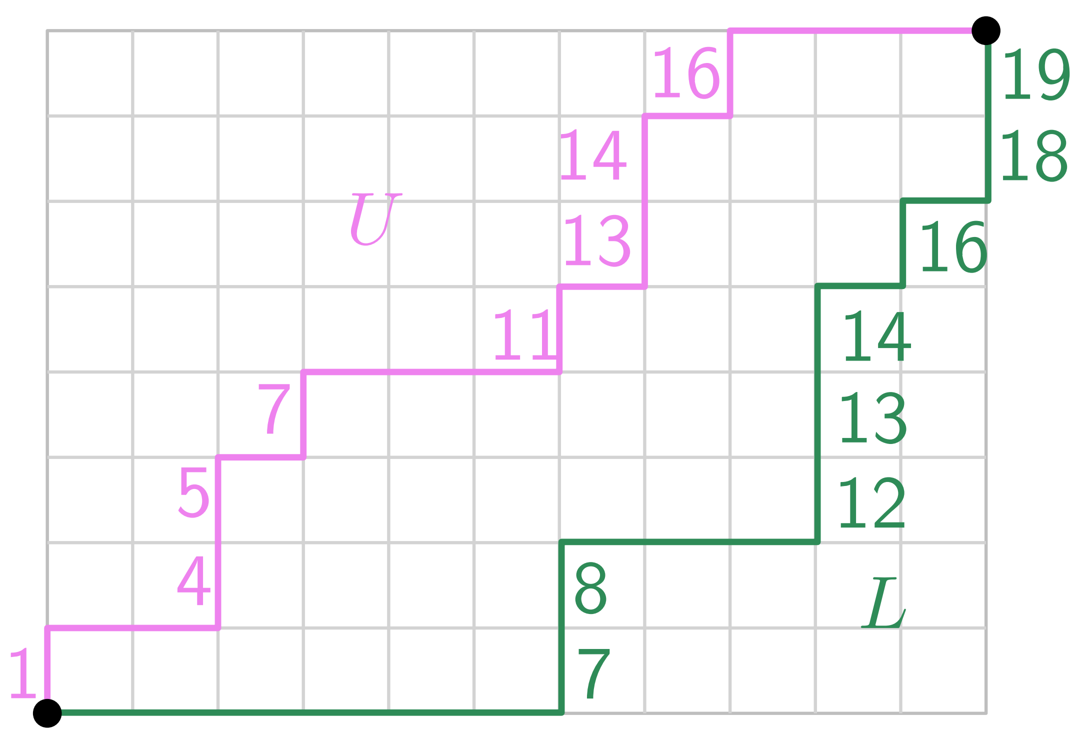

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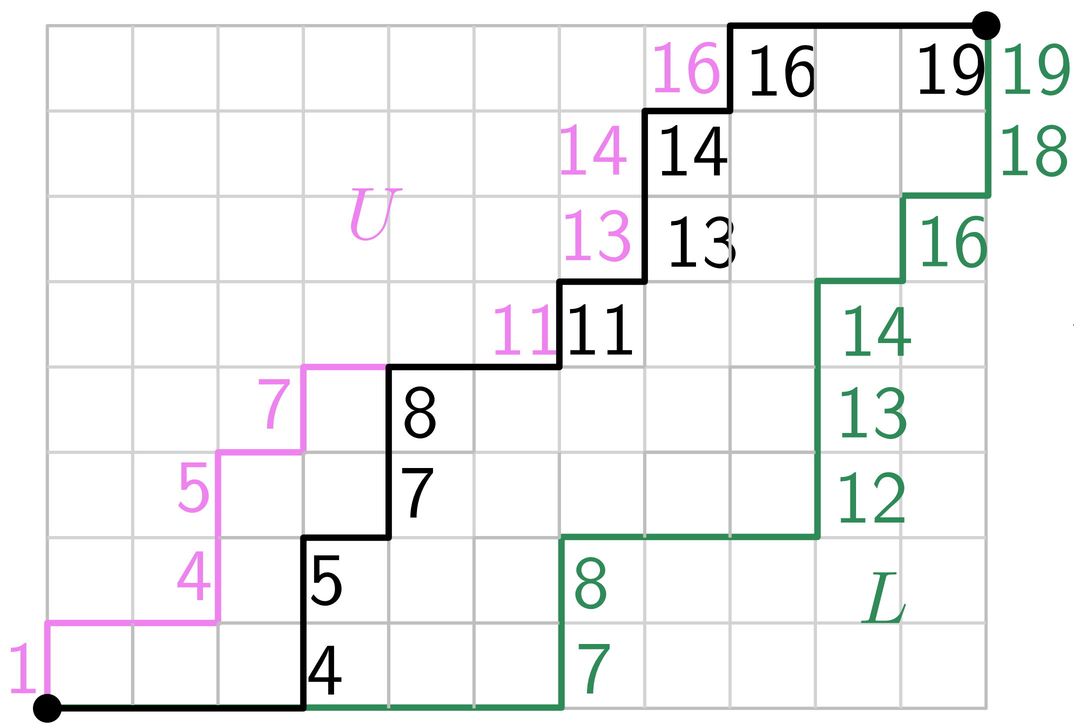


$$\mathcal{B} = \left\{ B \in \binom{[n]}{8} : U \leq_G B \leq_G L \right\}$$

$$M = M[U, L]$$

# Representable matroids

The (real) Grassmannian $Gr_{k,n}$: consists of all $k$-dim v.s.$V$ in $\mathbb{R}^n$.

- choose a basis $\{v_1, \cdots, v_k\}$ for such $V \rightsquigarrow A = \begin{pmatrix} -v_1- \\ \vdots \\ -v_k- \end{pmatrix}_{k \times n}$
- the set $\{I \in \binom{[n]}{k} : p_I \neq 0\}$ is the set of bases of a matroid $M = M_V$.

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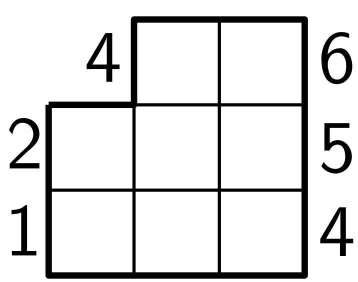
> A matroid $M$ over $[n]$ of rank $k$ is representable (over $\mathbb{R}$) if there is a rank $k$ matrix $A_{k \times n}$ that represents it.




$M = M[124, 456]$

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & \star & 0 & \star & \star \\ 0 & 1 & \star & 0 & \star & \star \\ 0 & 0 & 0 & 1 & \star & \star \end{pmatrix} \end{array}$$

$\star$'s are generic

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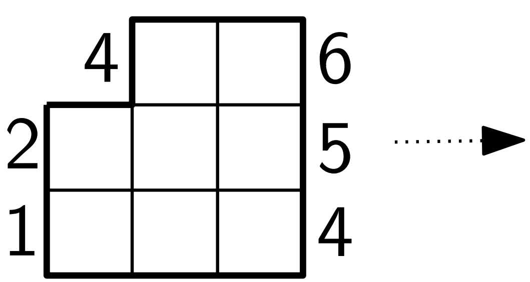


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$\star$'s are generic

$Gr_{k,n} = \cup X_U^L$ Richardson

$M[U,L]$ is generic in $X_U^L$

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# Positroids

The non-negative Grassmannian $Gr_{k,n}^{\geq 0}$ consists of all the $A_{k \times n} \in Gr_{k,n}$ s.t. $p_I \geq 0$ for all $I \in \binom{n}{k}$.

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$M : \{13, 14, 23, 24\}$

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$M' : \{12, 14, 23, 34\}$

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$p_{12} > 0$
$p_{14} > 0 : b > 0$
$p_{23} > 0 : a < 0$
$p_{34} > 0 : a > 0$

rep. matroid not positroid

# Matroids vs. positroids

Given $M = ([n], \mathcal{B})$ its matroid base polytope $P_M$ is

$$P_M := conv\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$$ [GGMS'84]

where $e_B = \sum_{i \in B} e_i$.

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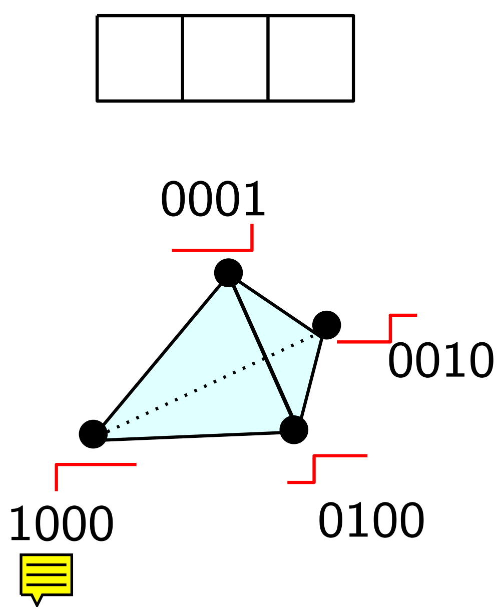


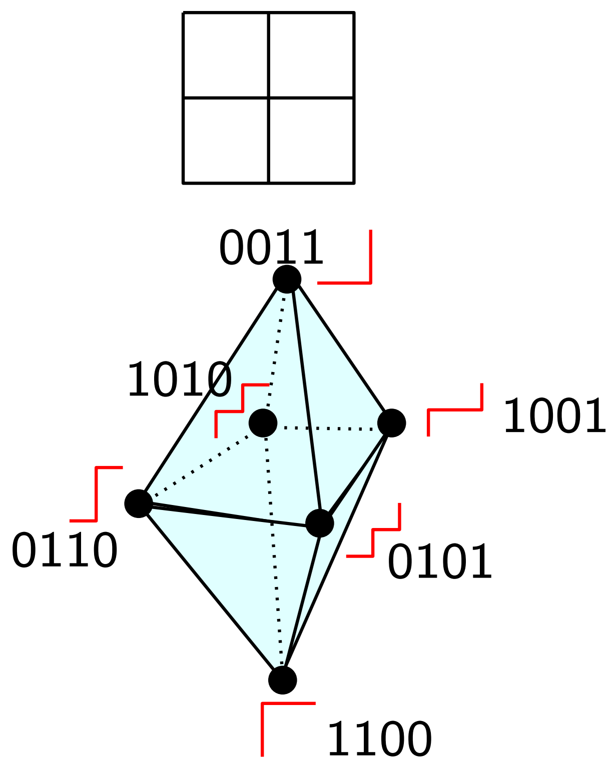

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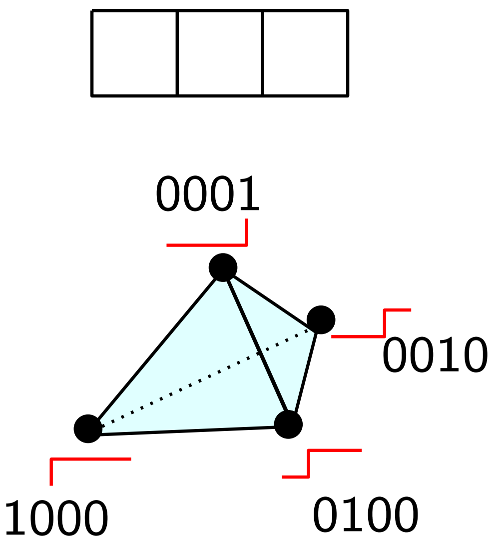


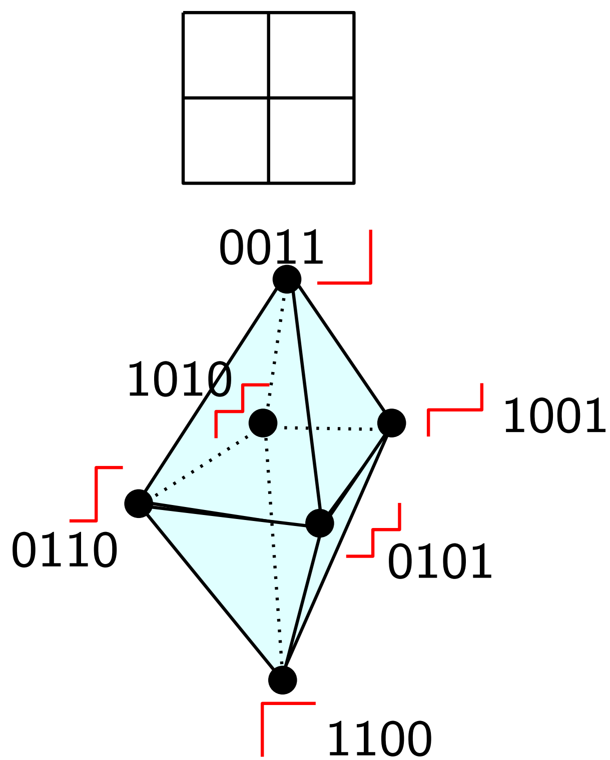


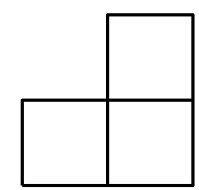

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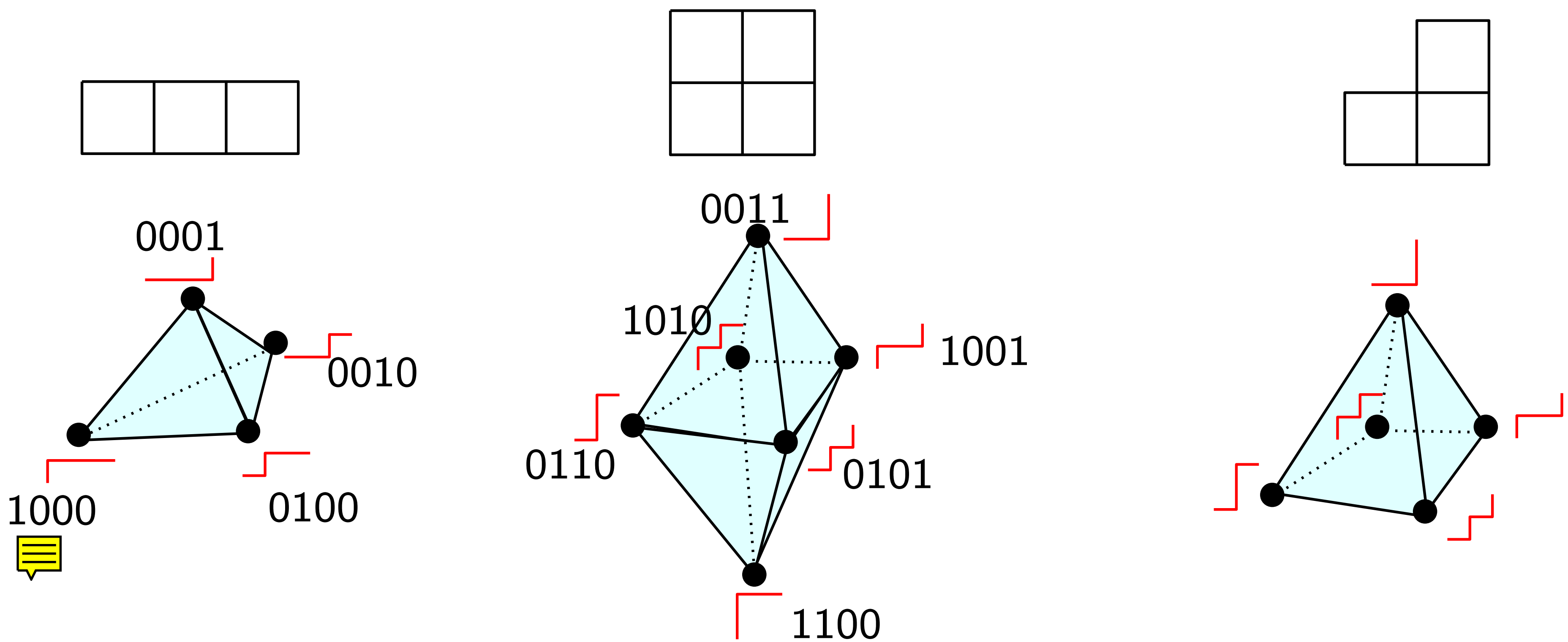

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Theorem [Gelfand, Goresky, Macpherson, Serganova]

Let $P = conv\{e_B : B \in \mathcal{B}\}$ be a $0/1$-polytope. Then $\mathcal{B}$ is the collection of bases of a matroid iff every edge of $P$ has direction $e_i - e_j$.

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$$\{Matroids\} = \{0/1 \; polytopes\} \cap \{gen. \; permutahedra\}$$

# Positroid (base) polytope

Alcoved polytope: $H$-description consists of $c_{ij} \leq x_i + x_{i+1} + \cdots + x_j \leq b_{ij}$

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An application:

- LPMs are positroids: alcoved description by [Knauer, Martinez, Ramírez '13]

$$P_M = \left\{ \vec{p} \in \mathbb{R}^n \;\middle|\; 0 \leq p_i \leq 1 \text{ and } \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j \text{ for all } i \in [n] \right\}$$

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∘ Alcoved pol. hace a canonical regular unimodular triangulation. [Lam, Postnikov]

# Combinatorics of positroids

- for each $i \in n$ define $<_i$ : $i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1$
- Grassmann necklace of $M$: $\mathcal{I} = (I_1, \ldots, I_n)$ where $I_s = \min_{<_s} \mathcal{B}$
- Envelope: $\mathcal{P}(\mathcal{I}) = \{B : B \geq_s I_s, \forall s\}$
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Example:

$\mathcal{B}(M) = \{13, 14, 23, 24\} \rightsquigarrow$

| | |
|---|---|
| $I_1$ | $1234$ |
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| $I_4$ | $4123$ |

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$$\mathcal{B}(M) = \{13, 14, 23, 24\} \rightsquigarrow \begin{array}{ll} I_1 = 13 & 1234 \\ I_2 = 23 & 2341 \\ I_3 = 31 & 3412 \\ I_4 = 41 & 4123 \end{array}$$

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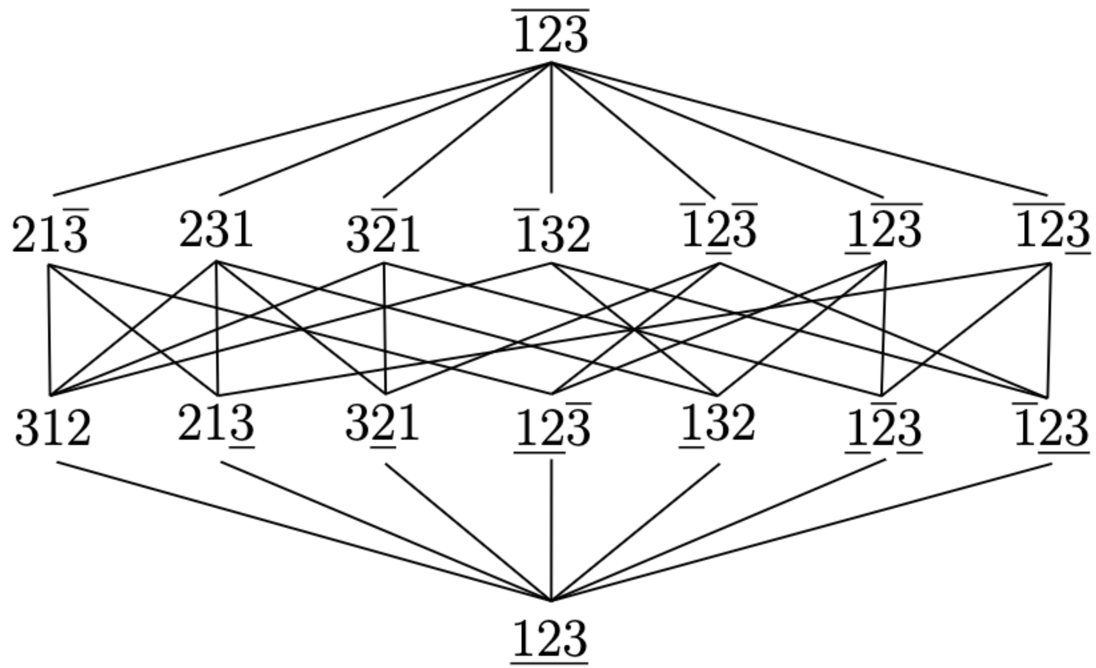
Grassmann necklaces
Le-diagrams
Eq. classes of plabic graphs...

[Postnikov'06]

# Positroids on $[3]$



# Flag matroids

$$M: \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \end{array}$$

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Circuits: minimal dependent sets

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Given matroids $N, M$ over $[n]$, we say $N$ is a quotient of $M$ if every circuit of $M$ is union of circuits of $N$ (denoted $N \leq_q M$).
In this case the sequence $(N, M)$ is a flag matroid.

# Flags of positroids

Question: Given positroids $N, M$ over $[n]$, is there a combinatorial rule to determine if $N \leq_q M$, or viceversa?

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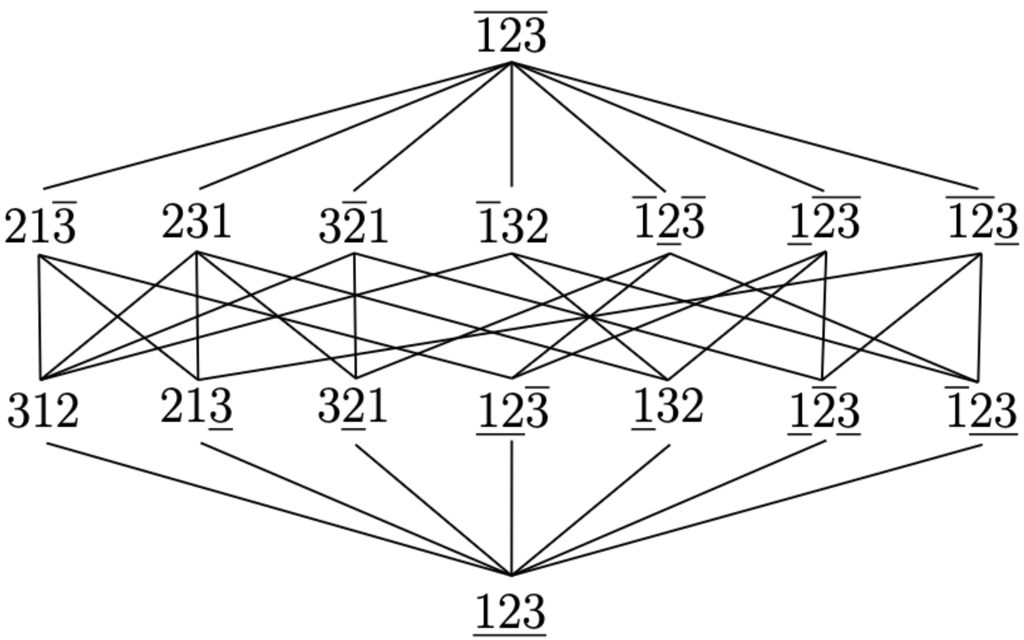

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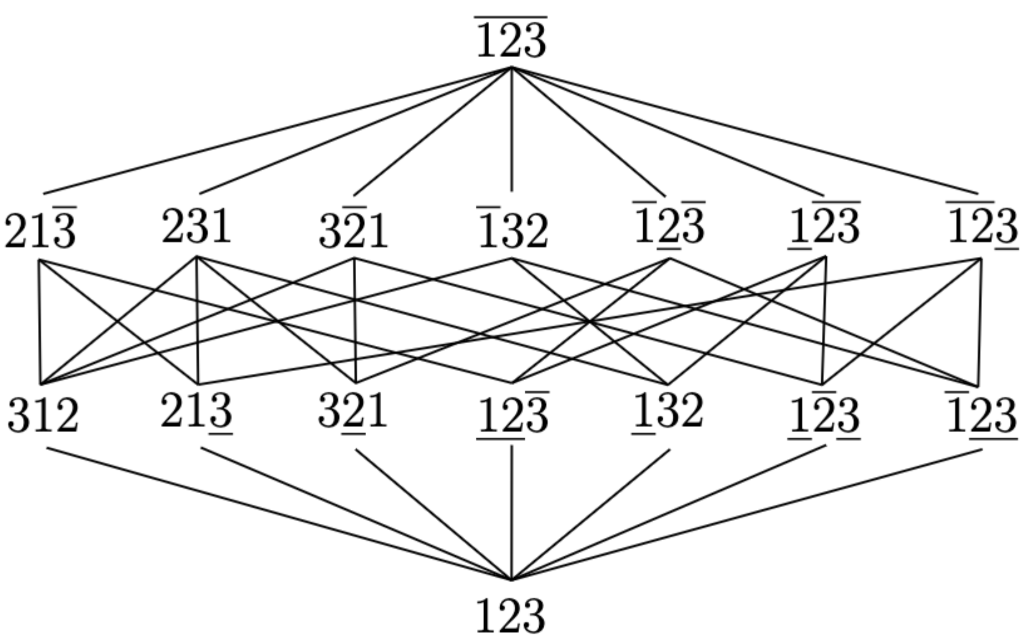


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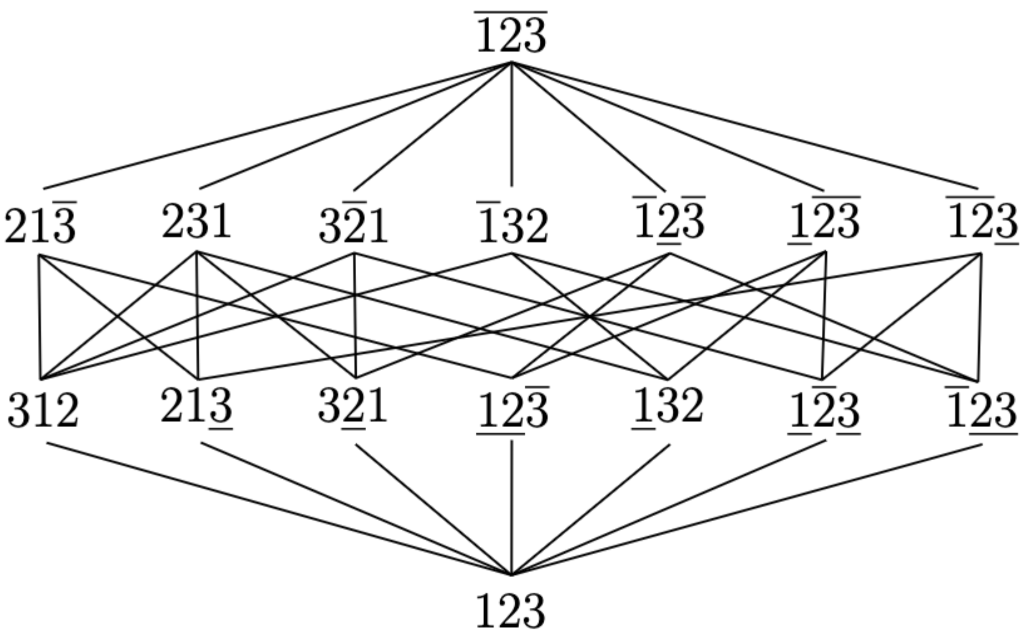

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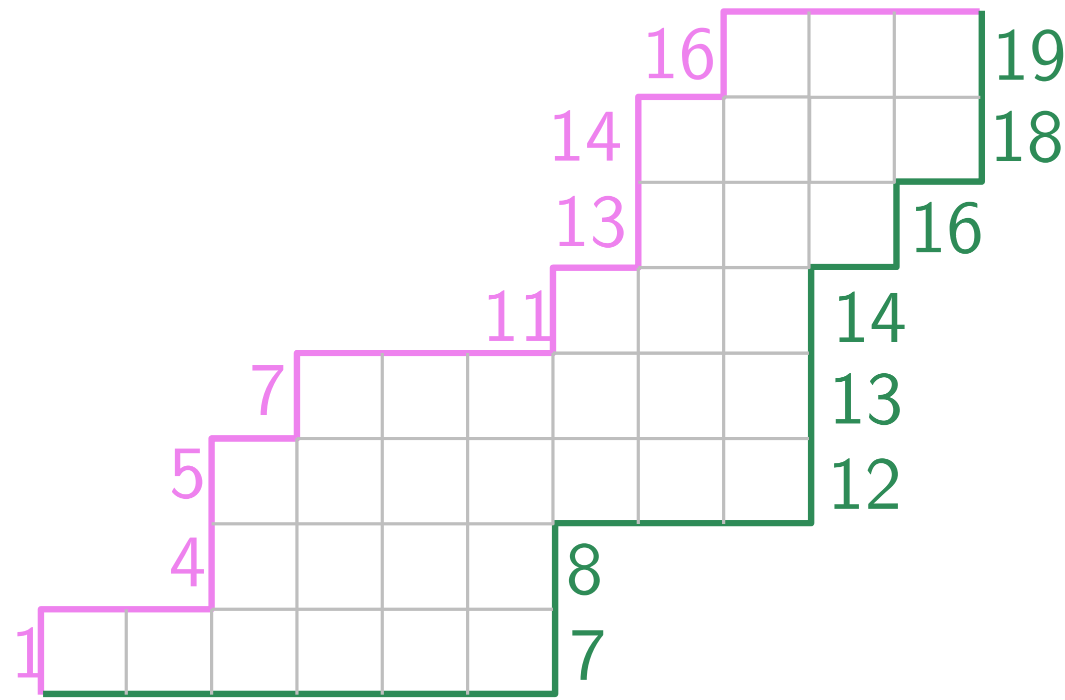

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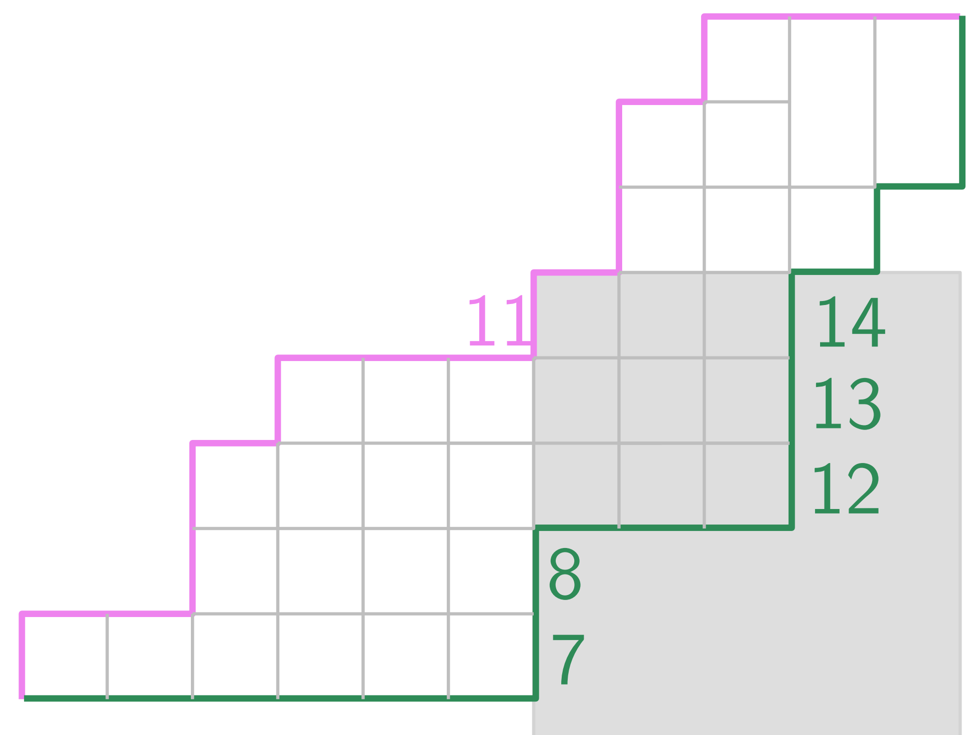

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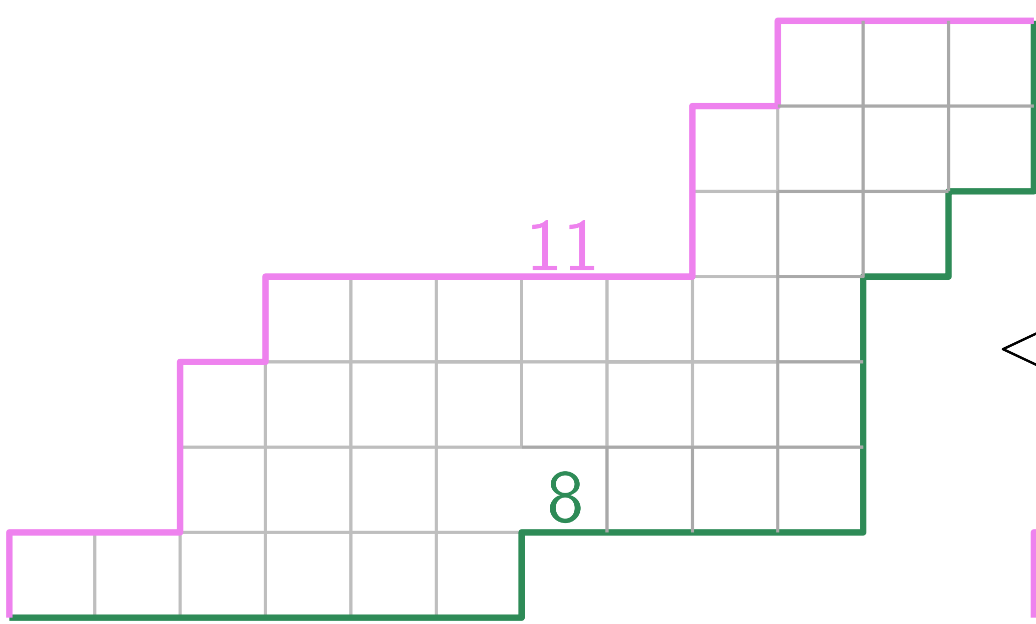
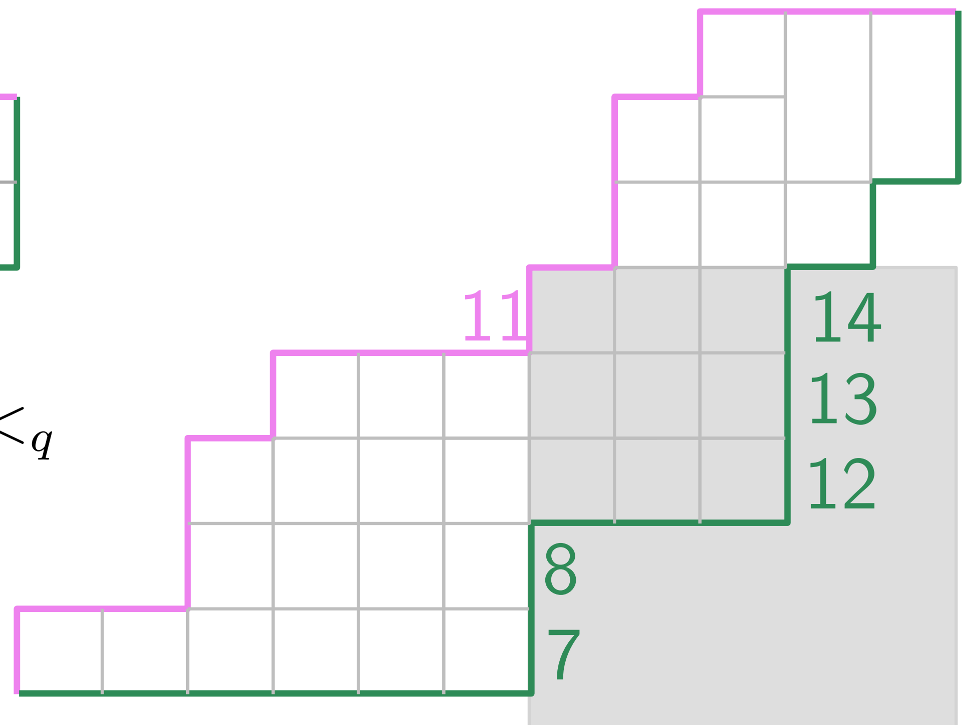



$<_q$



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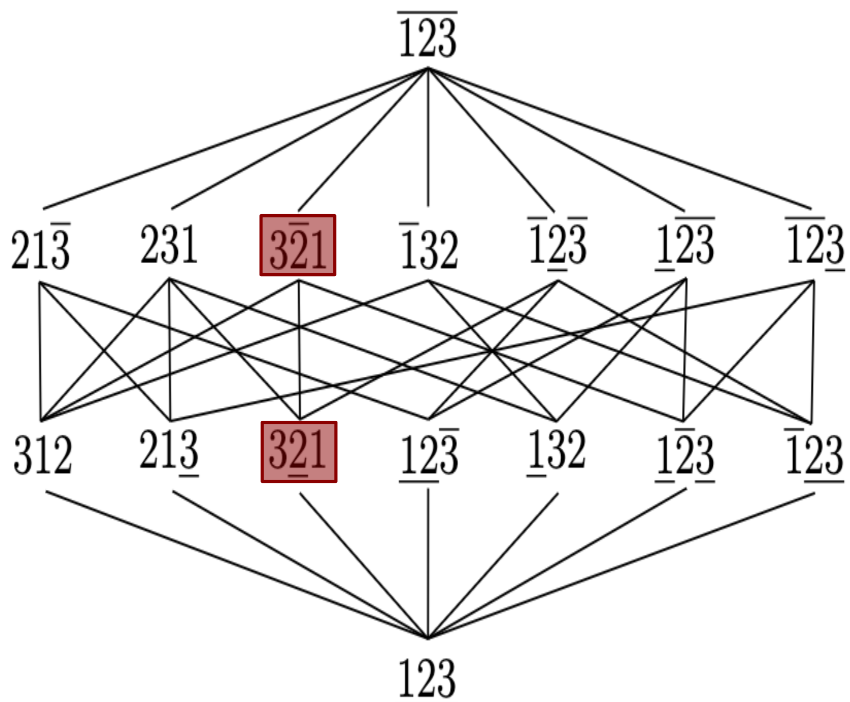


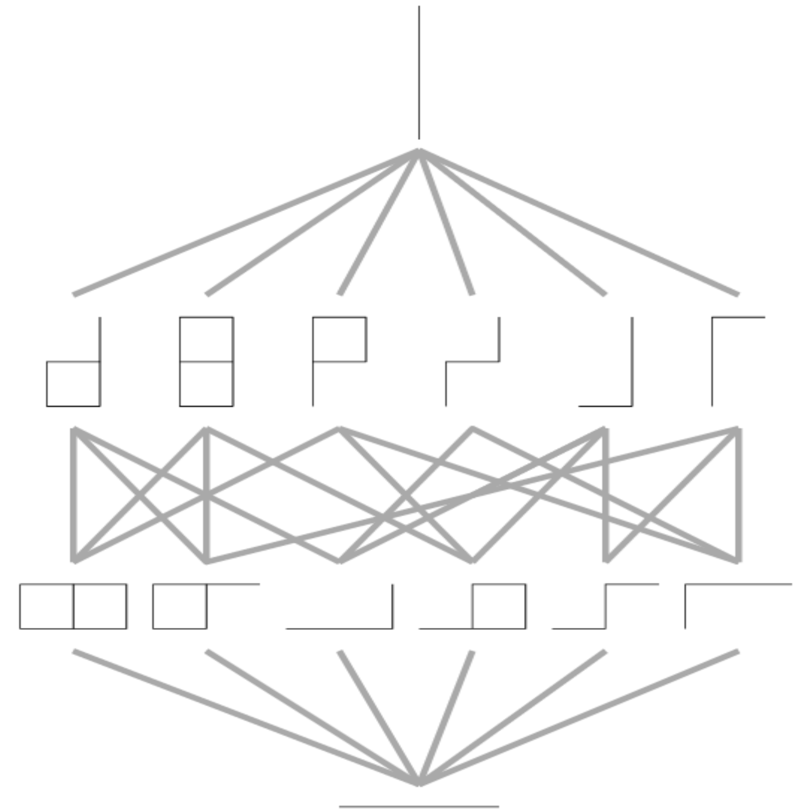

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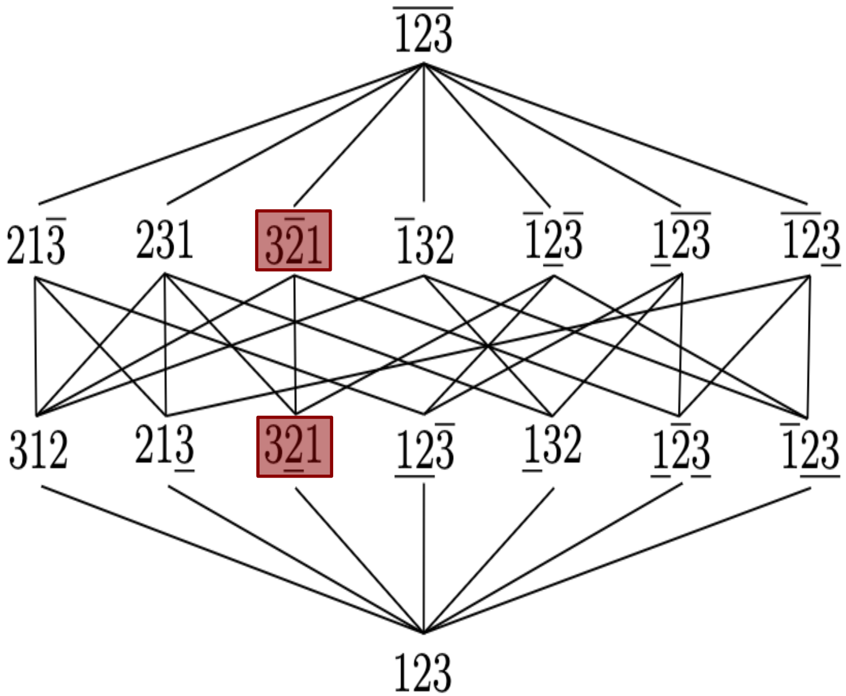


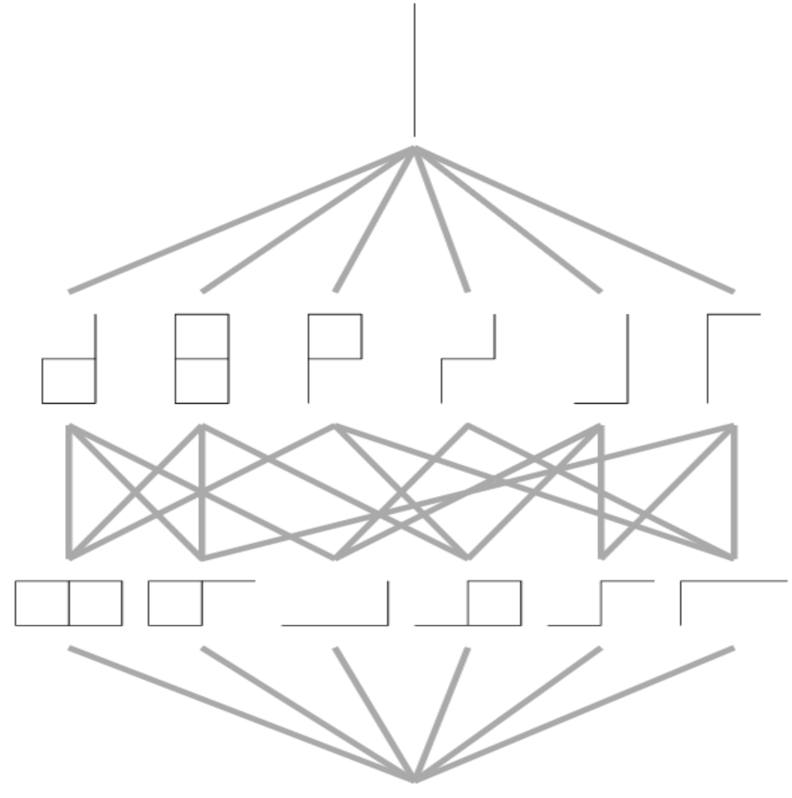

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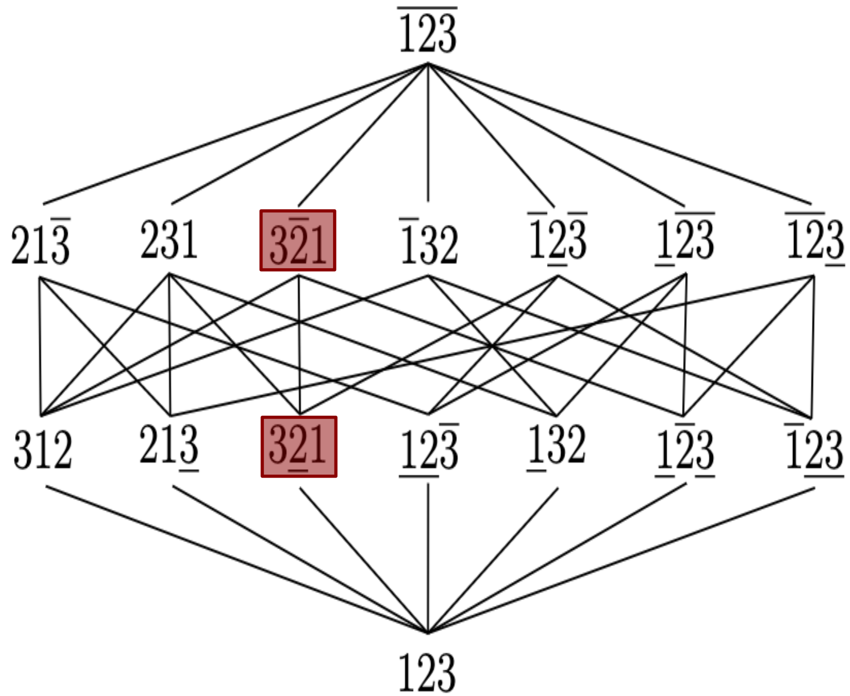

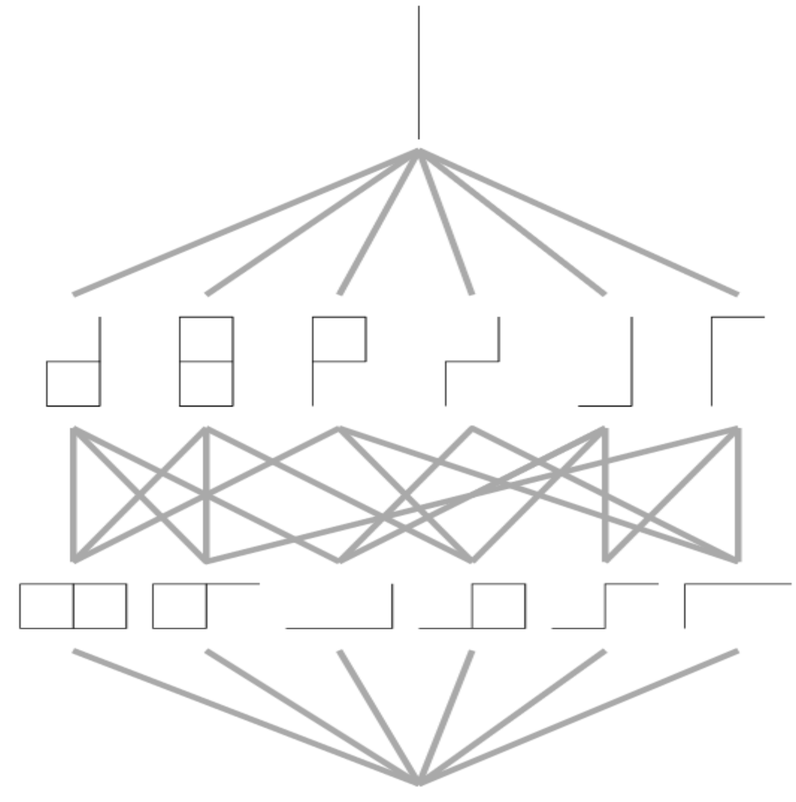

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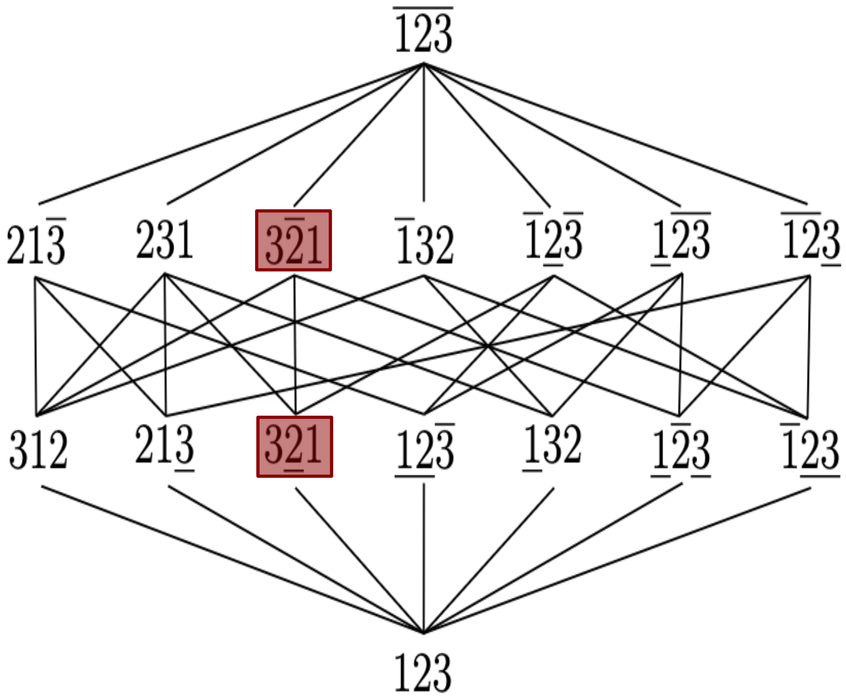


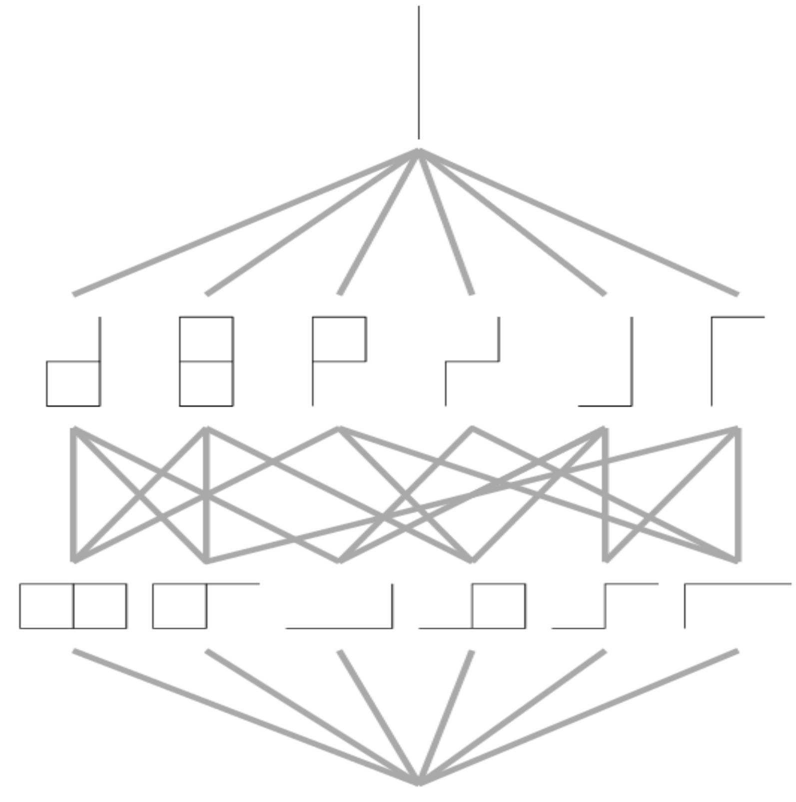

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- Right: every $F$ is a point in the nonnegative (full) flag variety $Fl_n^{\geq 0}$:

$\exists A \in Gr_{n,n}^{\geq 0}$ such that

$$A = \begin{pmatrix} -v_1- \\ \vdots \\ -v_i- \\ \vdots \\ -v_n- \end{pmatrix} \quad A_i \in Gr_{i,n}^{\geq 0} \text{ represents } M_i$$

# Flag matroid polytope

Given a flag $F : M_1 < \cdots < M_j$ its flag matroid polytope is

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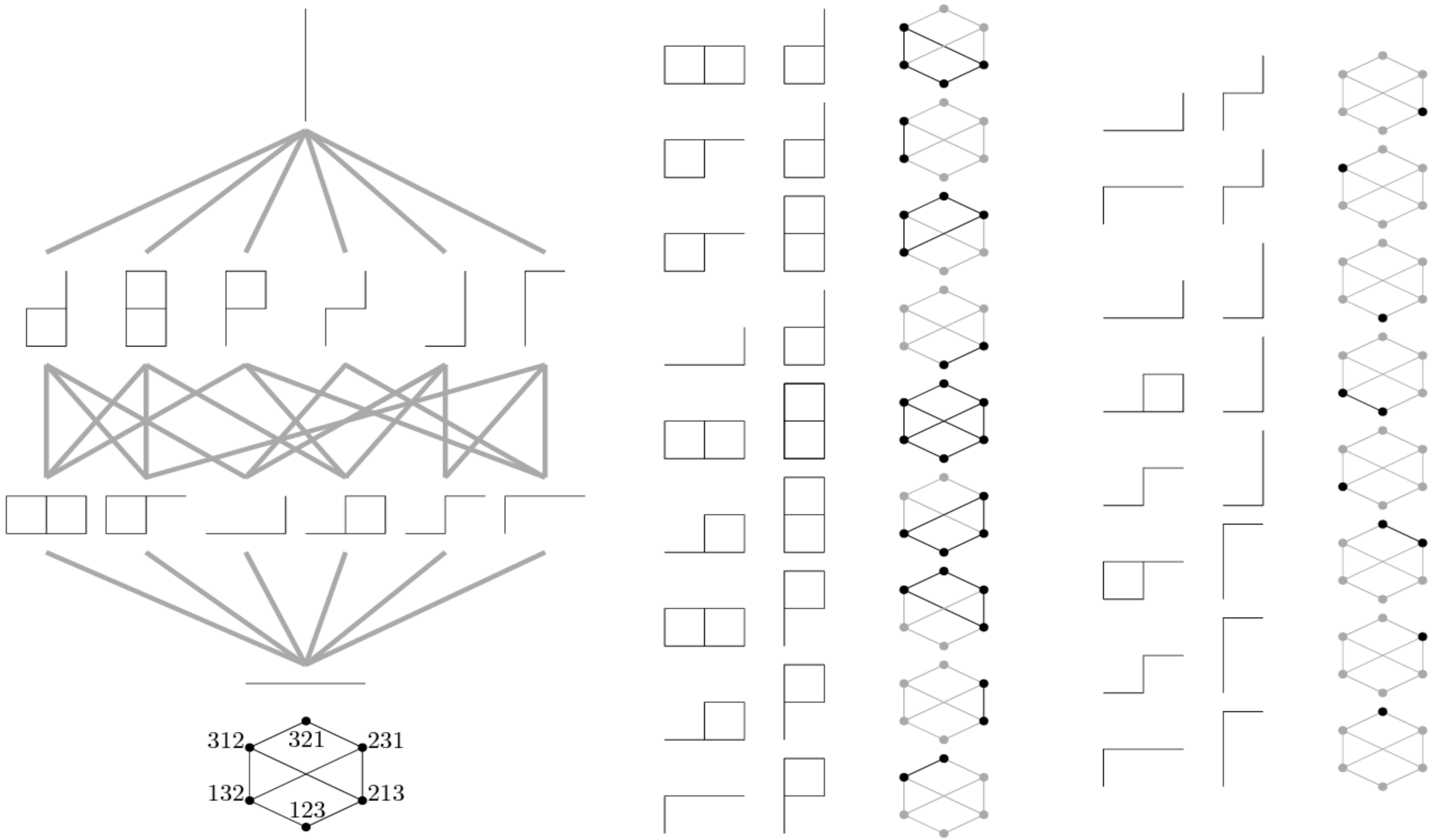
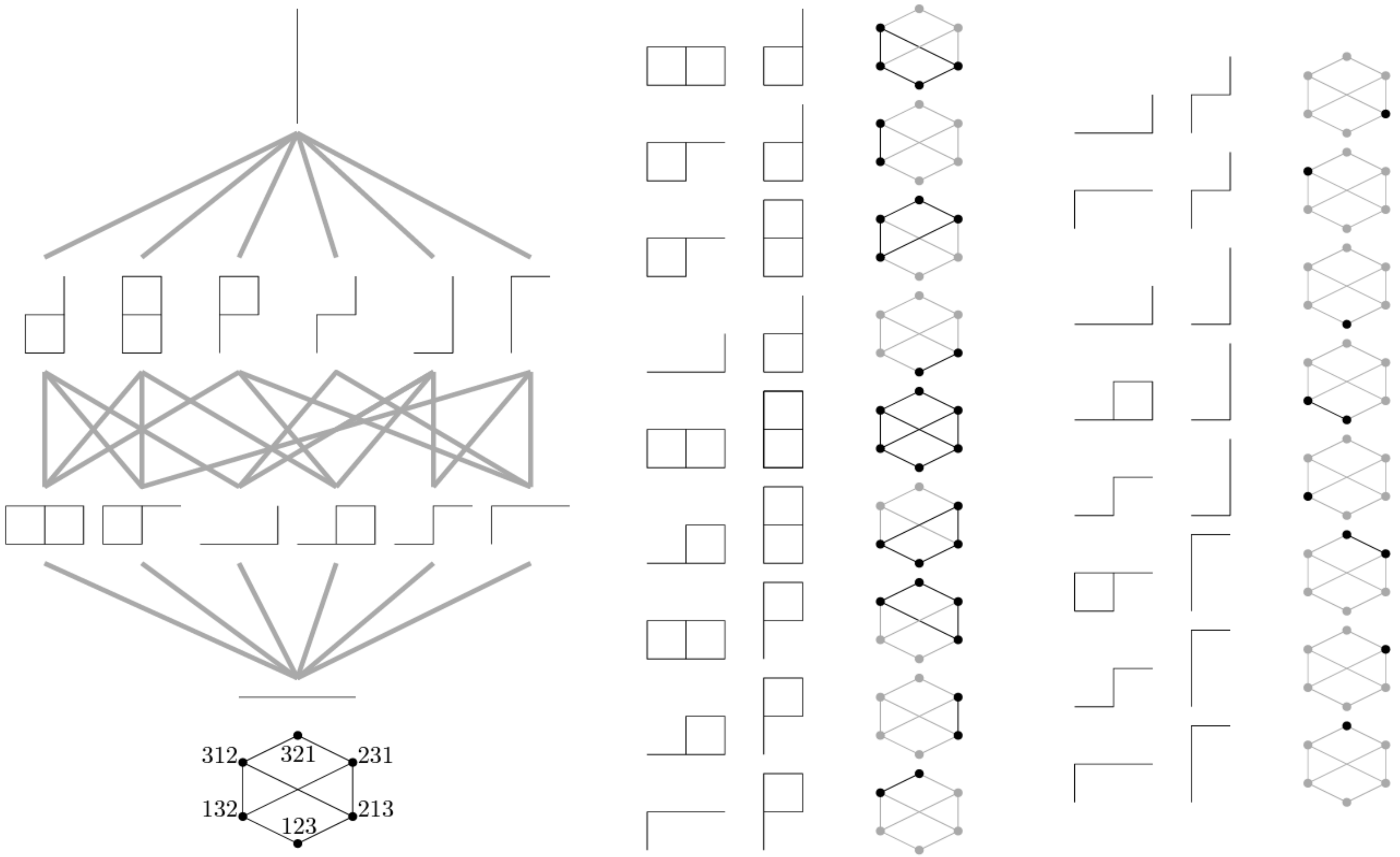


# Bruhat interval polytopes




[Williams et al.] Bijective correspondence between (polytopes of) $F \in \mathcal{F}\ell_n^{\geq 0}$ and intervals in $Bruhat_n$.

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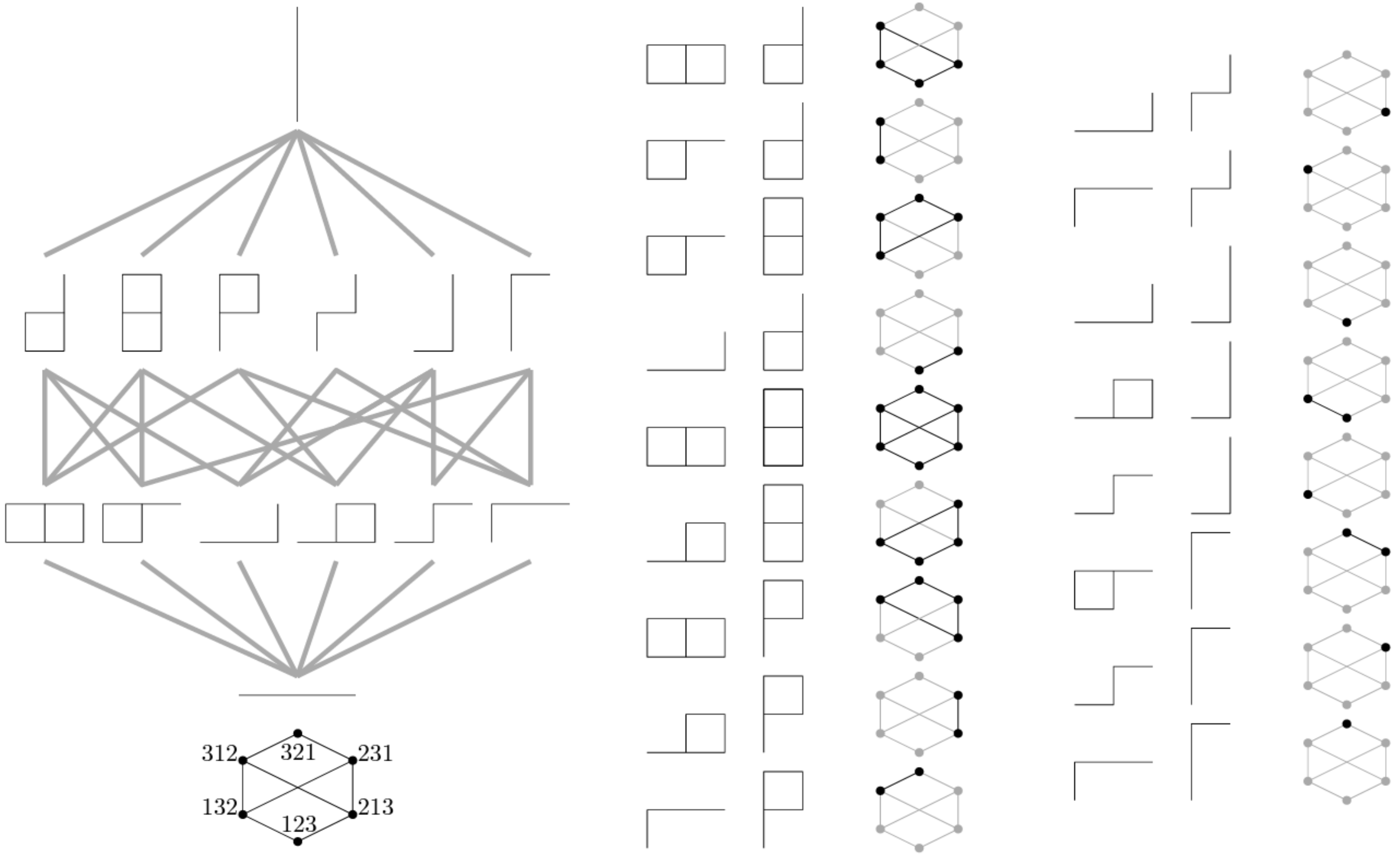


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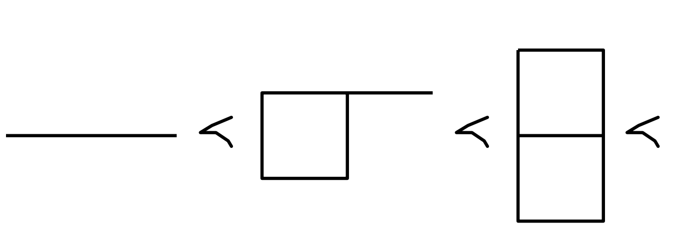
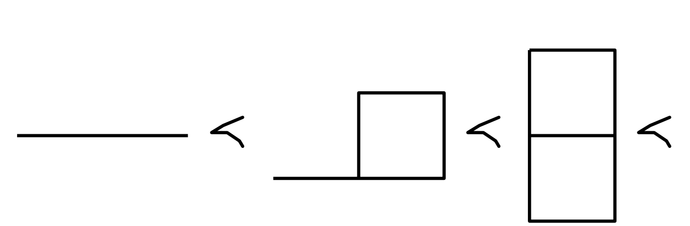
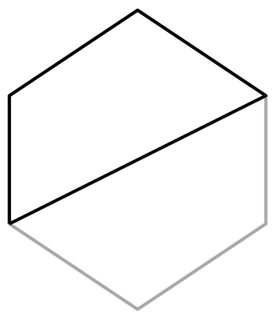
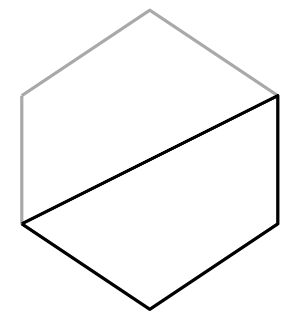
**Theorem** [B., Knauer'22]

Every full LPFM $F$ is an interval in $Bruhat_n$. Thus, $F$ corresponds to a point in $\mathcal{F}\ell_n^{\geq 0}$



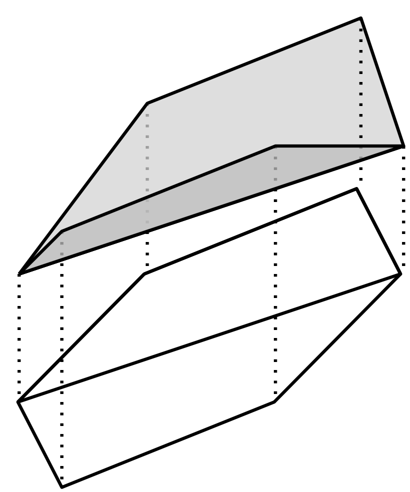






## Regular subdivision:

Comes from a height vector on the vertices



## $\Delta_{k,n}$

- Polytope of $U_{k,n}$

## $Perm_n$

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Theorem: [Lukowski,Parisi,Williams'20]

Let $P = (P_I)_I \in \mathbb{R}^{\binom{[n]}{k}}$. TFAE:
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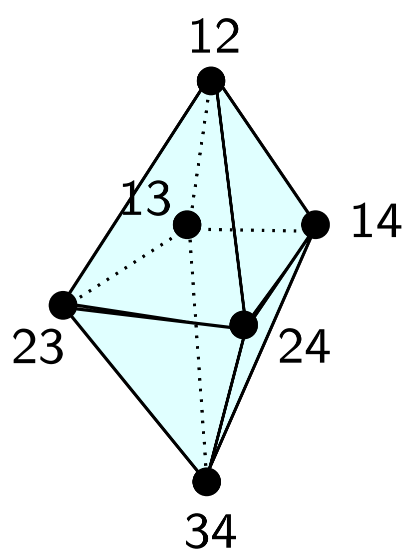


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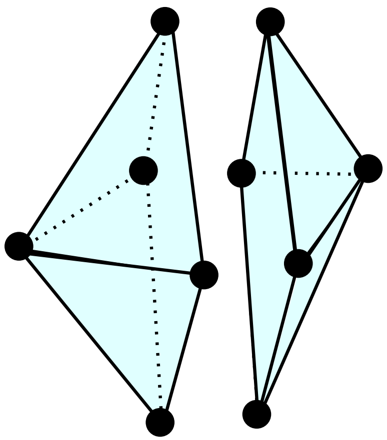
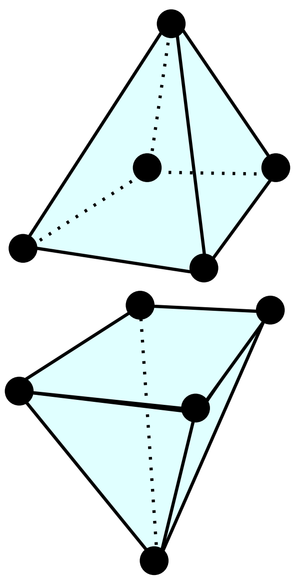


positroidal
not LPM



LPMs

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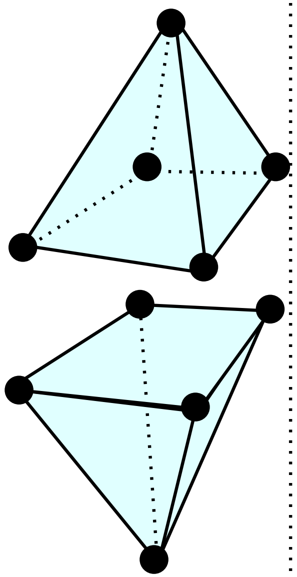

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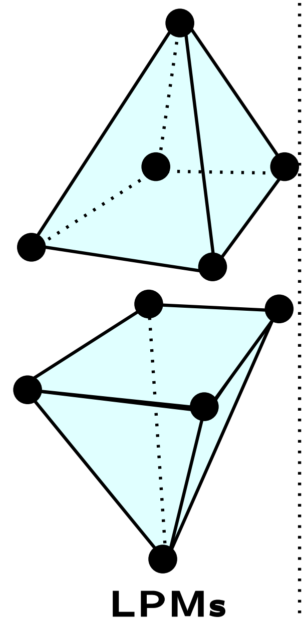


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- Polytope of $\mathcal{U} : U_{0,n} < U_{1,n} < \cdots < U_{n,n}$

**Theorem:** [Joswig,Loho,Luber,Olarte'22] Boretsky,Eur,Williams'22]
Let $\mu = (\mu_I)_I \in \prod_{k=1}^{n} \mathbb{R}^{\binom{[n]}{k}}$. TFAE:
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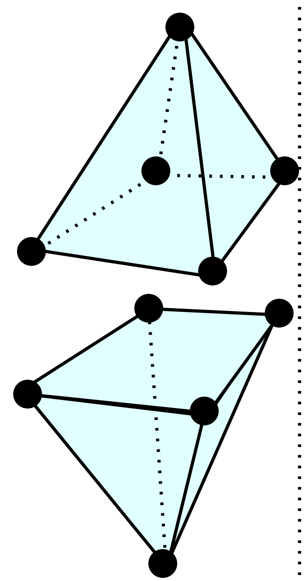

LPMs

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Let $\mu = (\mu_I)_I \in \prod_{k=1}^{n} \mathbb{R}^{\binom{[n]}{k}}$. TFAE:
(i) The subdivision of $Perm_n$ induced by $\mu$ consists of Bruhat interval polytopes.
(ii) $\mu$ is a point in $Tr^{\geq 0}\mathcal{F}\ell_n$.

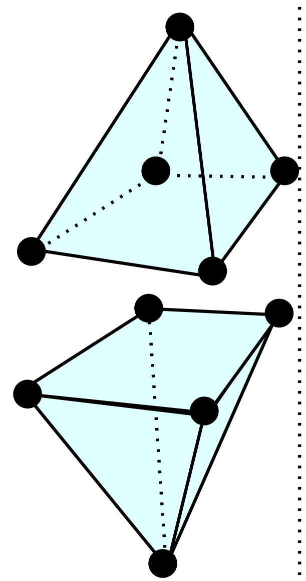
**Question:**
Are there subdivisions of $Perm_n$ into LPFMs?

## $\Delta_{k,n}$

○ Polytope of $U_{k,n}$

**Theorem:** [Lukowski,Parisi,Williams'20]
Let $P = (P_I)_I \in \mathbb{R}^{\binom{[n]}{k}}$. TFAE:
(i) The subdivision of $\Delta_{kn}$ induced by $P$ is positroidal.
(ii) $P$ is a point in $Tr^{\geq 0}Gr_{kn}$.

○ [Chatelain, Ramirez'11] finest subdiv. of LPMs in LPMs.
○ [Tewari, Ashraf'23] LPMs subdiv. using tropical geometry.



**LPMs**

## $Perm_n$

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**Question:**
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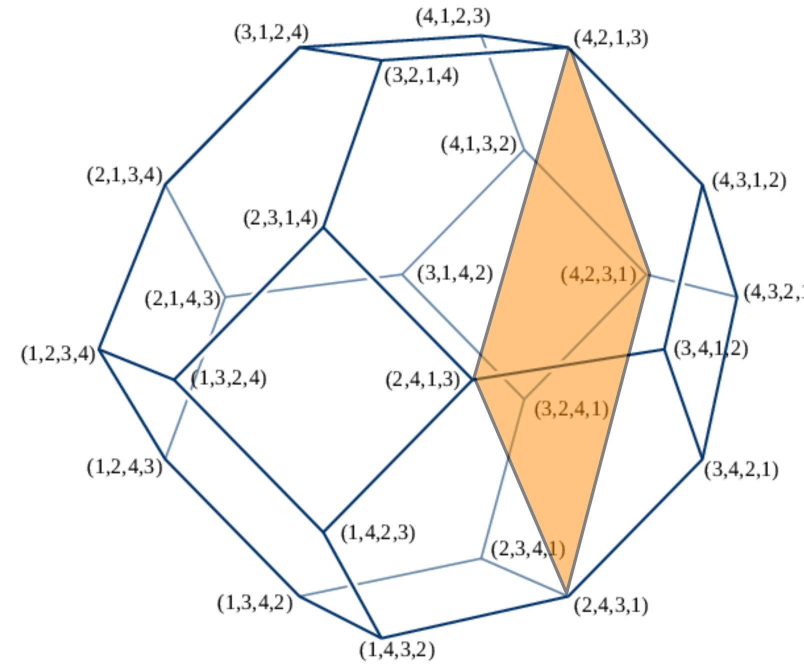
YES!


(4,1,2,3)
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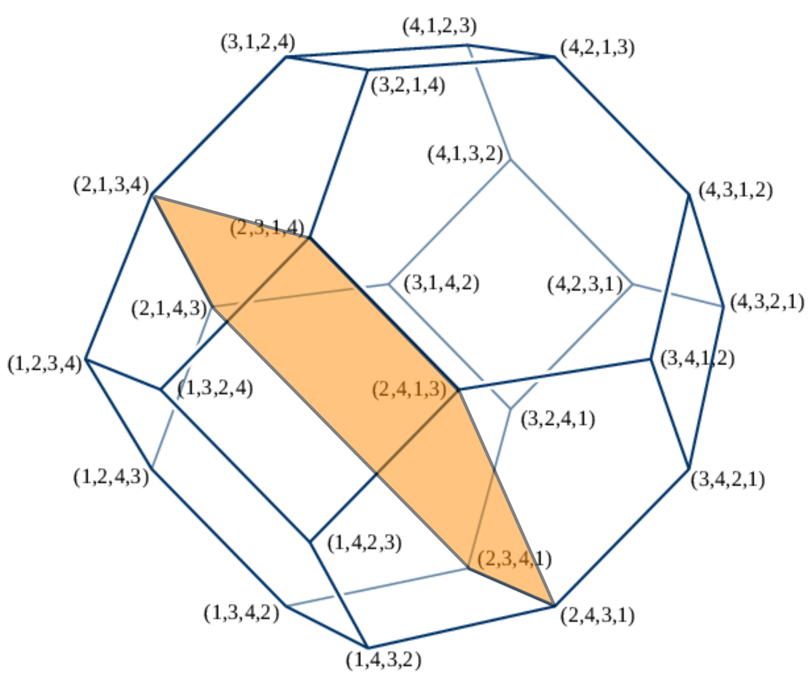
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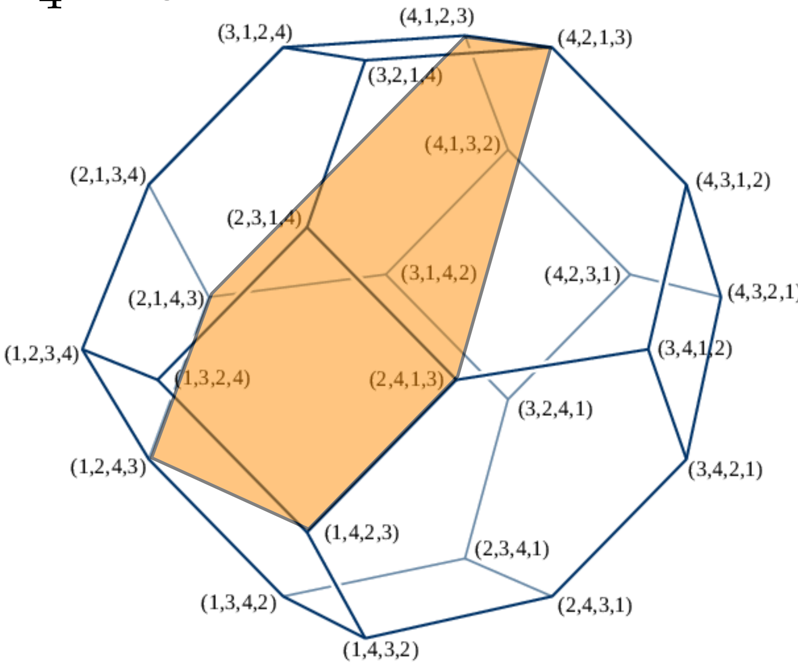
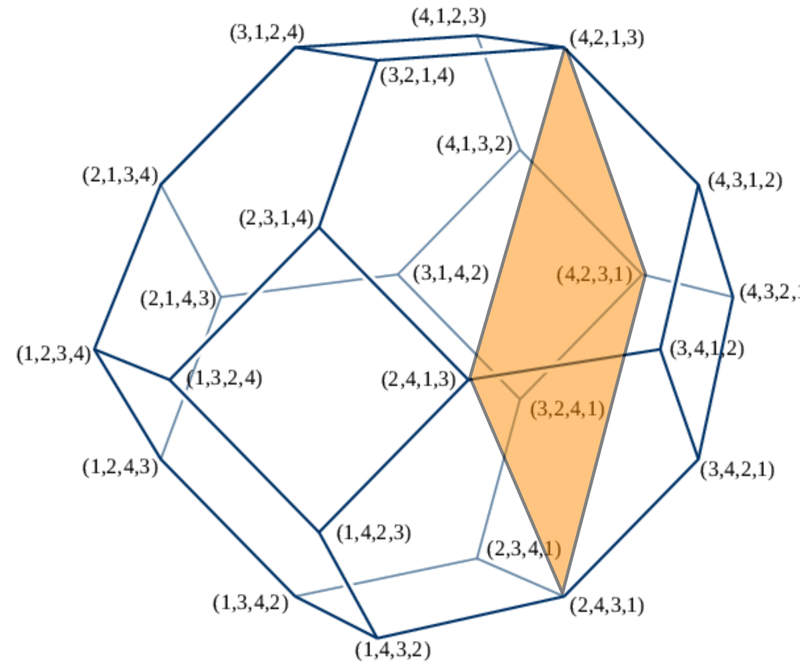
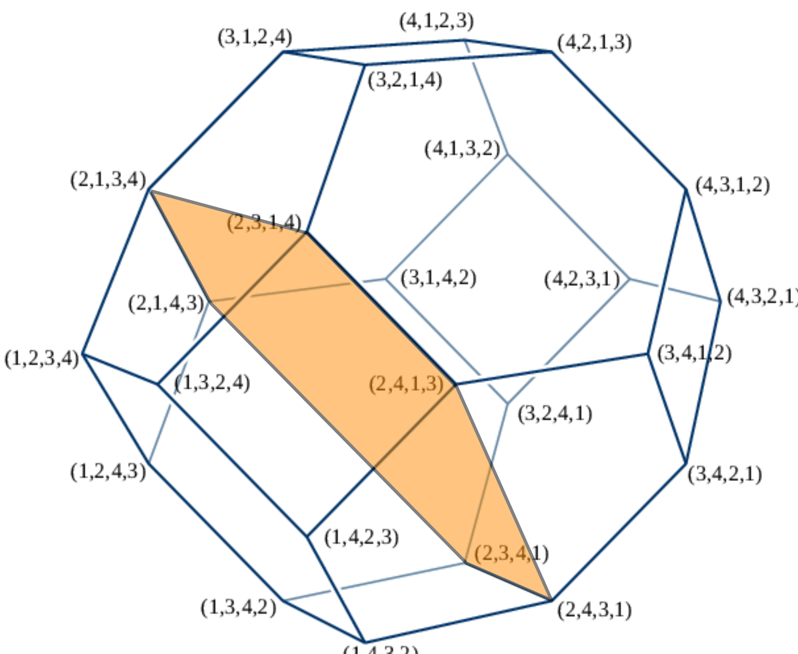
$x_4 = 3$

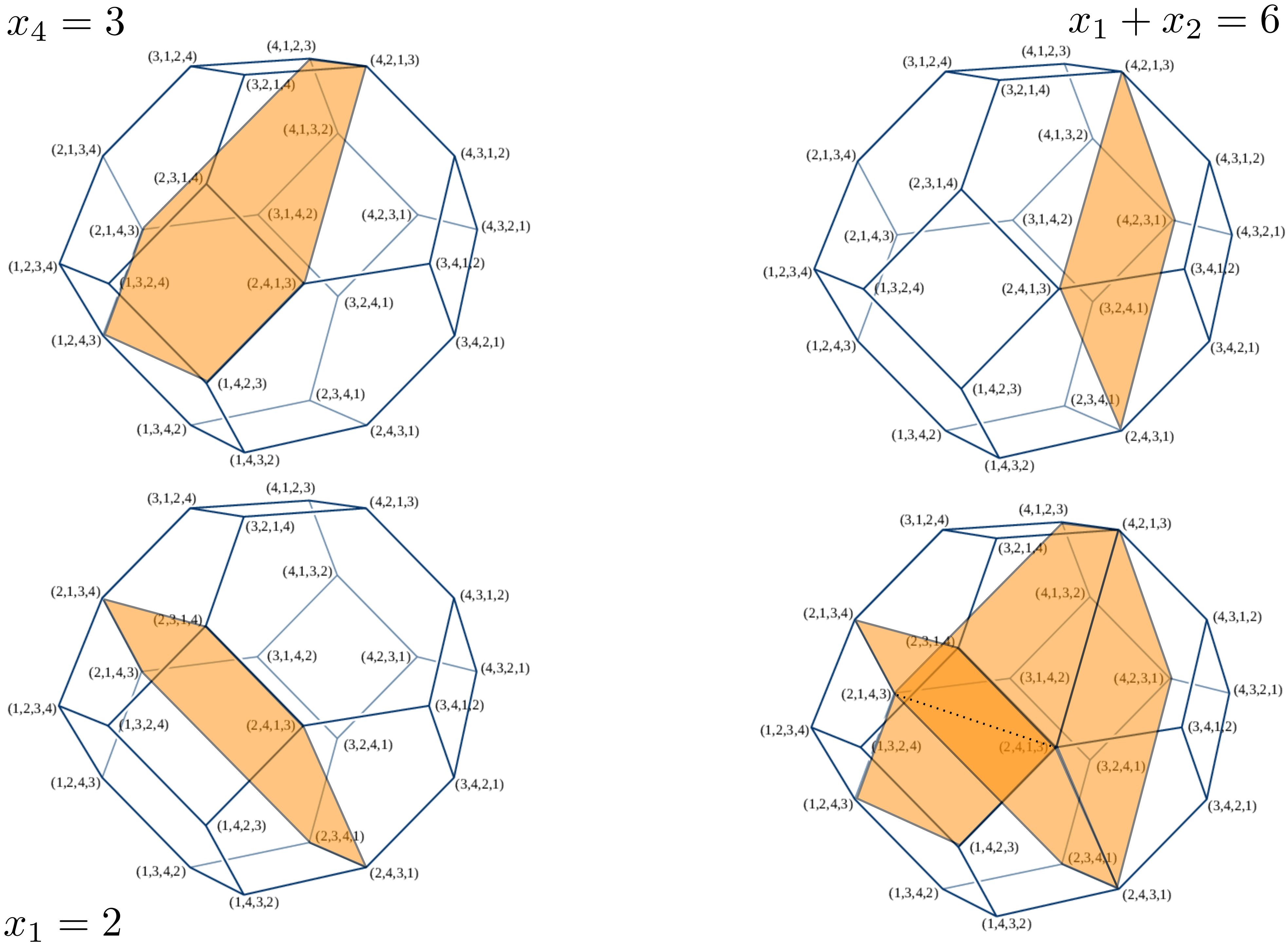



$x_1 + x_2 = 6$




$x_1 = 2$


$x_4 = 3$
(3,1,2,4) (4,1,2,3) (4,2,1,3) (3,2,1,4) (4,1,3,2) (2,1,3,4) (4,3,1,2) (2,3,1,4) (3,1,4,2) (4,2,3,1) (2,1,4,3) (4,3,2,1) (1,2,3,4) (1,3,2,4) (2,4,1,3) (3,4,1,2) (3,2,4,1) (1,2,4,3) (3,4,2,1) (1,4,2,3) (2,3,4,1) (1,3,4,2) (2,4,3,1) (1,4,3,2)
$x_1 + x_2 = 6$
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(3,1,2,4) (4,1,2,3) (4,2,1,3) (3,2,1,4) (4,1,3,2) (2,1,3,4) (4,3,1,2) (2,3,1,4) (3,1,4,2) (4,2,3,1) (2,1,4,3) (4,3,2,1) (1,2,3,4) (1,3,2,4) (2,4,1,3) (3,4,1,2) (3,2,4,1) (1,2,4,3) (3,4,2,1) (1,4,2,3) (2,3,4,1) (1,3,4,2) (2,4,3,1) (1,4,3,2)

# Coarsest LPFM subdivisions

Let $\omega = n\ n-1 \cdots 21$, $e = 12 \cdots n$
- $u_i\!\downarrow = i\hat{\omega}_i$
- $u_i\!\uparrow = i\hat{e}_i$

**Theorem** [B., Knauer'24]

Each of the following hyperplanes give a coarsest non-trivial subdivision of $Perm_n$ into LPFMs
- $x_1 = i$ for $i = 2, \ldots, n-1 \rightsquigarrow [e, u_i\!\downarrow] \cup [u_i\!\uparrow, \omega]$

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- $x_1 + x_2 = 4$
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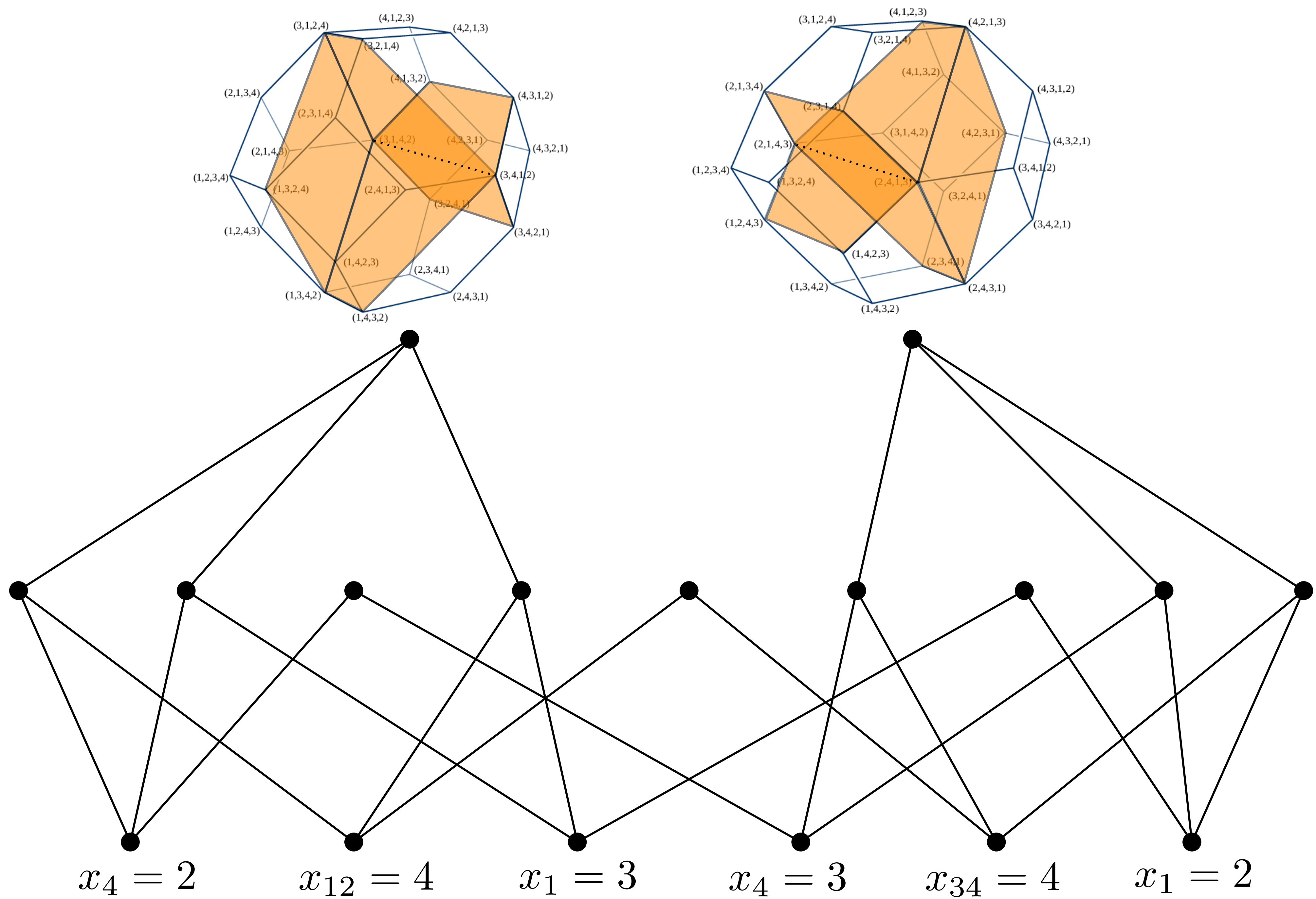
**Example:** $n = 5$

- $x_1 = 2 : [e, 25431] \cup [21345, \omega]$
- $x_1 = 3 : [e, 35421] \cup [31245, \omega]$
- $x_1 = 4 : [e, 45321] \cup [41235, \omega]$
- $x_1 + x_2 = 4 : [e, 31|542] \cup [13|245, \omega]$
- $x_1 + x_2 = 8 : [e, 53|421] \cup [35|124, \omega]$
- $x_5 = 2 : [e, 54312] \cup [13452, \omega]$
- $x_5 = 3 : [e, 54213] \cup [12453, \omega]$
- $x_5 = 4 : [e, 53214] \cup [12354, \omega]$
- $x_4 + x_5 = 4 : [e, 542|31] \cup [245|13, \omega]$
- $x_4 + x_5 = 8 : [e, 421|53] \cup [124|35, \omega]$

# LPFMs subdivisions of $Perm_4$



**Proposition** [B., Knauer'24]

Let $\mathcal{P} = I_1 \cup \cdots \cup I_m$ be an LPFM subdivision of $Perm_n$ and let $I_j = [u_j, v_j]$. Then $\mathcal{P}^* = I_1^* \cup \cdots \cup I_m^*$ is an LPFM subdivision of $Perm_n$ where $u_j^*(i) := n + 1 - u_j(i)$ (and similar for $v_j^*$).

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- $(x_i = a)^* = x_i = n - a + 1$ for $a = 2, \ldots, n-1$
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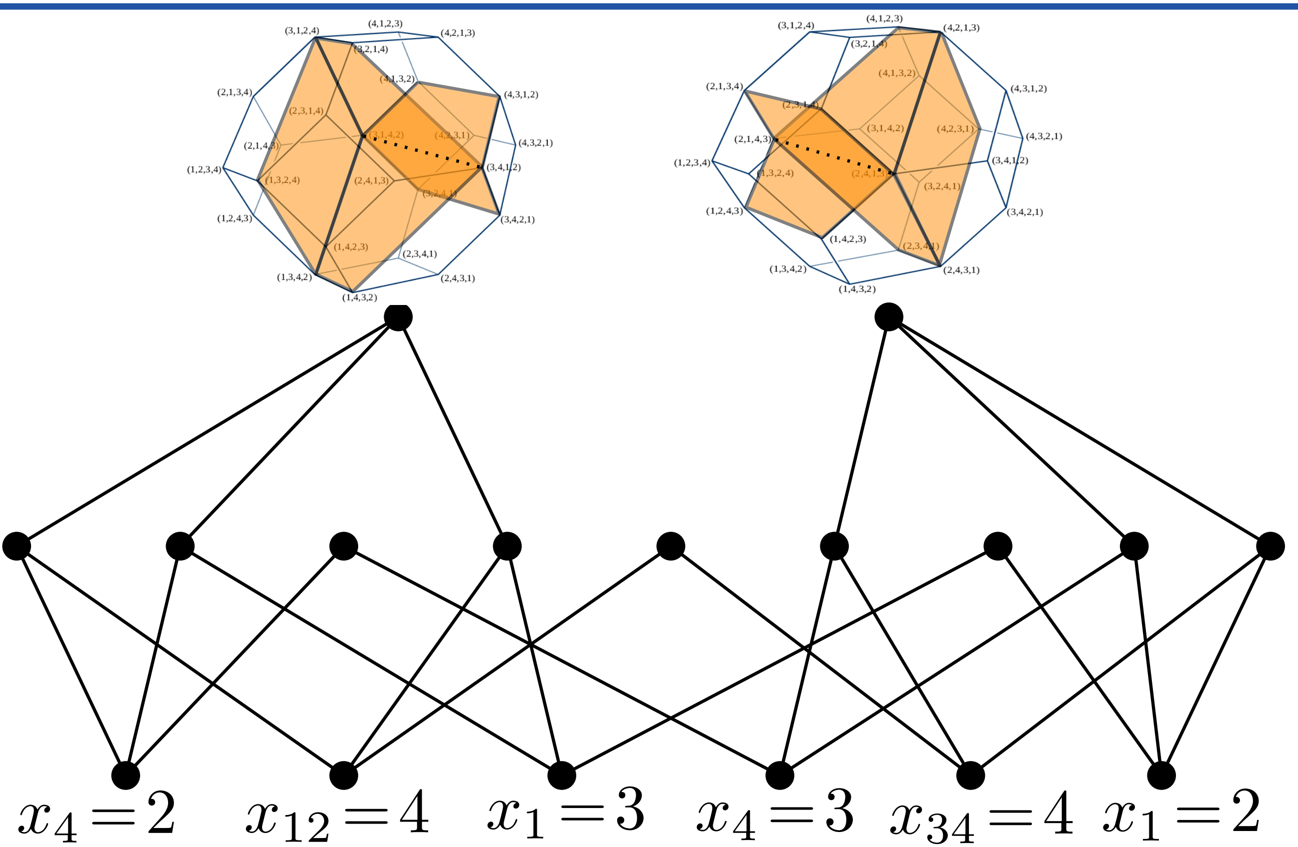

# Some current questions

∘ Let $F_1 \cup F_2$ be the subdivision of $Perm_n$ given by $x_i = a$. Deletion of $n$ in each constituent of $F_1$ and $F_2$ gives rise to the subdivision of $Perm_{n-1}$ via $x_i = a$.

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**Question:**

Does the subdivision given by $x_{ij} = a$ related to some operation(s) on matroids?.

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Question:

- Does the subdivision given by $x_{ij} = a$ related to some operation(s) on matroids?.

◦ A collection of hyperplanes is *compatible* if they give rise to an LPFM subdivision.

Questions:

- What are the compatible hyperplanes for $Perm_n$?.
- Are there more hyperplanes that give rise to LPFMs subdivisions?
- What are the finest subdivisions in LPFMs of $Perm_n$?
- What are their $f$-vectors?
- What weight vectors do LPFM subdivisions correspond to, as points in $Tr^{\geq 0}\mathcal{F}\ell_n$?

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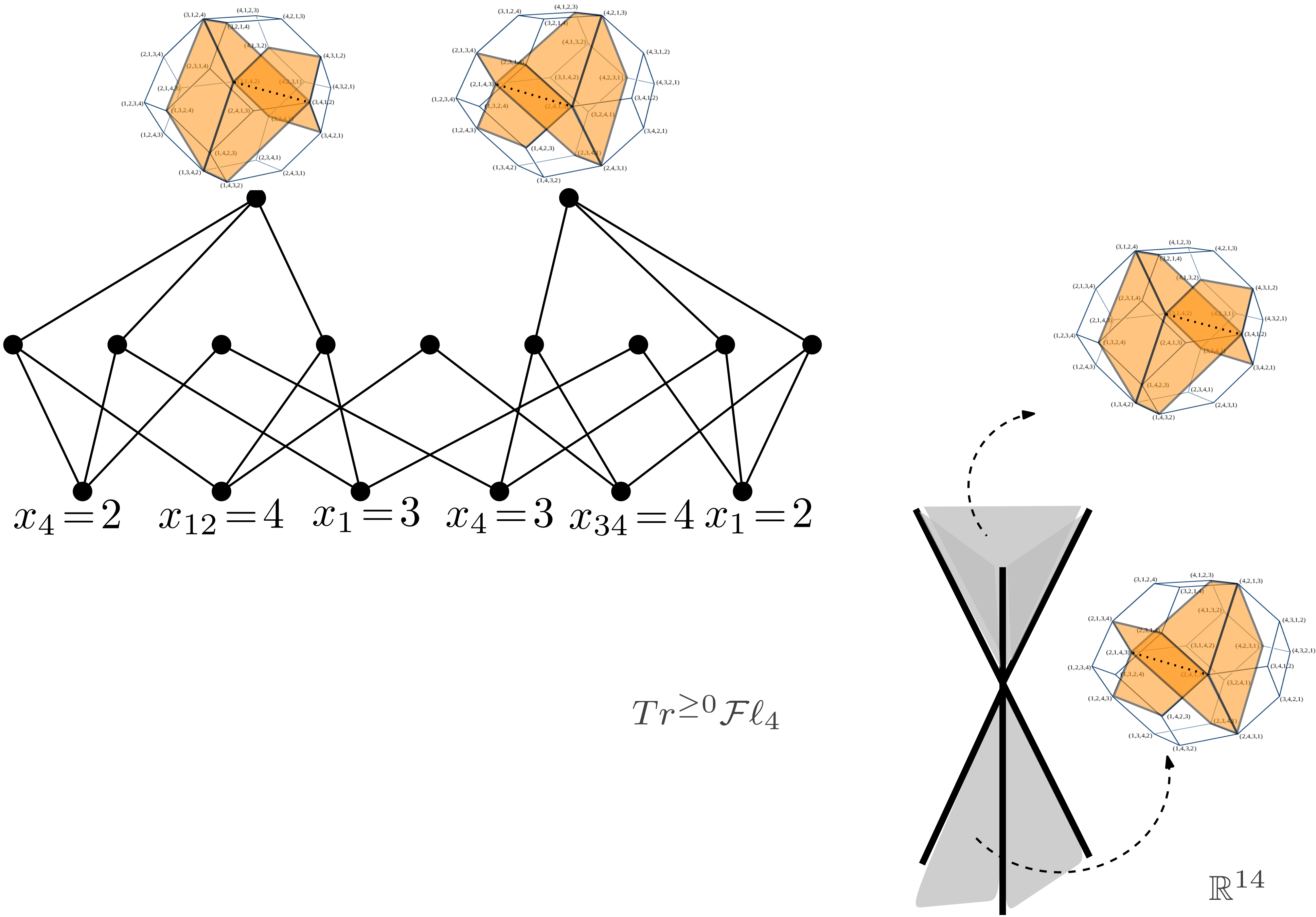
| Height function $(P_1,P_2,P_3,P_4;P_{12},P_{13},P_{14},P_{23},P_{24},P_{34};P_{123},P_{124},P_{134},P_{234})$ | Bruhat interval polytopes in subdivision | $f$-vector |
|---|---|---|
| $(-1,-1,-1,0;-1,-1,0,-1,0,0;0,0,0,0)$ | $P_{1243,4321}, P_{1234,4213}$ | $(24,39,18,2)$ |
| $(-1,-1,-1,0;0,0,0,0,0,0;0,0,0,0)$ | $P_{1342,4321}, P_{1234,4312}$ | |
| $(1,0,0,0;0,0,0,0,0,0;0,0,0,0)$ | $P_{2134,4321}, P_{1234,2431}$ | |
| $(1,0,0,0;0,0,0,1,1,1;0,0,0,0)$ | $P_{3124,4321}, P_{1234,3421}$ | |
| $(0,0,0,0;-1,-1,-1,-1,-1,0;0,0,0,0)$ | $P_{2413,4321}, P_{1234,4231}$ | $(24,40,19,2)$ |
| $(0,0,0,0;1,0,0,0,0,0;0,0,0,0)$ | $P_{1324,4321}, P_{1234,3142}$ | |
| $(-1,-1,0,0;-1,-1,-1,-1,-1,0;0,0,0,0)$ | $P_{1423,4321}, P_{1342,4231}, P_{1324,4213}, P_{1234,4132}$ | $(24,42,23,4)$ |
| $(0,-1,-1,0;0,0,1,0,0,0;0,0,0,0)$ | $P_{3142,4321}, P_{1243,3421}, P_{2134,4312}, P_{1234,2413}$ | |
| $(1,1,0,0;1,0,0,0,0,0;0,0,0,0)$ | $P_{2314,4321}, P_{1324,2431}, P_{3124,4231}, P_{1234,3241}$ | |

LPFMs coarsest subdivisions of $Perm_4$

**Polyhedral and Tropical Geometry of Flag Positroids**
Boretsky, Eur, Williams '22



$x_4=2$
$x_{12}=4$
$x_1=3$
$x_4=3$
$x_{34}=4$
$x_1=2$
$Tr^{\geq 0}\mathcal{F}\ell_4$
$\mathbb{R}^{14}$

# Acknowledgements


- Chris Eur: github code with $Tr^{\geq 0}\mathcal{F}\ell_5$
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- Johannes Rau: tropical mentor

Anastasia Chavez
St. Mary's College

Daniel Tamayo
Autobiz France

***Quotients of Uniform Positroids*** '22

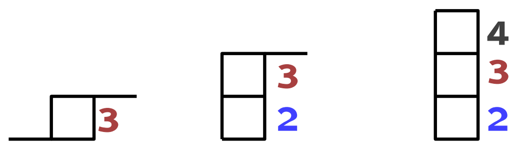
Kolja Knauer
U. of Barcelona

***LPM quotients*** '24

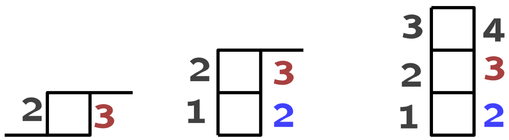
***LPFMs and subdivisions of*** $Perm_n$ >'24

Danke schön!

¡Gracias!




$$\begin{matrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 2\end{matrix}$$




| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 1 | 3 | 4 | 2 |

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 2 | 1 |

$$3 \subset 13 \subset 134 \rightsquigarrow e_3 + e_{13} + e_{134} = (3, 1, 4, 2)$$